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The Non-existence of Certain Regular Graphs of Girth 5

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For certain positive integers k it is shown that there is no k-regular graph with girth 5 having k^2+3 vertices. This provides a new lower bound for the number of vertices of girth 5 graphs with these valences.

Let $f(k, g)$ be the minimum number of vertices that a graph with valency k and girth g can have. The results of Hoffman and Singleton [3] included $f(k, 5) > k^2+1$ for all k, except $k = 2, 3, 7, 57$ (it is unknown if the statement is true for $k = 57$). Brown proved in [2] that $f(k, 5) > k^2+2$ for all k with the above exceptions; this statement is contained in a far more general theorem that has recently been proven by Bannai and Ito [1]. Robertson [5] constructed a 4-regular graph with girth 5 having $4^2+3 = 19$ vertices. Wegener [7] showed $f(5, 5) > 5^2+3$ and $f(6, 5) > 6^2+3$ was independently proved by O'Keefe and Wong [4] and by Spill [6]. In this paper we show $f(k, 5) > k^2+3$ for infinitely many valences k.

THEOREM. *Let $k \in \mathbb{N}$ be odd, $k \geqslant 3$, $k \neq l^2+l+3$ and $k \neq l^2+l-1$ for each nonnegative integer l. Then there is no k − regular graph with girth 5 and k^2+3 vertices.*

Proof. Assume there is a graph G contradicting the theorem. We construct a graph H with the same vertex set V as G such that two vertices of H are adjacent iff they have distance 3 in G. Let A be the adjacency matrix of G and B the adjacency matrix of H. For each vertex v of G there are exactly 2 vertices of distance 3 from v in G because there are exactly k vertices at distance 1 and $k(k-1)$ vertices at distance 2 from v; the remaining two vertices must have distance 3 from v. Consequently every component of H is a cycle. Let b be the number of these cycles and let a_i, $i = 1, \ldots, b$, be the length of these cycles ordered in an arbitrary manner.

Let I be the identity matrix and J be the all one matrix. Brown [2] showed $A^2 + A - (k-1)I = J - B$ and from this deduced that A must have the spectrum:

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(i) An eigenvalue k with multiplicity 1,

(ii) $b-1$ roots of the equation

$$r^2+r-(k-1)=-2, \tag{1}$$

(iii) one root of each of the equations

$$r^2+r-(k-1)=-2\cos(2\pi v/a_i), \tag{2}$$

where i ranges from 1 to b and v ranges from 1 to a_i-1 for each i.

The solutions of (1) are $r_1=-\frac{1}{2}+\frac{1}{2}\sqrt{4k-11}$, $r_2=-\frac{1}{2}-\frac{1}{2}\sqrt{4k-11}$. By writing $m(r)$ for the multiplicity of an eigenvalue r of A we get $m(r_1)+m(r_2)=b-1$ as r_1 and r_2 can only be solutions of (1), not of (2). For the general solution of (2) we get

$$r=-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{4k-3-8\cos(2\pi v/a_i)}.$$

We will be interested in the special case $\cos(2\pi v/a_i)=-1$ which occurs if and only if a_i is even and $v=a_i/2$, leading to the eigenvalues $r_3=-\frac{1}{2}+\frac{1}{2}\sqrt{4k+5}$, $r_4=-\frac{1}{2}-\frac{1}{2}\sqrt{4k+5}$. Let $\beta_e=m(r_3)+m(r_4)$ be the sum of the multiplicities of these eigenvalues. Then β_e is exactly the number of even cycles in H because (iii) provides for every even a_i exactly one eigenvalue that equals either r_3 or r_4.

We will now show that for those valences, k, mentioned in the theorem two of the eigenvalues $r_1,\dots,r_4$ cannot have integer multiplicities, thus proving the non-existence of the corresponding graphs.

LEMMA 1. *Let r_i and r_j be eigenvalues of a real square matrix A with rational entries and let $m(r_i)$ and $m(r_j)$ be their multiplicities. If r_i and r_j are algebraic conjugate over $\mathbb{Q}$ then $m(r_i)=m(r_j)$.*

Proof. The characteristic polynomial of A has a unique factorization into irreducible polynomials over $\mathbb{Q}$. Each eigenvalue of A is the root of exactly one of these polynomials and algebraic conjugate eigenvalues are roots of the same irreducible polynomial. Now assume that one of these roots, say r_i, is a multiple root of the corresponding irreducible polynomial $p(x)$. Then the derivative $p'(x)$ would also have r_i as a root and by applying the euclidean algorithm we would get a nontrivial GCD of $p(x)$ and $p'(x)$ in $\mathbb{Q}[x]$, contradicting the irreducibility of $p(x)$. Consequently p has only single roots and hence the multiplicities $m(r_i)$ and $m(r_j)$ equal the exponent of $p(x)$ in the characteristic polynomial of A. ∎

In the case $r_i\notin\mathbb{Q}$ for $i=1,\dots,4$ we can apply Lemma 1 for the two pairs of eigenvalues r_1,r_2 and r_3,r_4 and get $m(r_1)=m(r_2)=(b-1)/2$ and

$m(r_3) = m(r_4) = \beta_e/2$. The following lemma shows that either $b-1$ or β_e must be odd so that not all of the above multiplicities can be integers.

LEMMA 2. *If k is odd then $\beta_e \equiv b \bmod 2$.*

Proof. Let $\beta_0 = b - \beta_e$ be the number of cycles of odd length in H. For odd k the number of vertices of G, $k^2 + 3$ is even. Assume that β_0 is odd. Then all odd cycles together must have an odd number of vertices; consequently the number of vertices of H and hence G must be odd, in contradiction to the above statement. Thus β_0 is even and so $\beta_e \equiv (\beta_e + \beta_0) \bmod 2$. ∎

It only remains to show that none of the eigenvalues r_i, $i = 1, \ldots, 4$, are rational for $k \neq l^2 + l + 3$ and $k \neq l^2 + l - 1$.

The eigenvalues r_1 and r_2 are rational if and only if $4k - 11$ is a perfect square. If $4k - 11 = s^2$ with $s \in \mathbb{N}$, then $s = 2l + 1$ for some $l \in \mathbb{N}$. This means $4k - 11 = (2l + 1)^2$, which is equivalent to $k = l^2 + l + 3$ for arbitrary l. A similar observation in the case of r_3 and r_4 shows $\sqrt{4k + 5} \in \mathbb{Q}$ if and only if $k = l^2 + l - 1$ for some $l \in \mathbb{N}$. This proves the theorem.

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