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Almost Complete Intersections and Factorial Rings

CRAIG HUNEKE

Department of Mathematics, The University of Michigan, Ann Arbor, Michigan 48109

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INTRODUCTION

It is our purpose in this paper to investigate the relationship between two classes of ideals in regular local rings R: prime ideals P such that R/P is factorial, and prime ideals Q which are minimally generated by $htQ + 1$ elements. These latter ideals are called almost complete intersections.

Peskine and Szpiro [12] say two unmixed ideals I and J in a regular local ring R are (geometrically) linked if I and J have no common minimal prime ideals and if there is an R-sequence $x_1,..., x_n$ such that $(x_1 ,..., x_n) = I \cap J$. We will delete the word "geometricaly" throughout this paper; linkage will always be used in the sense above. Murthy [11] essentially showed that if R is a regular local ring and P is a prime ideal such that R/P is factorial, then P is linked to an almost complete intersection. In Section 1 we provide a new and simple proof of this result using a generalization of a theorem of Hartshorne [2] concerning the connectedness of Spec(R).

Recall that a ring R is said to satisfy Serre's condition S_d if the following condition holds:

$$\text{depth } R_J \geqslant \min[d, htJ]$$

for all prime ideals J of R. R is said to satisfy condition R_d if R_J is regular for all prime ideals J of height at most d. If R is a local ring satisfying S_2 and R_1, then R is an integrally closed domain [10]. Suppose R is a regular local ring and I is an ideal such that R/I satisfies R_j where $j \leqslant 3$. Then for "most" ideals J linked to I (see [12] for a precise definition of "most"), R/J will also satisfy R_j. Now let P be a prime ideal in a regular local ring R such that R/P is a factorial domain satisfying S_3. As R/P satisfies R_1 in this case, most ideals Q linked to P will have the property that R/Q satisfies S_2 and R_1. In particular, Q will be a prime ideal, and R/Q will be integrally closed. We observe that if R is complete with an algebraically closed residue field, then R/P factorial implies R/P satisfies S_3 [3].

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The above discussion shows that if P is a prime ideal in a regular local ring R such that R/P is a factorial ring satisfying S_3, then P is linked to a prime ideal Q such that Q is an almost complete intersection and R/Q is integrally closed. In Section 2 we reverse this question. If Q is a prime ideal in a regular local ring R such that R/Q is integrally closed and Q is an almost complete intersection, then is it linked to a prime ideal P such that R/P is factorial? Surprisingly we obtain a satisfactory answer.

THEOREM 2.3. *Suppose R is a regular local ring and P is a prime ideal such that P is an almost complete intersection and such that P_Q is a complete intersection for all prime ideals Q of R such that* $\dim(R/P)_Q \leqslant 1$. *Let k be the least number of generators of P and set $S = R(T_1,..., T_k)$ be the polynomial ring in k-variables over R localized at the maximal ideal of R. Then there exists a prime ideal Q in S, linked to P, such that S/Q is factorial.*

Observe that $\dim(S) = \dim(R)$ and S is a faithfully flat extension of R. From this comment and from Peskine–Szpiro [12] it follows that S/Q is Cohen–Macaulay if and only if R/P is Cohen–Macaulay.

If P' is a prime ideal in R such that R/P' is integrally closed and Gorenstein, then as is well known, the general ideal P geometrically linked to P' will be an almost complete intersection such that R/P is an integrally closed domain. By Theorem 2.3, there is a faithfully flat extension of R, S, with $\dim(S) = \dim(R)$, such that P is linked to an ideal Q such that S/Q is factorial. This discussion shows

COROLLARY 2.7. *Let R be a regular local ring and let P' be an ideal such that R/P' is an integrally closed Gorenstein domain. Then there is a faithfully flat extension S of R of the same dimension and an ideal Q of S in the same linkage class as $P'S$ such that S/Q is factorial Gorenstein ring.*

The study of almost complete intersections relates to the syzygy conjecture of Evans and Griffith. Suppose R is a regular local ring. A module K is said to be a kth syzygy if there is an exact sequence

$$0 \to K \to F_k \to \cdots \to F_1$$

where the F_i are free R-modules. Evans and Griffith conjecture there are no non-free kth syzygies of rank less than k. This question is still open even for the case $k = 3$ and rank $k = 2$. The syzygy conjecture has now been solved by Evans and Griffith. Bruns *et al.* [1] show if such a module K exists, then there is a hight two prime ideal Q in a regular local ring R, generated by three elements, such that R/Q is not Cohen–Macaulay. If, in addition, K_J is free for prime ideals $J \neq M$, the maximal ideal of R, then one can asume that Q_J is a complete intersection for all prime ideals Q such that $\dim(R/Q)_J \leqslant 1$. Bruns *et al.* also show that the existence of such a syzygy implies the existence of a height two prime ideal P such that R/P is a non-Cohen–Macaulay factorial ring. Theorem 2.3 shows the existence of such a prime ideal P in a more natural way, linked to Q, and allows one to exploit this linkage. Indeed, using the local duality theory developed by Hartshorne and Ogus [3] we are able to prove

THEOREM 2.3. *Let R be a regular local ring and P a prime almost complete intersection satisfying the conditons of Theorem* 3.1. *If* $\operatorname{depth}(R/P)_Q$ *is at least one-half the dimension of $(R/P)_Q$ for every prime ideal Q containing P, then R/P is Cohen–Macaulay.*

Unfortunately, this result does not help in the basic cases of the syzygy problem.

All rings in this paper will be commutative Noetherian with identity. If I is an ideal, grade I will denote the length of a maximal R-sequence in I. For additional terminology and notation, we refer the reader to Matsumura [10].

1. MURTHY'S THEOREM

In [2] Hartshorne proved the following theorem.

THEOREM 1.1. *Let R be a commutative Noetherian ring such that* $\operatorname{Spec}(R)$ *is connected. Let I be an ideal with* $\operatorname{grade}(I) \geqslant 2$. *Then* $U = \operatorname{Spec}(R) - V(I)$ *is connected.*

This result is essentially equivalent to the lemma below.

LEMMA 1.2. *Suppose R is a commutative ring, I and J two non-nilpotent ideals such that $IJ = 0$, and $I + J \neq R$. Then* $\operatorname{grade}(I + J) \leqslant 1$.

This lemma bounds the grade of the ideal $I + J$, but gives no information concerning the possible grade of prime ideals associated to $I + J$. However, provided $\operatorname{grade}(I + J) \neq 0$, more can be said.

THEOREM 1.3. *Let R be a commutative ring and let I and J be two non-nilpotent ideals. Suppose $I + J \neq R$, IJ is nilpotent, and* $\operatorname{grade}(I + J) \geqslant 1$. *Then,*

$$\operatorname{Ass}(R/I + J) \subseteq \operatorname{Ass}(R/I) \cup \operatorname{Ass}(R/J) \cup \{P \mid \operatorname{depth} R_P \leqslant 1\}.$$

Proof. We note that the assumption grade$(I+J) \geqslant 1$ forces grade$(I+J)=1$ by Lemma 1.2. However, we will recover Lemma 1.2 from the theorem.

By raising I and J to a suitable power, we may assume $IJ=0$. As $I+J$ and I^n+J^n have the same nilradical, this does not change the grade of $I+J$. Suppose P is associated to $I+J$ but P is not in $\operatorname{Ass}(R/I) \cup \operatorname{Ass}(R/J) \cup \{P \mid \operatorname{depth} R_P \leqslant 1\}$. We may localize at P and assume it is maximal. Since P is associated to $I+J$, $(I+J:P) \nsubseteq I+J$. Also grade$(I+J:P) \geqslant 1$ since grade$(I+J) \geqslant 1$. By the well-known prime avoidance lemma [8] we may conclude $(I+J:P)$ is not contained in the union of $I+J$ with all associated primes of (0). Choose an element w in $(I+J:P)$ which is not in this union. Then w is not a zero-dvisor and $w \notin I+J$. Hence $P \subseteq (I+J:w) \neq R$, and since P is maximal, $P=(I+J:w)$.

By assumption, the grade of P is at least two, and P is not associated to either I or J. Consequently, we may choose an R-sequence x, z such that neither x nor z is contained in any associated prime of I or J. By the discussion above, there are equations,

$$wx = i_1 + j_1$$

and

$$wz = i_2 + j_2.$$

In these equations, the i_k are in I and the j_k are in J. Since $IJ=0$, it is easily seen that

$$wxi_2 = wzi_1.$$

Since w is a non-zero-divisor, this shows

$$xi_2 = zi_1.$$

However, x and z form an R-sequence and so there is a y in R such that $xy = i_1$. Since x is not a zero-divisor on R/I, y must be in I. The equation $wx = xy + j_1$ shows $x(w-y) = j_1$ and this forces $w-y$ to be in J. But then $w = y + (w-y)$ is in $I+J$, which contradicts the choice of w.

Lemma 1.2 follows from this theorem. We may assume R is reduced and then replace I by $(0:(0:I))$ and J by $(0:I)$. Then $I+J$ is contained in $(0:(0:I))+(0:I)$ and this latter ideal has grade one. The conclusion of Theorem 1.1 shows that any minimal prime ideal containing this ideal has grade at most one, which is the required conclusion.

Let R be any commutative local ring and let P be a prime ideal in R such that R_P is regular. We will say an ideal I is *generically linked* to P if there exists an R-sequence $x_1,\dots,x_n$ in P such that $(x_1,\dots,x_n)R_p = PR_p$ and $I = ((x_1,\dots,x_n) : P)$. In this case, $I \cap P = (x_1,\dots,x_n)$ so that I and P are linked.

PROPOSITION 1.1. *Let R be a Cohen–Macaulay local ring and let P be a prime ideal such that R_P is regular, R/P is factorial, but P is not generated by an R-sequence. Let Q be generically linked to P. Then Q is generated by $k+1$ elements where $k = \text{ht}P$, and if $x_1,\dots,x_k$ are the R-sequence such that*

$$(x_1,\dots,x_k) = P \cap Q,$$

then $x_1,\dots,x_k$ are part of a minimal generating set of Q.

Proof. Set $I = (x_1,\dots,x_k)$. Let $\bar{R}$ denote the ring R/I and by the overbar the map from R to R/I. Clearly $\overline{PQ} = 0$. Neither $\bar{P}$ nor $\bar{Q}$ can be nilpotent since $I \neq P$ and $I_P = P_P$. By Theorem 1.1,

$$\text{Ass}(\bar{R}/\bar{P} + \bar{Q}) \subseteq \text{Ass}(\bar{R}/\bar{P}) \cup \text{Ass}(\bar{R}/\bar{Q}) \cup \{\bar{J} \mid \text{depth}\, \bar{R}_{\bar{J}} \leqslant 1\}.$$

We may read this back in R and obtain:

$$\text{Ass}(R/P + Q) \subseteq \text{Ass}(R/P) \cup \text{Ass}(R/Q) \cup \{J \mid \text{depth}\, R_J \leqslant k+1\}.$$

Since $Q = (I : P)$, $\text{Ass}(R/Q) \subseteq \text{Ass}(R/I) \subseteq \{J \mid \text{depth}\, R_J \leqslant k+1\}$. By assumption, R is Cohen–Macaulay so that

$$\text{Ass}(R/P + Q) \subseteq \{J \mid \text{ht}\, J \leqslant k+1\}.$$

However, $(I : P)Q \nsubseteq P$ and so $\text{ht}(P+Q) \geqslant k+1$. We conclude $P+Q$ is unmixed.

Consider the ideal $(P+Q)/P$ in R/P. This is a height one unmixed ideal. Since R/P is factorial, it must be principal. We may assume there is a y in Q such that

$$(P+Q)/P = (P, y)/P.$$

We claim $Q = (I, y)$ which will prove the proposition. For $Q \subseteq (P, y)$ shows

$$Q \subseteq (Q \cap P, y) = (I, y).$$

COROLLARY 1.1. *Let R be a regular local ring and let P be a prime ideal such that R/P is factorial. Then P cannot be an almost complete intersection.*

Proof. This is immediate from [9], where Kunz showed almost complete intersections of finite projective dimension cannot be generically linked to almost complete intersections.

THEOREM 1.4 [11]. *Let R be a Gorenstein local ring and P a prime such that R_P is regular and R/P is a Cohen–Macaulay factorial ring. Then R/P is Gorenstein.*

Proof. Set $A = R/P$. As is well known (see [14], for example) A is Gorenstein if and only if A is Cohen–Macaulay and $\mathrm{Ext}_R^s(A, R)$ is isomorphic to A, where $s = \mathrm{height}(P)$. As in Proposition 1.1, choose an R-sequence $x_1, \dots, x_s$ in P which generates P generically. Let the overbar denote reduction modulo the ideal generated by the x_i. Then, a result of Rees [13] shows

$$\mathrm{Ext}_R^s(A, R) \simeq \mathrm{Ext}_{\bar{R}}^0(A, \bar{R}) = \mathrm{Hom}_{\bar{R}}(A, \bar{R}).$$

As is easily seen,

$$\mathrm{Hom}_{\bar{R}}(A, \bar{R}) \simeq (I : P)/I.$$

By Proposition 1.1, $(I : P) = (I, y)$ for some y and so

$$\mathrm{Hom}_{\bar{R}}(A, \bar{R}) \simeq (I, y)/I \simeq (y)/(y) \cap I \simeq R/(I : y) = A.$$

By our comments above this finishes the proof, since if P is a complete intersection, then A is trivially Gorenstein.

3. ALMOST COMPLETE INTERSECTIONS

In this section we prove the results on almost complete intersections stated in the Introduction. The main tools we will use for this were proved in [6] by this author, relying on the theory of d-sequences developed in [5]. Before we state these results we recall some facts about the symmetric algebra of a module.

Suppose M is a finitely generated R-module. The symmetric algebra of M, denoted $S_R(M)$, is a non-negatively graded algebra with the following universal property: if S is an R-algebra and $f: M \to S$ is an R-module homomorphism, then there exists a unique $f^*: S_R(M) \to S$ extending the identification of M as $S^1(M)$. Recall that by definition $S_R(M)$ is defined to be the tensor algebra $T(M) = R \oplus M \oplus (M \otimes M) \oplus \cdots$ modulo the submodule generated by elements of the form $a \otimes b - b \otimes a$. We will denote the nth graded piece of $S_R(M)$ by $S^n(M)$.

There is an alternative description of the symmetric algebra which we shall use. If M has a presentation

$$R^m \xrightarrow{(a_{ij})} R^n \to M \to 0,$$

then $S_R(M)$ can be identified with the ring $R[T_1,..., T_n]/J$, where J is the ideal generated by the m linear forms $\sum_1^n a_{ij} T_j$.

Before stating the results of [6] which will be needed, we recall the definitions of two algebras.

Let R be a commutative ring and let I be an ideal of R. The *Rees algebra of* I, denoted $\mathscr{R}(I, R)$, is the subalgebra $R[It]$ of the polynomial ring $R[t]$. The *graded algebra* of I, denoted $\mathrm{gr}_I(R)$, is the non-negatively graded algebra,

$$R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots,$$

or equivalently, $\mathrm{gr}_I(R) = \mathscr{R}(I, R)/I\mathscr{R}(I, R)$.

It is clear there is always a surjective homomorphism from $S_R(I)$ onto $\mathscr{R}(I, R)$.

The following theorem can be found in [7].

THEOREM 2.1. *Suppose P is a prime ideal in a Cohen-Macaulay ring R such that R_P is regular and P is an almost complete intersection. Then $\mathscr{R}(P, R)$ is isomorphic to the symmetric algebra $S_R(P)$.*

THEOREM 2.2 [6]. *Suppose R is a Cohen–Macaulay local ring and P is a prime ideal such that R_P is regular and P is an almost complete intersection. Then the following are equivalent:*

(1) $\mathrm{gr}_P(R)$ *is a domain*

(2) $P^2 = P^{(2)}$, *where $P^{(n)} = n$th symbolic power of $P = P_P^n \cap R$.*

(3) $P^n = P^{(n)}$ *for all $n \geqslant 1$.*

(4) P_Q *is generated by an R_Q-sequence for all prime ideals Q containing P such that $\dim(R/P)_Q$ is at most one.*

Of the above conditions, (4) is generally the easiest to verify. We can now state and prove the promised theorem.

THEOREM 2.3. *Let R be a regular local ring and P a prime ideal such that P is an almost complete intersection and $P^2 = P^{(2)}$. Set height $(P) = k$, and set $S = R[T_1,..., T_{k+1}]_m$ where m is the maximal ideal of R. Then there exists a prime Q in S, linked to P(in S) such that S/Q is factorial.*

Proof. By Theorem 2.2, $\mathrm{gr}_P(R)$ is a domain. It is well known [4] this implies $\mathscr{R}(P, R)[t^{-1}]$ is factorial. Theorem 2.1 shows we may identify $\mathscr{R}(P, R)$ with $S_R(P) = R[T_1,..., T_{k+1}]/Q$, where Q is the ideal generated by the linear forms $\sum_{i=1}^{k+1} b_i T_i$ with $\sum b_i x_i = 0$, where $P = (x_1,..., x_{k+1})$. Under these identifications, T_i is identified with $x_i t$, consequently $x_i/T_i = t^{-1}$. Now $Q \subseteq mR[T_1,..., T_{k+1}] = m^*$ and m^* is a prime ideal of $R[T_1,..., T_{k+1}]$. Set

$S = R(T_1,\ldots, T_{k+1}) = R[T_1,\ldots, T_{k+1}]_{m^*}$. Hence $S/Q = S_R(P)_m$. Since $T_i \notin m^*$, $t^{-1} \in S/Q$. Thus S/Q is a localization of $\mathscr{R}(P, R)[t^{-1}]$, and S/Q is factorial. We will show Q is linked to P which will prove Theorem 2.4.

Set $L_i = x_iT_1 - x_1T_i$ for $i = 2,\ldots, k+1$. We may choose $x_1,\ldots, x_{k+1}$ in such a way that any k of them form an R-sequence and $(x_2,\ldots, x_{k+1})_P = P_P$. Then it is easy to check that $L_2,\ldots, L_{k+1}$ form an S-sequence. Notice $(L_2,\ldots, L_{k+1}, x_1) = (x_1,\ldots, x_{k+1}) = PS$. We claim $(L_2,\ldots, L_{k+1} : x_1) = Q$. To prove this it is enough to show Q is contained in $(L_1,\ldots, L_{k+1} : x_1)$ since Q is prime, S is Cohen–Macaulay (in fact regular!), and $L_2,\ldots, L_{k+1}$ is an S-sequence of length equal to ht Q.

Let $\sum_{i=1}^{k+1} b_iT_i$ be in Q. Then $\sum_{i=1}^{k+1} b_ix_i = 0$; hence $x_1(\sum_{i=1}^{k+1} b_iT_i) = \sum_{i=2}^{k+1} b_i(x_1T_i - x_iT_i)$ which is in $(L_2,\ldots, L_{k+1})$. Since $(x_2,\ldots, x_{k+1})_P = P_P$, it follows $(L_2,\ldots, L_{k+1})_P = P_P$. It is now immediate that

$$(L_2,\ldots, L_{k+1}) = P \cap Q.$$

COROLLARY 2.7. *Let R be a regular local ring and suppose P is an ideal such that R/P is an integrally closed Gorenstein domain. Then there is a faithfully flat extension S of R and an ideal Q of S such that S/Q is a Gorenstein factorial ring and Q and PS are in the same linkage class.*

Proof. We may find an ideal J in R, linked to P, such that R/J is an integrally closed domain. Since R/P is Gorenstein, J is an almost complete intersection and R/J is Cohen–Macaulay. Theorem 2.3 shows we may find a faithfuly flat extension S of R and an ideal Q of S linked to PS in such a way that S/Q is factorial. As S/PS is Cohen–Macaulay, so is S/Q [12]. By Murthy's theorem [11], S/Q is Gorenstein.

We will now apply the theorem of Hartshorne and Ogus [3] to this situation. Their result states

THEOREM 2.4. *Let R be a regular local ring and let P be a prime ideal such that $A = R/P$ is factorial. Suppose for every prime ideal Q in A,*

$$\operatorname{depth} A_Q \geqslant \min\{\dim A_Q, (\dim A_Q/2) + 1\}. \qquad (*)$$

Then A is Cohen–Macaulay.

To apply this result we need an easy lemma found in [6].

LEMMA 2.5. *Let R be a local ring of depth at least $k + n$. Suppose $x_1,\ldots, x_k$ is an R-sequence and c is an element of R satisfying*

$$\operatorname{depth} R/(x_1,\ldots, x_k, c) = m < n.$$

Then depth $R/(x_1,\ldots, x_k : c) \geqslant m + 1$.

THEOREM 2.6. *Let R be a regular local ring and let P be a prime ideal of R satisfying the conditions of Theorem 2.3. Set $A = R/P$ and suppose that for all prime ideals Q in A, the following condition holds*:

$$\operatorname{depth} A_Q \geqslant (\dim A_Q/2).$$

Then A is Cohen–Macaulay.

Proof. By induction on the dimension of A we may assume A_Q is Cohen– Macaulay for all prime ideals $Q \neq m_A$, the maximal ideal of A. The conditions on P show $\dim(A) \geqslant 2$ if A is not Cohen–Macaulay.

Let S and Q be as in Theorem 2.3, and let $I = PS \cap Q$. Since P is an almost complete intersection, $PS = (I, y)$ for some y in S. In this case, $Q = (I : y)$. Since P and Q are linked, I is generated by an S-sequence. Set k equal to the length of this sequence, and put $n = \dim S/PS$. Then $\operatorname{depth} S = k + n$ and by assumption $\operatorname{depth} S/PS = \operatorname{depth} A = m \geqslant n$. Using Lemma 2.5 we may conclude $\operatorname{depth}(S/Q) \geqslant m + 1$.

Peskine and Szpiro [12] show that if I and J are linked, then R/I is Cohen–Macaulay if and only if R/J is Cohen–Macaulay. It easily follows that $(R/Q)_J$ is Cohen–Macaulay for all prime ideals $J \neq m_R$, the maximal ideal of R. On the other hand,

$$\begin{aligned} \operatorname{depth}(S/Q) &\geqslant \operatorname{depth}(S/PS) + 1 \\ &\geqslant (\dim A/2) + 1 = ((\dim(S/Q))/2 + 1. \end{aligned}$$

It follows that $(*)$ of Theorem 2.4 is satisfied. Hence S/Q is Cohen–Macaulay and by Peskine–Szpiro, S/PS is also. Since $R \to S$ is faithfully flat, we conclude R/P is Cohen–Macaulay.

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