JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **81**, 48–62 (1981)

Steffensen Sequences Satisfying a Certain Composition Law

JAMES WARD BROWN

Department of Mathematics and Statistics, The University of Michigan, Dearborn, Michigan 48128

Submitted by R. P. Boas

1. INTRODUCTION

The *umbral composition* of polynomial sequences $\{p_n(x)\}$ $(n = 0, 1, 2,...)$ has been used for at least a century (see, for example, [1]). To define it, we assume that

$$\{p_n(x)\} \quad \text{and} \quad \{q_n(x)\} \tag{1.1}$$

are simple, the degree of the nth polynomial in each case being precisely n, and we write

$$p_n(x) = \sum_{k=0}^{n} a_{n,k} x^k \qquad (n = 0, 1, 2,...). \tag{1.2}$$

DEFINITION 1. The umbral composition $\{(pq)_n(x)\}$ of the sequences (1.1) is the sequence of polynomials

$$(pq)_n(x) = \sum_{k=0}^{n} a_{n,k} q_k(x) \qquad (n = 0, 1, 2,...),$$

where the coefficients $a_{n,k}$ are those in (1.2).

Umbral composition is evidently associative. In fact, simple polynomial sequences form a group under the operation

$$\{p_n(x)\} \circ \{q_n(x)\} = \{(pq)_n(x)\}, \tag{1.3}$$

the identity element being $\{x^n\}$ (see [8]).

Using umbral composition as a basic tool, Rota *et al.* [7] began a development of a class of polynomial sequences in two variables which they introduced via a pair of binomial-type identities and called *Steffensen*

0022-247X/81/050048–15$02.00/0

sequences. An equivalent generating function characterization is given in [3], and we take that as our starting point.

DEFINITION 2. A polynomial sequence $\{P_n^{(\alpha)}(x)\}$ is a Steffensen sequence if it is generated by a relation of the form

$$e^{\alpha K(t)}G(t)\,e^{xH(t)} = \sum_{n=0}^{\infty} P_n^{(\alpha)}(x)\,\frac{t^n}{n!}, \tag{1.4}$$

where

$$K(t) = k_1 t + k_2 t^2 + k_3 t^3 + \cdots \qquad (k_1 \neq 0), \tag{1.5}$$

$$H(t) = h_1 t + h_2 t^2 + h_3 t^3 + \cdots \qquad (h_1 \neq 0), \tag{1.6}$$

$$G(t) = g_0 + g_1 t + g_2 t^2 + \cdots \qquad (g_0 \neq 0). \tag{1.7}$$

We follow Rota *et al.* in treating the power series that appear as *formal* power series. Although the underlying field can be taken to be more general, it is understood here to be the field of complex or, when specifically noted, real numbers.

A Steffensen sequence is simple in each of the variables x and α (see [3]) and is a special type of *Sheffer sequence*, first studied to any great extent in [9]. To be precise, a polynomial sequence $\{P_n(x)\}$ is a Sheffer sequence if it is generated by a relation of the form

$$G(t)\,e^{xH(t)} = \sum_{n=0}^{\infty} P_n(x)\,\frac{t^n}{n!}, \tag{1.8}$$

where $H(t)$ and $G(t)$ are as in (1.6) and (1.7). Thus any Steffensen sequence $\{P_n^{(\alpha)}(x)\}$ is a Sheffer sequence in x (and also in α). Sheffer sequences form a subgroup of the group of simple sequences under the operation defined in (1.3). The identity element $\{x^n\}$ is, of course, generated by $\exp(xt)$. This group-theoretic approach to Sheffer sequences has already been exploited in [2, 4, 5] and will be touched on in our work here.

The chief aim of the present paper is to furnish a generating function characterization of those Steffensen sequences satisfying a certain composition law which we call *Property C*.

DEFINITION 3. A Steffensen sequence $\{P_n^{(\alpha)}(x)\}$ has Property C if

$$(P^{(\alpha_1)}P^{(\alpha_2)}P^{(\alpha_3)}\cdots P^{(\alpha_k)})_n(x) = \begin{cases} P_n^{(\alpha_1-\alpha_2+\alpha_3-\cdots+\alpha_k)}(x) & (k \text{ odd}), \\ (-1)^n\,P_n^{(-\alpha_1+\alpha_2-\alpha_3+\cdots+\alpha_k-n)}(x) & (k \text{ even}). \end{cases}$$

The problem is suggested to us by the fact that the sequence $\{L_n^{(\alpha)}(x)\}$ of Laguerre polynomials is, as Rota *et al.* showed [7, p. 729], a Steffensen sequence which has this "remarkable" property.

In Section 2 we state our main results with some discussion of them. Section 3 is devoted to the proofs of the two theorems in Section 2. Finally, in Section 4 we show that the sequence $\{L_n^{(\alpha)}(x)\}$ of Laguerre polynomials is essentially the only *orthogonal* Steffensen sequence which has Property C.

2. Statement of Main Results

We say that a Steffensen sequence $\{P_n^{(\alpha)}(x)\}$ has *Property A* if the identity in Definition 3 holds when k is odd $(k = 2j - 1)$,

$$(P^{(\alpha_1)}P^{(\alpha_2)}P^{(\alpha_3)} \cdots P^{(\alpha_{2j-1})})_n(x) = P_n^{(\alpha_1-\alpha_2+\alpha_3-\cdots+\alpha_{2j-1})}(x) \qquad (j = 1, 2,...), \tag{2.1}$$

and that it has *Property B* when k is even $(k = 2j)$,

$$(P^{(\alpha_1)}P^{(\alpha_2)}P^{(\alpha_3)} \cdots P^{(\alpha_{2j})})_n(x) = (-1)^n P_n^{(-\alpha_1+\alpha_2-\alpha_3+\cdots+\alpha_{2j}-n)}(x) \qquad (j = 1, 2,...). \tag{2.2}$$

Our characterization of those Steffensen sequences having Property C will actually be a corollary of the following two theorems, which are of interest in themselves. In each theorem $K^{-1}(t)$ denotes the formal power series inverse of the series

$$K(t) = k_1 t + k_2 t^2 + k_3 t^3 + \cdots \qquad (k_1 \neq 0),$$

defined by the equation

$$K(K^{-1}(t)) = t; \tag{2.3}$$

and, in the second theorem, $K'(t)$ denotes the formal derivative of $K(t)$.

Theorem 1. *Let $\{P_n^{(\alpha)}(x)\}$ be a Steffensen sequence, generated by* (1.4). *The following statements are equivalent*:

(A_1) *$\{P_n^{(\alpha)}(x)\}$ has Property* A;

(A_2) $(P^{(\alpha)}P^{(\alpha)})_n(x) = x^n$ $(n = 0, 1, 2,...)$;

(A_3) $H(t) = K^{-1}(-K(t))$ *and* $G(t) = s \exp[O(K(t))]$, *where* $s = \pm 1$ *and* $O(t)$ *is an odd power series.*

Note that if in statement (A_3) we put $K(t) = -\log(1-t)$ [so that $K^{-1}(t) = 1 - e^{-t}$], $s = 1$, and $O(t) = t$, generating relation (1.4) becomes

$$(1-t)^{-\alpha-1} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} P_n^{(\alpha)}(x) \frac{t^n}{n!}. \qquad (2.4)$$

Thus $P_n^{(\alpha)}(x) = L_n^{(\alpha)}(x)$, where $L_n^{(\alpha)}(x)$ are the Laguerre polynomials; and, as Rota *et al.* showed in [7], $\{L_n^{(\alpha)}(x)\}$ is in fact a Steffensen sequence with Property A. It should be emphasized that the notation for $L_n^{(\alpha)}(x)$ here and in [7] differs by a factor of $n!$ from that often used elsewhere.

Note too that contained in Theorem 1 is the fact that *the Steffensen sequences which have Property* A *are precisely the self-inverse Steffensen sequences*. That is, $\{P_n^{(\alpha)}(x)\}$ has Property A if and only if the product $\{P_n^{(\alpha)}(x)\} \circ \{P_n^{(\alpha)}(x)\}$ is the identity element $\{x^n\}$ in the group of Sheffer sequences.

THEOREM 2. *Let* $\{P_n^{(\alpha)}(x)\}$ *be a Steffensen sequence, generated by* (1.4). *The following statements are equivalent*:

(B_1) $\{P_n^{(\alpha)}(x)\}$ *has Property* B;

(B_2) $(P^{(\alpha)}P^{(\alpha)})_n(x) = (-1)^n P_n^{(-n)}(x)$ $(n = 0, 1, 2,...)$;

(B_3) $e^{t/2}K^{-1}(t)$ *is an odd power series*, $H(t) = K^{-1}(-K(t))$, *and* $G(t) = 1 + tK'(t)$.

Observe that if $K(t) = -\log(1-t)$, then $K^{-1}(t) = 1 - e^t$ and $e^{t/2}K^{-1}(t) = 2\sinh(t/2)$; and, with the corresponding $H(t)$ and $G(t)$ in statement (B_3), generating relation (1.4) becomes (2.4). So, once again as Rota *et al.* showed in [7], $\{L_n^{(\alpha)}(x)\}$ is a Steffensen sequence with Property B.

A Steffensen sequence $\{P_n^{(\alpha)}(x)\}$ may have Property A without having Property B. For $e^{t/2}K^{-1}(t)$ need not be odd in Theorem 1. Also, it is possible that $G(0) = -1$ in Theorem 1, while $G(0)$ is always unity in Theorem 2. It is true, however, that *if* $\{P_n^{(\alpha)}(x)\}$ *has Property* B, *it also has Property* A. To see that this is so, we assume that the conditions in statement (B_3) are satisfied. Evidently, all that actually needs to be shown is that $G(t)$ has the form in statement (A_3) of Theorem 1.

This is accomplished by first differentiating each side of identity (2.3) and writing $K'(K^{-1}(t)) = 1/(K^{-1}(t))'$. Then, since $G(t) = 1 + tK'(t)$, we have

$$G(K^{-1}(t)) = 1 + \frac{K^{-1}(t)}{(K^{-1}(t))'}. \qquad (2.5)$$

But $K^{-1}(t) = e^{-t/2}O_B(t)$, where $O_B(t)$ is an odd power series. Differentiating each side of this expression for $K^{-1}(t)$ and writing $K^{-1}(t)$ and $(K^{-1}(t))'$ in terms of $O_B(t)$ on the right-hand side of (2.5), we find that

$$G(K^{-1}(t)) = \frac{2O_B'(t) + O_B(t)}{2O_B'(t) - O_B(t)}. \tag{2.6}$$

From (2.5) we see that $\log[G(K^{-1}(t))]$ vanishes when $t = 0$; and, using (2.6), one can readily show that $\log[G(K^{-1}(t))]$ is odd. Thus

$$O_A(t) = \log[G(K^{-1}(t))] \tag{2.7}$$

is a well-defined odd power series. Finally, (2.7) is equivalent to

$$G(t) = e^{O_A(K(t))},$$

and $G(t)$ is of the required form (with $s = 1$) in statement (A_3) of Theorem 1.

An immediate consequence of the above remarks is the following corollary, which is the promised generating function characterization of Steffensen sequences having Property C.

COROLLARY 1. *A Steffensen sequence, generated by* (1.4), *has Property* C *if and only if* $e^{t/2}K^{-1}(t)$ *is an odd power series,* $H(t) = K^{-1}(-K(t))$, *and* $G(t) = 1 + tK'(t)$.

3. Proofs of Theorems 1 and 2

We preface our proofs of Theorems 1 and 2 with necessary background on Sheffer sequences. Recall from Section 1 that $\{P_n(x)\}$ is a Sheffer sequence if it is generated by

$$G(t)\, e^{xH(t)} = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}, \tag{3.1}$$

where $H(t)$ and $G(t)$ are of the forms (1.6) and (1.7), respectively. In accordance with terminology already used in [2], we shall refer to the $H(t)$ in (3.1) as the *basic function* and to the $G(t)$ as the *Appell function*. For a given Sheffer sequence these two functions are uniquely determined (see [5]).

We shall make extensive use of the fact that if $\{\tilde{P}_n(x)\}$ is a Sheffer sequence generated by

$$\tilde{G}(t)\, e^{x\tilde{H}(t)} = \sum_{n=0}^{\infty} \tilde{P}_n(x) \frac{t^n}{n!}, \tag{3.2}$$

the product

$$\{(P\tilde{P})_n(x)\} = \{P_n(x)\} \circ \{\tilde{P}_n(x)\}$$

of the sequence in (3.1) and this one is generated by (see [4] for a derivation)

$$G(t)\,\tilde{G}(H(t))\,e^{x\tilde{H}(H(t))} = \sum_{n=0}^{\infty} (P\tilde{P})_n(x)\,\frac{t^n}{n!}. \tag{3.3}$$

This result can be applied to Steffensen sequences, generated by relations of the form (1.4), since they are also Sheffer sequences with Appell functions $\exp(\alpha K(t))\,G(t)$, rather than simply $G(t)$. For example, if $\{P_n^{(\alpha)}(x)\}$ is generated by (1.4), the product

$$\{(P^{(\alpha_1)}P^{(\alpha_2)})_n(x)\} = \{P_n^{(\alpha_1)}(x)\} \circ \{P_n^{(\alpha_2)}(x)\}$$

is generated by

$$e^{\alpha_1 K(t) + \alpha_2 K(H(t))} G(t)\,G(H(t))\,e^{xH(H(t))} = \sum_{n=0}^{\infty} (P^{(\alpha_1)}P^{(\alpha_2)})_n(x)\,\frac{t^n}{n!}; \tag{3.4}$$

and, since $\{(P^{(\alpha_1)}P^{(\alpha_2)}P^{(\alpha_3)})_n(x)\} = \{(P^{(\alpha_1)}P^{(\alpha_2)})_n(x)\} \circ \{P_n^{(\alpha_3)}(x)\}$,

$$e^{\alpha_1 K(t) + \alpha_2 K(H(t)) + \alpha_3 K(H(H(t)))} G(t)\,G(H(t))\,G(H(H(t)))\,e^{xH(H(H(t)))}$$
$$= \sum_{n=0}^{\infty} (P^{(\alpha_1)}P^{(\alpha_2)}P^{(\alpha_3)})_n(x)\,\frac{t^n}{n!}. \tag{3.5}$$

From the among the known group-theoretic properties of Sheffer sequences, the characterization of self-inverse Sheffer sequences obtained in [5] is of particular interest to us here. The setting in [5] is considerably more general than is necessary for our present purposes, and we quote only what is needed. Suppose that the sequence generated by (3.1) is self-inverse; that is,

$$\{P_n(x)\} \circ \{P_n(x)\} = \{x^n\},$$

or

$$(PP)_n(x) = x^n \qquad (n = 0, 1, 2, \ldots).$$

In view of (3.3), this is equivalent to the pair of functional equations

$$H(H(t)) = t, \qquad G(t)\,G(H(t)) = 1. \tag{3.6}$$

One possibility for $H(t)$ is $H(t)=t$. This means that $G(t)=\pm 1$, in which case the trivial sequences $\{x^n\}$ and $\{-x^n\}$ are generated. It is shown in [5] that the only other possibility for $H(t)$ is

$$H(t)=K^{-1}(-K(t)), \tag{3.7}$$

where $K(t)$ is a series of the type

$$K(t)=k_1 t+k_2 t^2+k_3 t^3+\cdots \qquad (k_1\neq 0).$$

Moreover, $G(t)$ must be of the form

$$G(t)=se^{O(K(t))} \tag{3.8}$$

where $s=\pm 1$, $O(t)$ is an odd power series, and $K(t)$ is the same as in (3.7). Conversely, if $H(t)$ and $G(t)$ are of the type (3.7)–(3.8), $\{P_n(x)\}$ is self-inverse.

Finally, continuing to quote only the special cases of previous work that we actually need here, we mention that the modification $\{P_n(x-n)\}$ of a Sheffer sequence $\{P_n(x)\}$, generated by (3.1), remains a Sheffer sequence and is generated by [3]

$$\frac{G(u^{-1}(t))}{1+u^{-1}(t)\,H'(u^{-1}(t))}\,e^{xH(u^{-1}(t))}=\sum_{n=0}^{\infty} P_n(x-n)\frac{t^n}{n!}, \tag{3.9}$$

where $u^{-1}(t)$ is the formal power series inverse of $u(t)=t\exp H(t)$ and where $H'(t)$ denotes the formal derivative of $H(t)$. Observing from (1.4) that a Steffensen sequence $\{P_n^{(\alpha)}(x)\}$ is also a Sheffer sequence in α, we may use (3.9) to write the generating relation

$$e^{\alpha K(v^{-1}(-t))}\frac{G(v^{-1}(-t))}{1+v^{-1}(-t)\,K'(v^{-1}(-t))}\,e^{xH(v^{-1}(-t))}$$
$$=\sum_{n=0}^{\infty}(-1)^n P_n^{(\alpha-n)}(x)\frac{t^n}{n!}, \tag{3.10}$$

where $v^{-1}(t)$ is the formal inverse of $v(t)=t\exp K(t)$. We note from (3.10) that $\{(-1)^n P_n^{(\alpha-n)}(x)\}$ is itself a Steffensen sequence.

We turn now to the proof of Theorem 1, which is in three parts.

$(A_1)\Rightarrow(A_2)$: Putting $j=2$ and $\alpha_1=\alpha_2=\alpha_3=\alpha$ in (2.1), we have

$$\{P_n^{(\alpha)}(x)\}\circ\{P_n^{(\alpha)}(x)\}\circ\{P_n^{(\alpha)}(x)\}=\{P_n^{(\alpha)}(x)\}.$$

Then, by multiplying on each side by the group inverse of $\{P_n^{(\alpha)}(x)\}$, we obtain (A_2).

$(A_2) \Rightarrow (A_3)$: Referring to (3.4) and equating the uniquely determined basic and Appell functions for $\{(P^{(\alpha)}P^{(\alpha)})_n(x)\}$ and $\{x^n\}$, we find that

$$H(H(t)) = t,$$
$$e^{\alpha[K(t)+K(H(t))]}G(t)\,G(H(t)) = 1.$$

Putting $\alpha = 0$ in the second of these equations gives

$$G(t)\,G(H(t)) = 1;$$

and, consequently, $K(t) + K(H(t)) = 0$. Thus $H(t) = K^{-1}(-K(t))$; and, by earlier remarks in this section leading to (3.8), we see that $G(t)$ is as shown in statement (A_3).

$(A_3) \Rightarrow (A_1)$: Using induction on j, we first note that (2.1) is trivially true when $j = 1$.

The nature of our induction argument requires us to show that (2.1) is also true when $j = 2$, or that

$$\{P_n^{(\alpha_1)}(x)\} \circ \{P_n^{(\alpha_2)}(x)\} \circ \{P_n^{(\alpha_3)}(x)\} = \{P_n^{(\alpha_1-\alpha_2+\alpha_3)}(x)\}. \tag{3.11}$$

This is readily accomplished by observing from (3.5) that the basic and Appell functions for $\{(P^{(\alpha_1)}P^{(\alpha_2)}P^{(\alpha_3)})_n(x)\}$ are

$$H(H(H(t))), \tag{3.12}$$

$$e^{\alpha_1 K(t)+\alpha_2 K(H(t))+\alpha_3 K(H(H(t)))}G(t)\,G(H(t))\,G(H(H(t))), \tag{3.13}$$

and noting how it follows from statement (A_3) that

$$H(H(t)) = t, \qquad K(H(t)) = -K(t), \qquad G(t)\,G(H(t)) = 1. \tag{3.14}$$

With (3.14), the pair (3.12)–(3.13) easily reduces to

$$H(t), \qquad e^{(\alpha_1-\alpha_2+\alpha_3)K(t)}G(t),$$

which are the basic and Appell functions for $\{P_n^{(\alpha_1-\alpha_2+\alpha_3)}(x)\}$. Hence (3.11) is true.

To finish the induction proof, we assume (2.1) holds for any j and use (3.11) to write

$$\begin{aligned}
&\{P_n^{(\alpha_1)}(x)\} \circ \{P_n^{(\alpha_2)}(x)\} \circ \{P_n^{(\alpha_3)}(x)\} \circ \cdots \circ \{P_n^{(\alpha_{2j-1})}(x)\} \\
&\qquad \circ \{P_n^{(\alpha_{2j})}(x)\} \circ \{P_n^{(\alpha_{2j+1})}(x)\} \\
&\qquad = \{P_n^{(\alpha_1-\alpha_2+\alpha_3-\cdots+\alpha_{2j-1})}(x)\} \circ \{P_n^{(\alpha_{2j})}(x)\} \circ \{P_n^{(\alpha_{2j+1})}(x)\} \\
&\qquad = \{P_n^{(\alpha_1-\alpha_2+\alpha_3-\cdots+\alpha_{2j-1}-\alpha_{2j}+\alpha_{2j+1})}(x)\},
\end{aligned}$$

or

$$(P^{(\alpha_1)}P^{(\alpha_2)}P^{(\alpha_3)}\ldots P^{(\alpha_{2j+1})})_n(x)=P_n^{(\alpha_1-\alpha_2+\alpha_3-\cdots+\alpha_{2j+1})}(x).$$

The proof of Theorem 1 is now complete.

The following two lemmas will make our proof of Theorem 2 more efficient. In these lemmas $K(t)$ and $H(t)$ denote power series of the types (1.5) and (1.6).

LEMMA 1. *$K^{-1}(-K(t))=-te^{K(t)}$ if and only if $e^{t/2}K^{-1}(t)$ is an odd power series.*

To verify this, we need only replace t by $K^{-1}(t)$ in the above equation and multiply through the result by $e^{-t/2}$, thus writing that equation in the equivalent form

$$e^{-t/2}K^{-1}(-t)=-e^{t/2}K^{-1}(t).$$

LEMMA 2. *If $H(t)=K^{-1}(-K(t))$ and $H(t)=-te^{K(t)}$, then*

$$\frac{1}{1+H(t)\,K'(H(t))}=1+tK'(t). \tag{3.15}$$

To start the verification, rewrite the first expression for $H(t)$ as $K(H(t))=-K(t)$ and differentiate each side to get

$$K'(H(t))=-\frac{K'(t)}{H'(t)}.$$

Substituting this into the left-hand side of (3.15) and then using the second expression for $H(t)$ to substitute for $H(t)$ and $H'(t)$, we arrive at the desired result.

$(B_1)\Rightarrow(B_2)$: Simply put $j=1$ and $\alpha_1=\alpha_2=\alpha$ in (2.2).

$(B_2)\Rightarrow(B_3)$: Referring to (3.4) and (3.10) and equating the basic and Appell functions for the sequences $\{(P^{(\alpha)}P^{(\alpha)})_n(x)\}$ and $\{(-1)^n P_n^{(-n)}(x)\}$, we have

$$H(H(t))=H(v^{-1}(-t)), \tag{3.16}$$

$$e^{\alpha[K(t)+K(H(t))]}G(t)\,G(H(t))=\frac{G(v^{-1}(-t))}{1+v^{-1}(-t)\,K'(v^{-1}(-t))}, \tag{3.17}$$

where $v^{-1}(t)$ is the formal inverse of

$$v(t)=te^{K(t)}. \tag{3.18}$$

According to (3.16),

$$H(t) = v^{-1}(-t); \tag{3.19}$$

and (3.17) thus reduces to

$$e^{\alpha[K(t)+K(H(t))]}G(t) = \frac{1}{1 + H(t)\,K'(H(t))}. \tag{3.20}$$

Setting $\alpha = 0$ here, we have

$$G(t) = \frac{1}{1 + H(t)\,K'(H(t))}; \tag{3.21}$$

and it follows from (3.20) that $K(t) + K(H(t)) = 0$, or

$$H(t) = K^{-1}(-K(t)). \tag{3.22}$$

Evidently, then,

$$H(H(t)) = t; \tag{3.23}$$

and, from (3.19),

$$H(-v(t)) = t. \tag{3.24}$$

So, by (3.23) and (3.24), $H(t) = -v(t)$. Combining this with (3.18), we find that

$$H(t) = -te^{K(t)}. \tag{3.25}$$

In view of (3.22) and (3.25),

$$K^{-1}(-K(t)) = -te^{K(t)};$$

and, by Lemma 1, we see that $e^{t/2}K^{-1}(t)$ is odd. Also, because of expressions (3.22) and (3.25), we can appeal to Lemma 2 and write (3.21) as $G(t) = 1 + tK'(t)$.

$(B_3) \Rightarrow (B_1)$: We shall use induction on j obtain (2.2). First we show that (2.2) is true when $j = 1$, or that

$$(P^{(\alpha_1)}P^{(\alpha_2)})_n(x) = (-1)^n\,P_n^{(-\alpha_1+\alpha_2-n)}(x). \tag{3.26}$$

We need only refer to (3.4) and (3.10) to see that, in terms of basic and Appell functions, (3.26) is equivalent to the pair of equations

$$H(H(t)) = H(v^{-1}(-t)), \tag{3.27}$$

$$e^{\alpha_1 K(t) + \alpha_2 K(H(t))} G(t)\, G(H(t)) = e^{(-\alpha_1 + \alpha_2) K(v^{-1}(-t))} \frac{G(v^{-1}(-t))}{1 + v^{-1}(-t)\, K'(v^{-1}(-t))}, \tag{3.28}$$

where $v^{-1}(t)$ is the formal inverse of

$$v(t) = te^{K(t)}. \tag{3.29}$$

We must show that (3.27) and (3.28) hold. By Lemma 1, the expression for $H(t)$ in (B_3) can be written $H(t) = -t \exp K(t)$; and from (3.29) it follows that $H(t) = -v(t)$. Replacing t by $v^{-1}(-t)$ in this last equation gives $H(v^{-1}(-t)) = t$. But $H(t) = K^{-1}(-K(t))$ means that $H(H(t)) = t$, and so

$$H(t) = v^{-1}(-t). \tag{3.30}$$

Also, since $H(t) = K^{-1}(-K(t))$ and (3.30) holds,

$$K(v^{-1}(-t)) = -K(t); \tag{3.31}$$

and, by Lemma 2,

$$G(t) = \frac{1}{1 + H(t)\, K'(H(t))}. \tag{3.32}$$

In view of (3.30), (3.31), and (3.32), it is now a simple matter to see that (3.27)–(3.28) holds. That is, (3.26) is true.

To complete our induction argument, we assume that (2.2) is true for any j and note from the comments following the statement of Theorem 2 in Section 2 that since conditions (B_3) are satisfied, $\{P_n^{(\alpha)}(x)\}$ has Property A. This observation and (3.26) allow us to write

$$\begin{aligned}
&\{P_n^{(\alpha_1)}(x)\} \circ \{P_n^{(\alpha_2)}(x)\} \circ \{P_n^{(\alpha_3)}(x)\} \circ \cdots \circ \{P_n^{(\alpha_{2j})}(x)\} \\
&\quad \circ \{P_n^{(\alpha_{2j+1})}(x)\} \circ \{P_n^{(\alpha_{2j+2})}(x)\} \\
&\quad = \{P_n^{(\alpha_1 - \alpha_2 + \alpha_3 - \cdots + \alpha_{2j+1})}(x)\} \circ \{P_n^{(\alpha_{2j+2})}(x)\} \\
&\quad = \{(-1)^n P_n^{(-\alpha_1 + \alpha_2 - \alpha_3 + \cdots - \alpha_{2j+1} + \alpha_{2j+2} - n)}(x)\},
\end{aligned}$$

or

$$(P^{(\alpha_1)} P^{(\alpha_2)} P^{(\alpha_3)} \ldots P^{(\alpha_{2j+2})})_n(x) = (-1)^n P_n^{(-\alpha_1 + \alpha_2 - \alpha_3 + \cdots + \alpha_{2j+2} - n)}(x).$$

This finishes the proof of Theorem 2.

4. On Orthogonal Steffensen Sequences

Using rather different techniques, Sheffer [9] and Meixner [6] classified the Sheffer sequences which are orthogonal on some interval of the real line. To be precise, they identified those sequences generated by (1.8) which satisfy the classical three-term recurrence relation and did this by showing that a Sheffer sequence $\{P_n(x)\}$ is orthogonal if and only if the pair of basic and Appell functions in (1.8) is of one of the four special types mentioned below. It is, of course, now assumed that the underlying field is the field of real numbers.

The first type is

$$H(t) = \frac{at}{1-bt},$$
$$G(t) = \mu(1-bt)^{-c} \exp\left(\frac{d}{1-bt}\right) \qquad (ab\mu \neq 0, c > 0). \tag{4.1}$$

Since the Appell functions for the other three types will not be needed, we list here only the basic functions that occur. They are

$$H(t) = at \qquad (a \neq 0), \tag{4.2}$$

$$H(t) = a\log(1-bt) \qquad (ab \neq 0), \tag{4.3}$$

$$H(t) = a[\log(1-bt) - \log(1-ct)] \qquad (a \neq 0, bc > 0, b \neq c). \tag{4.4}$$

In (4.4), a can be pure imaginary, in which case b and c are complex conjugates; still $H(t)$ has real coefficients.

We shall use these four categories to characterize the orthogonal Steffensen sequences which have Property C; but first we give two lemmas.

Lemma 3. *If*

$$H(t) = h_1 t + h_2 t^2 + h_3 t^3 + \cdots \qquad (h_1 \neq 0),$$
$$K(t) = k_1 t + k_2 t^2 + k_3 t^3 + \cdots \qquad (k_1 \neq 0),$$

and $H(t) = K^{-1}(-K(t))$, *then* $h_1 = -1$ *and* $h_3 = -h_2^2$.

To see that $h_1 = -1$, write $K(H(t)) = -K(t)$, differentiate each side with respect to t, and set $t = 0$. The result is $k_1 h_1 = -k_1$ $(k_1 \neq 0)$. To obtain the relation $h_3 = -h_2^2$, note that $H(H(t)) = t$ and substitute

$$H(t) = -t + h_2 t^2 + h_3 t^3 + \cdots$$

into this identity, arriving at

$$t + 0t^2 - 2(h_2^2 + h_3)\,t^3 + \cdots = t + 0t^2 + 0t^3 + \cdots.$$

Equating the coefficients of t^3 gives the desired result.

Turning to the next lemma, we recall that a Steffensen sequence is a Sheffer sequence with basic function $H(t)$ and Appell function $\exp(\alpha K(t))\,G(t)$.

LEMMA 4. *If a Steffensen sequence* $\{P_n^{(\alpha)}(x)\}$ *is orthogonal for some value of* α *and has Property* C, *then its basic function must be of the type*

$$H(t) = \frac{-t}{1 - bt} \qquad (b \neq 0). \tag{4.5}$$

To show this, we assume that $\{P_n^{(\alpha)}(x)\}$ has the stated properties and note from Corollary 1 that $H(t) = K^{-1}(-K(t))$. Inasmuch as $\{P_n^{(\alpha)}(x)\}$ is an orthogonal *Sheffer* sequence for some value of α, our proof is based on an examination of the four possibilities for $H(t)$ that are listed just before Lemma 3.

In case (4.1), $h_1 = H'(0) = a$; and Lemma 3 tells us that $a = -1$. Thus $H(t)$ might be as shown in (4.5).

If $H(t)$ is as in (4.2), $h_1 = H'(0) = a$; and, again by Lemma 3, $H(t) = -t$. Now by Corollary 1 and Lemma 1, $H(t) = -t \exp K(t)$. Therefore,

$$K(t) = \log\left(\frac{H(t)}{-t}\right) = \log 1 = 0;$$

but this cannot be since $K(t)$ is not identically zero. The possibility that $H(t)$ is as in (4.2) is thus eliminated.

The possibilities of cases (4.3) and (4.4) are also readily eliminated with the aid of Lemma 3, as we now show.

In case (4.3), $h_1 = H'(0) = -ab$. So $ab = 1$, and $H(t)$ is the series

$$H(t) = -t - \frac{b}{2}t^2 - \frac{b^2}{3}t^3 - \cdots.$$

But $h_3 = -h_2^2$, which means that $b = 0$. This is a contradiction to the condition on b in (4.3).

In case (4.4), $h_1 = H'(0) = -a(b - c)$. So $a = 1/(b - c)$, and it follows that

$$H(t) = -t - \frac{b + c}{2}t^2 - \frac{b^2 + bc + c^2}{3}t^3 - \cdots.$$

Again since $h_3 = -h_2^2$, we have $b = c$. This contradiction eliminates the possibility of case (4.4), and Lemma 4 is proved.

The theorem presented below uses the following terminology, which defines an equivalence relation on the class of all simple polynomial sequences.

DEFINITION 4. Two simple polynomial sequences $\{p_n(x)\}$ and $\{q_n(x)\}$ are equivalent if there is a nonzero constant b such that $p_n(x) = b^n q_n(x/b)$ $(n = 0, 1, 2,...)$.

THEOREM 3. *A Steffensen sequence $\{P_n^{(\alpha)}(x)\}$ is orthogonal for some value of α and has Property C if and only if it is equivalent to the sequence $\{L_n^{(\alpha)}(x)\}$ of Laguerre polynomials.*

To prove this, we first assume that $\{P_n^{(\alpha)}(x)\}$ is equivalent to $\{L_n^{(\alpha)}(x)\}$. Evidently, then, it is generated by

$$(1 - bt)^{-\alpha - 1} \exp \left(\frac{-xt}{1 - bt}\right) = \sum_{n=0}^{\infty} P_n^{(\alpha)}(x) \frac{t^n}{n!} \qquad (b \neq 0). \tag{4.6}$$

The basic and Appell functions here are of the type (4.1) when $\alpha > -1$, and so $\{P_n^{(\alpha)}(x)\}$ is orthogonal for those values of α. It is easy to show that (4.6) is a special case of (1.4) when $K(t) = -\log(1 - bt)$ [the inverse being $K^{-1}(t) = (1 - e^{-t})/b$], $H(t) = K^{-1}(-K(t))$, and $G(t) = 1 + tK'(t)$; also, $e^{t/2}K^{-1}(t)$ is odd. Hence, by Corollary 1, $\{P_n^{(\alpha)}(x)\}$ has Property C.

Conversely, let us assume that $\{P_n^{(\alpha)}(x)\}$ is orthogonal for some α and that it has Property C. By Lemma 4, its basic function must be of the type

$$H(t) = \frac{-t}{1 - bt} \qquad (b \neq 0);$$

also, it follows from Corollary 1 and Lemma 1 that $H(t) = -t \exp K(t)$. So

$$K(t) = \log \left(\frac{H(t)}{-t}\right) = -\log(1 - bt).$$

Moreover, by Corollary 1,

$$G(t) = 1 + tK'(t) = (1 - bt)^{-1}.$$

The sequence $\{P_n^{(\alpha)}(x)\}$ is then generated by (4.6), which means that $\{P_n^{(\alpha)}(x)\}$ is equivalent to $\{L_n^{(\alpha)}(x)\}$. This completes the proof of Theorem 3.

ACKNOWLEDGMENT

Part of this work was supported by a Campus Grant from The University of Michigan–Dearborn.

REFERENCES

1. P. APPELL, Sur une classe de polynômes, *Ann. Sci. École Norm. Sup. (2)* **9** (1880), 119–144.
2. J. W. BROWN, Generalized Appell connection sequences, II, *J. Math. Anal. Appl.* **50** (1975), 458–464.
3. J. W. BROWN, On multivariable Sheffer sequences, *J. Math. Anal. Appl.* **69** (1979), 398–410.
4. J. W. BROWN AND J. L. GOLDBERG, Generalized Appell connection sequences, *J. Math. Anal. Appl.* **46** (1974), 242–248.
5. J. W. BROWN AND M. KUCZMA, Self-inverse Sheffer sequences, *SIAM J. Math. Anal.* **7** (1976), 723–728.
6. J. MEIXNER, Orthogonale Polynomsysteme mit einer besonderen Gestalt der erzeugenden Funktion, *J. London Math. Soc.* **9** (1934), 6–13.
7. G.-C. ROTA, D. KAHANER, AND A. ODLYZKO, On the foundations of combinatorial theory. VIII. Finite operator calculus, *J. Math. Anal. Appl.* **42** (1973), 684–760.
8. I. M. SHEFFER, On sets of polynomials and associated linear functional operators and equations, *Amer. J. Math.* **53** (1931), 15–38.
9. I. M. SHEFFER, Some properties of polynomial sets of type zero, *Duke Math. J.* **5** (1939), 590–622.