JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **84**, 614–630 (1981)

On the Existence of Global Smooth Solutions for a Model Equation for Fluid Flow in a Pipe

MITCHELL LUSKIN*

Department of Mathematics, The University of Michigan, Ann Arbor, Michigan 48109

Submitted by C. S. Morawetz

We model the flow of a fluid in a pipe by a first-order nonlinear hyperbolic system with zero-order nonlinear dissipation. We prove that a unique, global smooth solution exists if the initial data are in an appropriate invariant region and if the first derivatives of the initial data are sufficiently small.

1. INTRODUCTION

We consider here the existence of a global smooth (continuous) solution to the initial value problem for the quasilinear hyperbolic system modeling fluid flow in a pipe

$$\begin{aligned} &\rho_t + G_x = 0, \qquad (x,t) \in \mathbb{R} \times [0, \infty), \\ &G_t + (Gv)_x + p_x = -\frac{f|G|\,G}{2D\rho}, \qquad (1.1) \\ &\rho(x,0) = \rho_0(x), \qquad G(x,0) = G_0(x), \qquad x \in \mathbb{R}, \end{aligned}$$

where ρ is mass density, G is momentum density, $v = G/\rho$ is velocity, $p = p(\rho)$ is pressure, D is pipe diameter, and f is the "Moody" friction factor [10, pp. 288–289]. The friction factor is a function of the Reynolds number, $\mathrm{Re} \equiv |G|\, D/\mu$, where μ is the viscosity. Since D and μ are to be treated as constants, we will consider f to be a function of $|G|$. We shall show that under realistic hypotheses on the equation of state, $p \equiv p(\rho)$, and the friction factor, $f \equiv f(|G|)$, the system (1.1) has global smooth solutions if the initial data lie in appropriate invariant regions and if the first derivatives of the initial data are sufficiently small. This result is in contrast to the situation when there is no friction term present in (1.1). In that case, it is now well-

* Supported by AFOSR under Contract F49620-79-C-0149 and by a visiting professorship at the École Polytechnique Fédérale, Lausanne, Switzerland.

0022-247X/81/120614-17$02.00/0
Copyright © 1981 by Academic Press, Inc.
All rights of reproduction in any form reserved.

known that discontinuities inevitably occur unless the initial data satisfy very restrictive conditions [6].

There is considerable practical interest in obtaining numerical approximations to the solution of (1.1). Knowing that the solution is smooth allows one to take advantage of efficient, high-order schemes which may be inappropriate for solutions with discontinuities. Previous analyses of numerical methods for (1.1) have assumed the existence of smooth solutions [3, 7]. In fact, the global existence of the approximate finite element solutions proposed in [3, 7] has been obtained by showing that the approximate solution is always in a neighborhood of a smooth solution to the differential problem.

Earlier work on the existence of global smooth solutions for quasilinear hyperbolic equations with lower-order dissipation has been done by Nishida [8]. Our work generalizes the form of the dissipation considered by Nishida. Nishida's theory allowed only linear dissipation, i.e., $-f|G|\,G/2D\rho$ in (1.1) is replaced by $-\hat{f}G$ where $\hat{f}$ is a positive constant. Since the "Moody" friction factor is discontinuous (multiple valued) when the flow changes from laminar to turbulent, we treat the case of discontinuous (multiple valued) $f(|G|)$.

Also, we are able to replace Nishida's requirement that the initial data lie in appropriate "sufficiently small" neighborhoods by the requirement that the initial data need only lie in an appropriate invariant region. This improvement in the analysis allows the application of our results to problems where the initial data can vary over ranges which are realistic for fluid flow problems of practical interest in engineering, such as the modeling of gas and oil pipelines.

2. Definitions and Statement of Theorem

The definitions and hypotheses for our theorem given in this section are illustrated by the example in Section 5. We assume that $p(\rho) \in C^2(\mathbb{R}^+)$. Furthermore, we guarantee the hyperbolicity of (1.1) by assuming that $p'(\rho) > 0$ for $\rho \in \mathbb{R}^+$ and we set $\gamma(\rho) = \sqrt{p'(\rho)}$.

We turn to the friction factor and set $F(G) \equiv f|G|\,G/2D$. For fixed $G_c > 0$ (the critical flow rate at which the transition from laminar to turbulent takes place), we assume that F is a maximal monotone function such that $F \in C^1((-\infty, -G_c]) \cap C^1([-G_c, G_c]) \cap C^1([G_c, \infty))$. More simply, F is a single valued, monotonic C^1 function on $\mathbb{R} - \{-G_c, G_c\}$. At $G = \pm G_c$, F is multiple valued and takes the values

$$F(\pm G_c) = [F(\pm G_c -), F(\pm G_c +)]. \tag{2.1}$$

We also assume that there exist positive constants δ and C_F, independent of G, such that

$$F(0)=0, \tag{2.2}$$

$$F' \geqslant \delta > 0, \tag{2.3}$$

$$\frac{F}{GF'}, \left|\frac{2F}{GF'}-1\right| \leqslant C_F < \infty. \tag{2.4}$$

We next introduce the Riemann invariants

$$w(\rho, G) \equiv G/\rho + \int_{\rho_*}^{\rho} \gamma(s)/s\, ds,$$

$$z(\rho, G) \equiv G/\rho - \int_{\rho_*}^{\rho} \gamma(s)/s\, ds$$

for a fixed $\rho_* \in \mathbb{R}^+$. We note that the map $(\rho, G) \to (w, z)$ is a C^2 diffeomorphism of $\mathbb{R}^+ \times \mathbb{R}$ onto a domain in $\mathbb{R}^2$. For a fixed positive constant M, we define the set

$$\Omega = \{(\rho, G) \mid |w(\rho, G)|, |z(\rho, G)| \leqslant M\}. \tag{2.5}$$

(See Fig. 1 in Section 5). We assume for the initial data that $\rho_0, G_0 \in C^1(\mathbb{R})$ and that

$$\{(\rho_0(x), G_0(x)) \mid x \in \mathbb{R}\} \subset \Omega.$$

Define the constants

$$\begin{aligned}
C_s &= \max_{\Omega} \frac{|v|}{\gamma}, \\
C_\gamma &= \max_{\Omega} \left|\frac{\gamma' \rho}{\gamma}\right|, \\
D_0 &= \sup_{x \in \mathbb{R}} \{|w(\rho_0, G_0)_x|, |z(\rho_0, G_0)_x|\}, \\
a &= \left[\min_{\Omega} \frac{F'}{2\rho}\right]^{-1}, \\
b &= C_F C_s C_\gamma + C_F C_s - 1, \\
c &= D_0 + 3 \max_{\Omega} \left[\frac{F}{2G\rho} \cdot \frac{|v|}{\gamma}\right].
\end{aligned} \tag{2.6}$$

We assume for simplicity that $C_\gamma \leqslant 1$. This hypothesis is valid for most real problems.

THEOREM. *Suppose that* $b < 0$ *and* $b^2 - 4ac > 0$. *Then there exists a unique solution* $\rho, G \in W^{1,\infty}(\mathbb{R} \times \mathbb{R}^+)$ *such that*

$$\begin{aligned} &\rho_t + G_x = 0, && (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\ &G_t + (Gv)_x + p(\rho)_x \in F(G)/\rho, && \\ &\rho(x, 0) = \rho_0(x), \quad G(x, 0) = G_0(x), && x \in \mathbb{R}. \end{aligned} \tag{2.7}$$

We note that since $F(G)$ is a discontinuous function, we cannot expect the solution ρ, G of (2.7) to be in $C^1(\mathbb{R} \times \mathbb{R}^+)$.

3. Bounds for Smooth Solutions

In this section we assume that $F \in C^1(\mathbb{R})$. We first generalize a result of Dupont [4] for semilinear equations to show that the sets Ω are "invariant regions" for smooth solutions. We note that Chueh, Conley, and Smoller [1] have shown that discontinuous solutions must lie in certain unbounded invariant regions which also have a boundary defined by constant values of Riemann invariants. A simple calculation shows that (ρ, G) is a $C^1(\mathbb{R} \times [0, T])$ solution of (1.1) if and only if (w, z) is a $C^1(\mathbb{R} \times [0, T])$ solution of

$$\begin{aligned} w_t + \lambda w_x &= -F/\rho^2, \qquad (x, t) \in \mathbb{R} \times [0, T], \\ z_t + \nu z_x &= -F/\rho^2, \end{aligned} \tag{3.1}$$

where $\lambda = v + \gamma$, $\nu = v - \gamma$.

LEMMA 1. *If* (w, z) *is a* $C^1(\mathbb{R} \times [0, T])$ *solution of* (3.1), *then*

$$\sup_{(x,t) \in \mathbb{R} \times [0,T]} \{|w(x, t)|, |z(x, t)|\} \leqslant \sup_{x \in \mathbb{R}} \{|w(x, 0)|, |z(x, 0)|\}. \tag{3.2}$$

Proof. We define the differential operators

$$\frac{d}{d\lambda} = \frac{\partial}{\partial t} + \lambda \frac{\partial}{\partial x} \tag{3.3}$$

and

$$\frac{d}{d\nu} = \frac{\partial}{\partial t} + \nu \frac{\partial}{\partial x}. \tag{3.4}$$

We note that since $v = G/\rho = (w + z)/2$, we have that

$$\begin{aligned} \frac{dw}{d\lambda} &= -\sigma(w + z), \\ \frac{dz}{dv} &= -\sigma(w + z), \end{aligned} \tag{3.5}$$

where $\sigma = F/(2G\rho) \in C^1(\mathbb{R} \times [0, T])$ is positive.

For $x_0 \in \mathbb{R}$ and $t_0 > 0$ define the characteristic curves $x^{\lambda}(t, x_0, t_0)$ and $x^{v}(t, x_0, t_0)$ by

$$\frac{\partial}{\partial t} x^{\lambda} = \lambda, \qquad x^{\lambda}(t_0, x_0, t_0) = x_0, \tag{3.6}$$

and

$$\frac{\partial}{\partial t} x^{v} = v, \qquad x^{v}(t_0, x_0, t_0) = x_0. \tag{3.7}$$

Also, define the sets (for fixed x_0 and t_0)

$$\Gamma = \{(x, t) \mid x^{\lambda}(t, x_0, t_0) \leqslant x \leqslant x^{v}(t, x_0, t_0)\}$$

and for $0 \leqslant t \leqslant t_0$

$$\Gamma_t = \{(x, s) \in \Gamma \mid 0 \leqslant s \leqslant t\}.$$

Let

$$M(t) = \max_{\Gamma_t} \{w(x, s), z(x, s)\}.$$

We note that $M(t)$ is a nondecreasing function since the sets Γ_t are growing. We shall show that $M(t) \leqslant M(0)$; hence, $M(t)$ is constant.

If we integrate (3.5) along the characteristic curve x^{λ} we obtain for $(x, s) \in \Gamma$

$$\begin{aligned} w(x, s) = {} & \exp(-A(s, x, s))\, w(x^{\lambda}(0, x, s), 0) \\ & + \int_0^s \exp(-A(s, x, s) + A(\tau, x, s)) \\ & \times \sigma(x^{\lambda}(\tau, x, s), \tau)\, z(x^{\lambda}(\tau, x, s), \tau)\, d\tau, \end{aligned} \tag{3.8}$$

where

$$A(\tau, x, s) = \int_0^{\tau} \sigma(x^{\lambda}(\zeta, x, s), \zeta)\, d\zeta.$$

Hence

$$w(x, s) \leqslant \exp(-A(s, x, s))\, M(0) + [1 - \exp(-A(s, x, s))]\, M(s). \tag{3.9}$$

Now suppose that $(x, s) \in \Gamma_t$ and

$$w(x, s) = M(t). \tag{3.10}$$

Then from (3.9) we obtain

$$M(t) \leqslant \exp(-A(s, x, s))\, M(0) + [1 - \exp(-A(s, x, s))]\, M(t). \tag{3.11}$$

Hence, since $\exp(-A(s, x, s)) > 0$, we can conclude that $M(t) \leqslant M(0)$. A similar argument applies when $\max_{\Gamma_t} z > \max_{\Gamma_t} w$. The proof is completed by applying the above arguments to $(-w, -z)$. Q.E.D.

Now define

$$D(t) = \sup_{\mathbb{R} \times [0,\, t]} \{|w_x(x, s)|, |z_x(x, s)|\}.$$

We can derive the following bound for $D(t)$.

LEMMA 2. *If (w, z) is a $C^1(\mathbb{R} \times [0, t])$ solution of* (3.1), *then*

$$D(t) \leqslant c + (b + 1)\, D(t) + aD(t)^2. \tag{3.12}$$

Proof. We note that as a result of Lemma 1 we can assume that $\{(w(x, s), z(x, s)) \mid (x, s) \in \mathbb{R} \times [0, t]\} \subset \Omega$. Let $k = F'/2\rho$. Since $w \in C^1(\mathbb{R} \times [0, t])$, it follows from (3.1) that $dw_x/d\lambda \in C(\mathbb{R} \times [0, t])$ and $dw_x/d\lambda = -\lambda_x w_x - (F/\rho^2)_x$.

We show in the Appendix that

$$(F/\rho^2)_x = kw_x + \frac{d}{d\lambda}\left(\frac{F}{2G\rho} \cdot \frac{v}{\gamma}\right) + \frac{kF}{G\rho} \cdot \frac{v}{\gamma} - \frac{kF}{GF'} \cdot \frac{v}{\gamma} \cdot \frac{\gamma'\rho}{\gamma} \cdot z_x - k\left[\frac{2F}{GF'} - 1\right] \frac{v}{\gamma} w_x \tag{3.13}$$

so it follows that

$$\frac{d}{d\lambda} w_x + kw_x = -\frac{d}{d\lambda}\left(\frac{F}{2G\rho} \cdot \frac{v}{\gamma}\right) - \frac{kF}{G\rho} \cdot \frac{v}{\gamma} + \frac{kF}{GF'} \cdot \frac{v}{\gamma} \frac{\gamma'\rho}{\gamma} \cdot z_x + k\left[\frac{2F}{GF'} - 1\right] \frac{v}{\gamma} w_x - k\left[\frac{\lambda_x}{k} w_x\right].$$

We then integrate along the λ-characteristic curve to obtain the integral equation for w_x,

$$\begin{aligned} w_x(x,t) &= e^{-\int_0^t k\,d\tau} w_x(x^\lambda(0,x,t),0) \\ &\quad - \left[\frac{F}{2G\rho}\cdot\frac{v}{\gamma}(x,t) - e^{-\int_0^t k\,d\tau}\frac{F}{2G\rho}\cdot\frac{v}{\gamma}(x^\lambda(0,x,t),0)\right] \\ &\quad - \int_0^t e^{-\int_\tau^t k\,ds}\, k\frac{F}{2G\rho}\cdot\frac{v}{\gamma}\,d\tau \\ &\quad + \int_0^t e^{-\int_\tau^t k\,ds}\, k\frac{F}{GF'}\cdot\frac{v}{\gamma}\left(\frac{\gamma'\rho}{\gamma}\right) z_x\,d\tau \\ &\quad + \int_0^t e^{-\int_\tau^t k\,ds}\, k\left[\frac{2F}{GF'}-1\right]\frac{v}{\gamma} w_x\,d\tau \\ &\quad - \int_0^t e^{-\int_\tau^t k\,ds}\, k\left[\frac{\lambda_x}{k} w_x\right] d\tau. \end{aligned} \tag{3.14}$$

Now $\lambda_x = \lambda_w w_x + \lambda_z z_x$. Since (see Appendix)

$$\lambda_w = \frac{1}{2} + \frac{1}{2}\frac{\gamma'\rho}{\gamma}, \qquad \lambda_z = \frac{1}{2} - \frac{1}{2}\frac{\gamma'\rho}{\gamma}, \tag{3.15}$$

it follows from the hypothesis $C_\gamma \leqslant 1$ that

$$|\lambda_x| \leqslant D(t). \tag{3.16}$$

Also, note that

$$\int_0^t e^{-\int_\tau^t k\,ds} k\,d\tau = 1 - e^{-\int_0^t k\,d\tau} \leqslant 1. \tag{3.17}$$

The estimate (3.12) now follows easily from repeated use of the triangle inequality. Q.E.D.

LEMMA 3. *Suppose that $b<0$ and $b^2-4ac>0$. Then a $C^1(\mathbb{R}\times\mathbb{R}^+)$ solution to* (1.1) *exists which satisfies the estimate*

$$D(t) \leqslant \frac{-b-\sqrt{b^2-4ac}}{2a}, \qquad t\in\mathbb{R}^+. \tag{3.18}$$

Proof. The local existence theorem of Douglis [2] and Hartman and Winter [5] guarantees that a $C^1(\mathbb{R}\times[0,t_1])$ solution to (3.1) exists for some $t_1>0$ and allows us to continue the C^1 solution as long as uniform bounds on the solution and its first derivatives hold. Uniform bounds on the solution follow from Lemma 1.

Let $L(y) = ay^2 + by + c$. Since $b < 0$ and $b^2 - 4ac > 0$, it follows that if $L(y) \geqslant 0$ and $y \leqslant -b/2a$, then $y \leqslant -b/2a - \sqrt{b^2 - 4ac}/2a$. Now

$$D(0) = D_0 \leqslant c < \frac{b^2}{4a} = -\frac{b}{2a}\left(-\frac{b}{2}\right) < -\frac{b}{2a}$$

and $L(D(0)) \geqslant 0$. Also, by Lemma 2, $L(D(t)) \geqslant 0$ for t such that a $C^1(\mathbb{R} \times [0, t])$ solution exists. Hence, it follows by the continuity of $D(t)$ that the uniform bound

$$D(t) \leqslant -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

is valid as long as the C^1 solution exists. Thus, we can conclude from the local existence theorem that a $C^1(\mathbb{R} \times \mathbb{R}^+)$ solution exists. Q.E.D.

4. Proof of Theorem

We assume again that F is a maximal monotone function such that $F \in C^1((-\infty, -G_c]) \cap C^1([-G_c, G_c]) \cap C^1([G_c, \infty))$. Let $J \in C_0^\infty((-1, 1))$ be such that $J \geqslant 0$ and $\int_{-1}^{1} J(Y)\, dY = 1$, and set $J_n(Y) = nJ(nY)$ for $n \in \mathbb{N}$. We can then define the functions $F_n(G) \in C^1(\mathbb{R})$ by

$$F_n(0) = 0,$$

$$F_n'(G) = F'(G) + \alpha J_n(G - G_c) + \beta J_n(G + G_c), \qquad G \neq \pm G_c,$$

where

$$\alpha = F(G_c+) - F(G_c-),$$

$$\beta = F(-G_c+) - F(-G_c-).$$

It is easily checked that $F_n' \geqslant \delta$ and that constants C_{F_n} (corresponding to C_F) can be found so that $C_{F_n} \to C_F$. It also follows that the constants a_n, b_n, c_n (corresponding to a, b, c) satisfy $a_n \to a$, $b_n \to b$, $c_n \to c$. Let (ρ_n, G_n) be the solution of (1.1) corresponding to F_n. Then Lemma 3 guarantees that for n sufficiently large there exist $C^1(\mathbb{R} \times \mathbb{R}^+)$ solutions (ρ_n, G_n) to (1.1) corresponding to F_n. Furthermore, Lemmas 1 and 3 give uniform bounds for the values and first derivatives of (ρ_n, G_n) which are independent of n.

Hence, we can conclude from the Ascoli–Arzelà theorem that there exist functions $\rho, G \in C(\mathbb{R} \times \mathbb{R}^+)$ such that $\rho_{n_j} \to \rho$, $G_{n_j} \to G$ uniformly on compact subsets of $\mathbb{R} \times \mathbb{R}^+$ for a subsequence $\{n_j\}$. Also, the Banach–Alaoglu theorem implies that we may also assume that $\rho_{n_j} \rightharpoonup \rho$, $G_{n_j} \rightharpoonup G$ weakly in $W^{1,\infty}(\mathbb{R} \times \mathbb{R}^+)$. It is straightforward to show that

$$\rho_t + G_x = 0 \qquad \text{a.e.} \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+.$$

In order to demonstrate that

$$G_t + (Gv)_x + p(\rho)_x \in -\frac{F(G)}{\rho} \qquad \text{a.e.} \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+,$$

it is sufficient to show that $F_{n_j}(G_{n_j}) \rightharpoonup \sigma$ weakly in $L^\infty(\mathbb{R} \times \mathbb{R}^+)$ where $\sigma \in F(G)$ a.e. $(x, t) \in \mathbb{R} \times \mathbb{R}^+$.

We follow an argument due to Rauch [9]. Define the functions

$$\bar{F}_\epsilon(G) = \operatorname*{ess\,sup}_{|G-Y|<\epsilon} F(Y),$$

$$\underline{F}_\epsilon(G) = \operatorname*{ess\,inf}_{|G-Y|<\epsilon} F(Y).$$

Let K be a compact subset of $\mathbb{R} \times \mathbb{R}^+$. Then for any $\varepsilon > 0$ there is an $n_0 > 4/\varepsilon$ such that $n \geqslant n_0$ implies that $|G(x,t) - G_n(x, t)| < \varepsilon/2$ for $(x, t) \in K$. Therefore,

$$\begin{aligned} \underline{F}_\epsilon(G(x, t)) &\leqslant \underline{F}_{\epsilon/2}(G_n(x, t)) \leqslant F_n(G_n(x, t)) \\ &\leqslant \bar{F}_{\epsilon/2}(G_n(x, t)) \leqslant \bar{F}_\epsilon(G(x, t)) \qquad (x, t) \in K. \end{aligned}$$

Hence, if $h \in L^1(K)$, $h \geqslant 0$, we have

$$\int_K \underline{F}_\epsilon(G)\, h\, dx\, dt \leqslant \int_K F_n(G_n)\, h\, dx\, dt \leqslant \int_K \bar{F}_\epsilon(G)\, h\, dx\, dt.$$

Let $n \to \infty$ and use the weak convergence of $F_n(G_n)$ to obtain

$$\int_K \underline{F}_\epsilon(G)\, h\, dx\, dt \leqslant \int_K \sigma h\, dx\, dt \leqslant \int_K \bar{F}_\epsilon(G)\, h\, dx\, dt.$$

Next, use Lebesgue's theorem above to let $\varepsilon \to 0$ and conclude that

$$\int_K \underline{F}(G)\, h\, dx\, dt \leqslant \int_K \sigma h\, dx\, dt \leqslant \int_K \bar{F}(G)\, h\, dx\, dt,$$

where

$$\underline{F}(G) = \lim_{\epsilon \to 0} \underline{F}_\epsilon(G) = F(G-), \qquad \bar{F}(G) = \lim_{\epsilon \to 0} \bar{F}_\epsilon(G) = F(G+).$$

Since $h \geqslant 0$ was arbitrary, we can conclude that

$$\underline{F}(G) \leqslant \sigma \leqslant \bar{F}(G) \qquad \text{a.e.} \quad (x, t) \in K.$$

This concludes the proof of existence.

We now turn to the proof of uniqueness. So, let (ρ_1, G_1) and (ρ_2, G_2) be two solutions of (2.7) in $W^{1,\infty}(\mathbb{R} \times \mathbb{R}^+)$, i.e.,

$$\begin{aligned} &\rho_{i,t} + G_{i,x} = 0 \qquad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\ &G_{i,t} + (\gamma(\rho_i)^2 - v_i^2)\, \rho_{i,x} + 2v_i G_{i,x} = \sigma_i/\rho_i, \\ &\rho_i(x, 0) = \rho_0(x), \qquad G_i(x, 0) = G_0(x), \qquad x \in \mathbb{R}, \end{aligned} \tag{4.1}$$

where $\sigma_i \in F(G_i)$ a.e. $(x, t) \in \mathbb{R} \times \mathbb{R}^+$. Letting $\tilde{\rho} = \rho_1 - \rho_2$, $\tilde{G} = G_1 - G_2$, we obtain from (4.1)

$$\begin{aligned} &\tilde{\rho}_t + \tilde{G}_x = 0 \qquad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\ &\tilde{G}_t + (\gamma(\rho_1)^2 - v_1^2)\, \tilde{\rho}_x + 2v_1 \tilde{G}_x \\ &\quad = [(\gamma(\rho_2)^2 - v_2^2) - (\gamma(\rho_1)^2 - v_1^2)]\, \rho_{2,x} + 2(v_1 - v_2)\, G_{2,x} \\ &\quad\quad - \rho_1^{-1}[\sigma_1 - \sigma_2] - [\rho_1^{-1} - \rho_2^{-1}]\, \sigma_2, \\ &\tilde{\rho}(x, 0) = 0, \qquad \tilde{G}(x, 0) = 0. \end{aligned} \tag{4.2}$$

Multiply the first equation in (4.2) by $(\gamma(\rho_1)^2 - v_1^2)\, \tilde{\rho}$ and integrate over $[-y, y]$ for $y \in \mathbb{R}^+$ to obtain

$$\int_{-y}^{y} \tilde{\rho}_t \tilde{\rho} [\gamma(\rho_1)^2 - v_1^2]\, dx + \int_{-y}^{y} \tilde{G}_x \tilde{\rho} [\gamma(\rho_1)^2 - v_1^2]\, dx = 0.$$

Hence, we obtain from integration by parts that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{-y}^{y} \tilde{\rho}^2 [\gamma(\rho_1)^2 - v_1^2]\, dx - \int_{-y}^{y} \tilde{G} \tilde{\rho}_x [\gamma(\rho_1)^2 - v_1^2]\, dx \\ &\quad \leqslant \kappa \left[\int_{-y}^{y} \tilde{\rho}^2\, dx + \int_{-y}^{y} \tilde{G}^2\, dx \right] - \tilde{G}\tilde{\rho}[\gamma(\rho_1)^2 - v_1^2] \Big|_{-y}^{y}, \end{aligned} \tag{4.3}$$

where κ is a constant independent of t (since $(\rho_i, G_i) \in W^{1,\infty}(\mathbb{R} \times \mathbb{R}^+)$).

Next, we obtain from multiplying the second equation in (4.2) by $\tilde{G}$ and integrating over $[-y, y]$ that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{-y}^{y} \tilde{G}^2\, dx + \int_{-y}^{y} [\gamma(\rho_1)^2 - v_1^2]\, \tilde{\rho}_x \tilde{G}\, dx + \int_{-y}^{y} 2v_1 \tilde{G}_x \tilde{G}\, dx \\ &\quad \leqslant \kappa \left[\int_{-y}^{y} \tilde{\rho}^2\, dx + \int_{-y}^{y} \tilde{G}^2\, dx \right] - \int_{-y}^{y} \rho_1^{-1}\, [\sigma_1 - \sigma_2]\, \tilde{G}\, dx. \end{aligned} \tag{4.4}$$

It follows from integration by parts that

$$-\int_{-y}^{y} 2v_1 \tilde{G}_x \tilde{G}\, dx = \int_{-y}^{y} v_{1,x} \tilde{G}^2\, dx - v_1 \tilde{G}^2 \Big|_{-y}^{y}.$$

Also, it follows from the monotonicity of F that

$$[\sigma_1 - \sigma_2]\, \tilde{G} \geqslant 0.$$

Hence, we obtain from adding (4.3) and (4.4)

$$\frac{d}{dt}\left[\int_{-y}^{y} \tilde{\rho}^2 [\gamma(\rho_1)^2 - v_1^2]\, dx + \int_{-y}^{y} \tilde{G}^2\, dx\right]$$
$$\leqslant \kappa \left[\int_{-y}^{y} \tilde{\rho}^2\, dx + \int_{-y}^{y} \tilde{G}^2\, dx\right] - [\tilde{G}\tilde{\rho}[\gamma(\rho_1)^2 - v_1^2] + \tilde{G}^2 v_1]\Big|_{-y}^{y}. \quad (4.5)$$

Now recall that the hypothesis $b < 0$ implies that $\gamma(\rho_1)^2 - v_1^2$ is positive and bounded away from zero. Also, note that the hypothesis $(\rho_i, G_i) \in W^{1,\infty}(\mathbb{R} \times \mathbb{R}^+)$ guarantees that the boundary term in (4.5) is bounded independent of $y, t \in \mathbb{R}^+$. Hence, we can conclude from (4.5) and Gronwall's lemma that for $T > 0$ we have $(\tilde{\rho}, \tilde{G}) \in L^\infty(0, T; L^2(\mathbb{R}))$.

Next, since $(\tilde{\rho}, \tilde{G}) \in W^{1,\infty}(\mathbb{R} \times \mathbb{R}^+) \cap L^\infty(0, T; L^2(\mathbb{R}))$, we can conclude that

$$\tilde{G}\tilde{\rho}[\gamma(\rho_1)^2 - v_1^2] + \tilde{G}^2 v_1 \,|_{-y}^{y} \to 0 \quad (4.6)$$

as $y \to +\infty$ uniformly for $t \in [0, T]$. Thus, since $\tilde{\rho}(x, 0) = \tilde{G}(x, 0) = 0$, it follows from (4.5), (4.6), and Gronwall's lemma that $\tilde{\rho} = \tilde{G} = 0$ for $(x, t) \in \mathbb{R} \times [0, T]$. Finally, $\tilde{\rho} = \tilde{G} = 0$ in $\mathbb{R} \times \mathbb{R}^+$ since $T > 0$ was arbitrary.

Q.E.D.

5. An Example

In this section, we present an example with a hypothetical friction factor which has been constructed to resemble the Moody friction factor [10]. First, we assume that $\gamma(\rho)$ is a constant and consider the structure of the invariant regions, Ω (see Fig. 1). It is easy to check that the curve $z = M$ can be represented by the function $G(\rho)$ where $G' = \gamma + v$. Similarly, the curve $w = M$ is represented by the function $G(\rho)$ where $G' = -\gamma + v$. For this example, $C_\gamma = 0$ and $C_s = \max_\Omega(|v|/\gamma) = M/\gamma = G_*/(\rho_* \gamma)$.

Next, we consider the friction factor

$$F(G) = f_0 G, \qquad |G| < G_c$$
$$= f_0 |G|\, G/G_c, \qquad |G| > G_c,$$

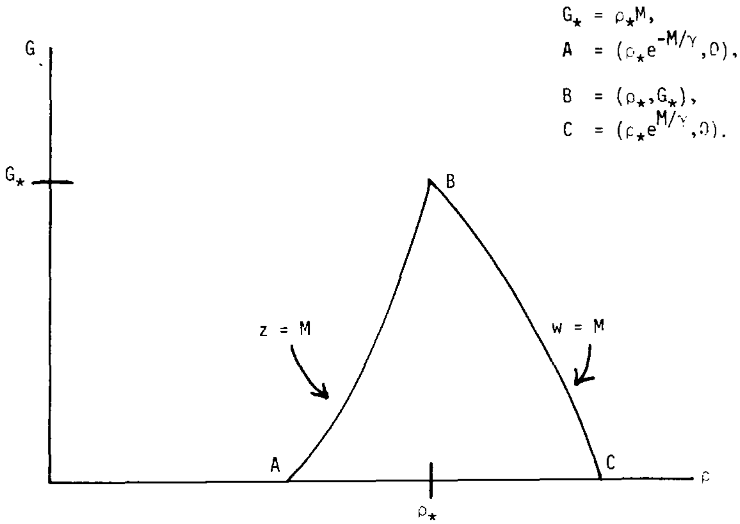

FIGURE 1

where f_0 is a positive constant. A calculation establishes that

$$\begin{aligned} F' &= f_0, & |G| &< G_c \\ &= 2f_0 |G|/G_c, & |G| &> G_c. \end{aligned}$$

Thus, $\delta = f_0$. Also,

$$\begin{aligned} \frac{F}{GF'} &= 1, & |G| &< G_c \\ &= \frac{1}{2}, & |G| &> G_c, \end{aligned}$$

so $C_F = 1$.

We now wish to determine regions Ω so that $b < 0$ and

$$6 \max_{\Omega} \frac{F}{G\rho} \cdot \frac{|v|}{\gamma} < b^2 \min_{\Omega} \frac{F'}{2\rho}. \tag{5.1}$$

It follows from our theorem that if the initial data lie in such a region and if the first derivatives of the initial data are sufficiently small, then a unique global smooth solution exists.

A calculation establishes that the maximum of $(F/G\rho) \cdot (v/\gamma)$ in Ω is attained at (ρ_*, G_*) and

$$\max_\Omega \frac{F}{G\rho}\cdot\frac{v}{\gamma}=\frac{f_0 C_s}{\rho_*} \qquad \text{if} \quad G_* < G_c$$

$$=\frac{f_0 C_s}{\rho_*}\cdot\frac{G_*}{G_c} \qquad \text{if} \quad G_* > G_c.$$

To see this, note that

$$\frac{F}{G\rho}\cdot\frac{v}{\gamma}=\frac{f_0}{\gamma}\frac{v}{\rho}=\frac{f_0 G}{\gamma\rho^2} \qquad \text{if} \quad G < G_c$$

$$=\frac{f_0}{\gamma G_c}v^2=\frac{f_0}{\gamma G_c}\frac{G^2}{\rho^2} \qquad \text{if} \quad G > G_c.$$

Thus, it is easy to see that the maximum of $(F/G\rho)\cdot(v/\gamma)$ on Ω lies on the curve $z=M$. Now if $z=M$ is represented by the function $G(\rho)$ such that $G'=\gamma+v$, we see that

$$\left(\frac{f_0 G(\rho)}{\gamma\rho^2}\right)'=\frac{f_0}{\gamma}\left[\frac{\rho G'-G}{\rho^3}\right]=\frac{f_0}{\gamma}\left[\frac{\rho\gamma}{\rho^3}\right]>0,$$

$$\left(\frac{f_0}{\gamma G_c}\frac{G^2}{\rho^2}\right)'=\frac{f_0}{\gamma G_c}\left[\frac{2\rho G G'-G^2}{\rho^3}\right]=\frac{f_0}{\gamma G_c}\left[\frac{2\rho G\gamma}{\rho^3}\right]>0.$$

Hence, the maximum is taken at the point $B=(\rho_*, G_*)$. Also, it is easy to check that the minimum of $F'/2\rho$ in Ω is attained at $(\rho_* e^{M/\gamma}, 0)$ and

$$\min_\Omega \frac{F'}{2\rho}=\frac{f_0}{2\rho_*}e^{-C_s}.$$

Thus, (5.1) is equivalent to

$$1<\frac{(1-C_s)^2}{12C_s}e^{-C_s} \qquad \text{if} \quad G_* < G_c$$

and

$$\frac{\gamma\rho_*}{G_c}<\frac{(1-C_s)^2}{12C_s^2}e^{-C_s} \qquad \text{if} \quad G_* > G_c.$$

So, consider the curve defined by the relation

$$1=\frac{(1-C_s)^2}{12C_s}e^{-C_s} \qquad \text{if} \quad G < G_c$$

and

$$\frac{\gamma\rho}{G_c}=\frac{(1-C_s)^2}{12C_s^2}e^{-C_s} \qquad \text{if} \quad G > G_c.$$

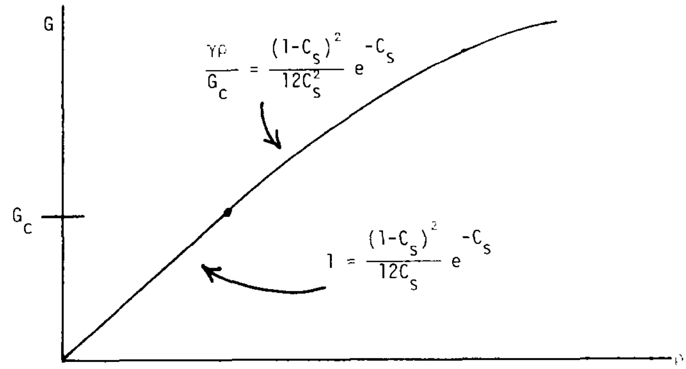

FIGURE 2

Our theorem asserts that if the initial data lie in an invariant region Ω such that the point $B = (\rho_*, \rho_* M)$ lies below this curve and if the first derivatives of the initial data are sufficiently small, then a unique global smooth solution exists.

APPENDIX

PROPOSITION 1. *If* $F \in C^1(\mathbb{R})$ *and* $\rho, G \in C^1(\mathbb{R} \times [0, t])$, *then*

$$\left(\frac{F}{\rho^2}\right)_x = kw_x + \frac{d}{d\lambda}\left(\frac{F}{2G\rho} \cdot \frac{v}{\gamma}\right) + \frac{kF}{G\rho} \cdot \frac{v}{\gamma}$$
$$- \frac{kF}{GF'} \cdot \frac{v}{\gamma} \cdot \frac{\gamma'\rho}{\gamma} \cdot z_x - k\left[\frac{2F}{GF'} - 1\right]\frac{v}{\gamma} w_x. \tag{6.1}$$

Proof. We have

$$\left(\frac{F}{\rho^2}\right)_x = \frac{F'G_x}{\rho^2} - \frac{2F}{\rho^3}\rho_x = F'\left[\frac{\rho v_x + \rho_x v}{\rho^2}\right] - \frac{2F}{\rho^3}\rho_x. \tag{6.2}$$

Now

$$\frac{w-z}{2} = \int_{\rho^*}^{\rho} \frac{\gamma(s)}{s}\, ds, \qquad \frac{w+z}{2} = v$$

so

$$\frac{(w-z)_x}{2} = \frac{\gamma}{\rho}\rho_x, \qquad \frac{(w+z)_x}{2} = v_x. \tag{6.2}$$

Hence, we can use (6.2) in (6.1) to obtain (recall that $k = F'/(2\rho)$),

$$\left(\frac{F}{\rho^2}\right)_x = kw_x + kz_x - \left[F' - \frac{2F}{G}\right]\frac{v}{\rho\gamma}\frac{(z-w)_x}{2}. \tag{6.3}$$

Next, note that

$$\frac{dz}{d\lambda} = z_t + \lambda z_x = \frac{dz}{dv} + (\lambda - v)\, z_x = -\frac{F}{\rho^2} + 2\gamma z_x, \tag{6.4}$$

so

$$\begin{aligned} z_x &= \frac{1}{2\gamma}\frac{dz}{d\lambda} + \frac{F}{2\gamma\rho^2} = \frac{1}{2\gamma}\frac{d}{d\lambda}(z+w) - \frac{1}{2\gamma}\frac{dw}{d\lambda} + \frac{F}{2\gamma\rho^2} \\ &= \frac{1}{\gamma}\frac{dv}{d\lambda} + \frac{F}{\gamma\rho^2} = \frac{1}{\gamma}\frac{dv}{d\lambda} + \frac{F}{G\rho}\frac{v}{\gamma}. \end{aligned} \tag{6.5}$$

Thus,

$$\begin{aligned} \left(\frac{F}{\rho^2}\right)_x &= kw_x + \frac{k}{\gamma}\frac{dv}{d\lambda} + \frac{kF}{G\rho}\cdot\frac{v}{\gamma} - \left[F' - \frac{2F}{G}\right]\frac{v}{\rho\gamma}\frac{z_x}{2} \\ &\quad - k\left[\frac{2F}{GF'} - 1\right]\frac{v}{\gamma}w_x. \end{aligned} \tag{6.6}$$

Now

$$\frac{dv}{d\lambda} = \frac{d}{d\lambda}(G/\rho) = \frac{1}{\rho}\frac{dG}{d\lambda} - \frac{G}{\rho^2}\frac{d\rho}{d\lambda}.$$

However, by (6.2),

$$\begin{aligned} \frac{d\rho}{d\lambda} &= \rho_t + \lambda\rho_x = -G_x + \lambda\rho_x = -\rho v_x - \rho_x v + \lambda\rho_x \\ &= \rho_x(\lambda - v) - \rho v_x = \rho_x\gamma - \rho v_x \\ &= \frac{\rho}{2}(w-z)_x - \frac{\rho}{2}(w+z)_x = -\rho z_x. \end{aligned} \tag{6.7}$$

Thus,

$$\frac{dv}{d\lambda} = \frac{1}{\rho}\frac{dG}{d\lambda} + \frac{G}{\rho}z_x$$

and by (6.7)

$$\begin{aligned}
\frac{k}{\gamma}\frac{dv}{d\lambda} &= \frac{F'}{2\rho^2\gamma}\frac{dG}{d\lambda}+\frac{kv}{\gamma}z_x \\
&= \frac{1}{2\rho^2\gamma}\frac{d}{d\lambda}F+\frac{kv}{\gamma}z_x \\
&= \frac{d}{d\lambda}\left(\frac{F}{2\rho^2\gamma}\right)+\frac{F}{(2\rho^2\gamma)^2}\frac{d}{d\lambda}(2\rho^2\gamma)\frac{kv}{\gamma}z_x \\
&= \frac{d}{d\lambda}\left(\frac{F}{2G\rho}\cdot\frac{v}{\gamma}\right)+\frac{F}{(2\rho^2\gamma)^2}\left[4\rho\gamma\frac{d\rho}{d\lambda}+2\rho^2\gamma'\frac{d\rho}{d\lambda}\right]+\frac{kv}{\gamma}z_x \\
&= \frac{d}{d\lambda}\left(\frac{F}{2G\rho}\cdot\frac{v}{\gamma}\right)-\frac{F}{G\rho}\frac{v}{\gamma}\left[1+\frac{1}{2}\left(\frac{\gamma'\rho}{\gamma}\right)\right]z_x+\frac{kv}{\gamma}z_x \\
&= \frac{d}{d\lambda}\left(\frac{F}{2G\rho}\cdot\frac{v}{\gamma}\right)+\left[F'-\frac{2F}{G}\right]\frac{v}{\rho\gamma}\frac{z_x}{2}-\frac{kF}{GF'}\frac{v}{\gamma}\frac{\gamma'\rho}{\gamma}z_x.
\end{aligned} \tag{6.8}$$

The estimate (6.1) now follows from substituting (6.8) in (6.6). Q.E.D.

PROPOSITION 2. *If* $\rho, G \in C^1(\mathbb{R}\times[0,t])$, *then*

$$\lambda_w = \frac{1}{2}+\frac{1}{2}\frac{\gamma'\rho}{\gamma}, \qquad \lambda_z = \frac{1}{2}-\frac{1}{2}\frac{\gamma'\rho}{\gamma}. \tag{6.9}$$

Proof. We have

$$\lambda = v+\gamma, \tag{6.10}$$

$$v = \frac{w+z}{2}, \tag{6.11}$$

$$\int_{\rho_*}^{\rho}\frac{\gamma}{s}\,ds = \frac{w-z}{2}. \tag{6.12}$$

From (6.11), it follows that

$$v_w = \frac{1}{2}. \tag{6.13}$$

Also,

$$\gamma_w = \gamma'\rho_w \tag{6.14}$$

and from differentiating (6.12) with respect to w,

$$\frac{\rho_w \gamma}{\rho} = \frac{1}{2}. \tag{6.15}$$

Thus, we obtain from (6.14) and (6.15)

$$\gamma_w = \frac{\gamma' \rho}{2\gamma}. \tag{6.16}$$

The first relation in (6.9) now follows from (6.13) and (6.16). The second relation in (6.9) is proved similarly. Q.E.D.

REFERENCES

1. K. CHUEH, C. CONLEY, AND J. SMOLLER, Positively invariant regions for systems of nonlinear diffusion equations, *Indiana Univ. Math. J.* **26** (1977), 373–392.
2. A. DOUGLIS, Some existence theorems for hyperbolic systems of partial differential equations in two independent variables, *Commun. Pure Appl. Math.* **5** (1952), 119–154.
3. T. DUPONT, Galerkin methods for modeling gas pipelines, *in* "Constructive and Computational Methods for Differential and Integral Equations," Lecture Notes in Mathematics No. 430, Springer-Verlag, Heidelberg, 1974.
4. T. DUPONT, Decay properties of some pipeline flow equations, preprint.
5. P. HARTMAN AND A. WINTER, On hyperbolic partial differential equations, *Amer. J. Math.* **74** (1952), 834–864.
6. P. LAX, Development of singularities of solutions on nonlinear hyperbolic partial differential equations, *J. Math. Phys.* **5** (1964), 611–613.
7. M. LUSKIN, A finite element method for first order hyperbolic systems, *Math. Comp.* **35** (1980), 1093–1112.
8. T. NISHIDA, Quasilinear wave equations with the dissipation, *Publ. Math. Orsay* (1977).
9. J. RAUCH, Discontinuous semilinear differential equations and multiple valued maps, *Proc. Amer. Math. Soc.* **64** (1977), 277–282.
10. V. STREETER, "Fluid Mechanics," 5th ed., McGraw–Hill, New York, 1971.