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3-MANIFOLDS WHOSE UNIVERSAL COVERINGS ARE LIE GROUPS*

To R.H. Bing on the occasion of his 65th birthday

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We classify those closed 3-manifolds whose universal covering space naturally admits the structure of a Lie group.

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Introduction

Our objective is to describe the closed 3-manifolds M whose universal covering naturally admits the structure of a Lie group G. Consequently, G contains a uniform discrete subgroup Γ and M may be identified with $\Gamma\backslash G$. The map $G \to M$ is a covering map and Γ is the group of covering transformations.

A convenient list of the 3-dimensional Lie groups that have uniform discrete subgroups can be found in [1] or in [4]. Our problem was treated in [1] and rather complete descriptions are obtained for the nilpotent and solvable cases. The two simple cases are partially described but are seriously incomplete and erroneous, e.g. [1; Theorem 4.6].

Our method, in the simple cases, is to describe M as an SO(2) manifold, and hence a Seifert manifold, and to determine from Γ its explicit Seifert invariants. This method turns out to work for all the Lie groups except one class of solvable groups (which are treated separately).

A rough summary of our results can be gleaned from Table 1 on the following page. Of course, more explicit results are to be found in the text along with appropriate references and carefully defined terminology.

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Table 1

Lie algebra	Simply connected form of the associated Lie group	Γ	$\Gamma\backslash G$, compact
1. Abelian	$\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$	$Z\oplus Z\oplus Z$	T^3, the 3-dimensional torus
2. Nilpotent	$\begin{pmatrix}1 & x_1 & x_2\\ 0 & 1 & x_3\\ 0 & 0 & 1\end{pmatrix}$ = "Heisenberg group" $x_1, x_2, x_3 \in R$	$\begin{pmatrix}1 & n_1 & n_2/k\\ 0 & 1 & n_3\\ 0 & 0 & 1\end{pmatrix}$ $n_1, n_2, n_3 \in Z$, k a fixed positive integer. The group is of the form $(Z\times Z)\circ_\phi Z$.	All principal S^1-bundles over the 2-torus with Euler class = $k \in H^2(T^2, Z)$. Fibers over S^1 with fiber a torus and geometric monodromy $\begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix}$.
3. Solvable (not nilpotent), 2-types (a) Two structure constants have the same sign, third is 0.	The universal covering of the Euclidean group (*orientation preserving* isometries of $\mathbb{R}^2$) $E^+(2)$ = Translations ∘ SO(2). $= \begin{pmatrix}\cos 2\pi\theta & -\sin 2\pi\theta & x_1\\ \sin 2\pi\theta & \cos 2\pi\theta & x_2\\ 0 & 0 & 1\end{pmatrix}$	$(Z\times Z)\circ_\phi Z$, with cyclic monodromy. (All possibilities appear, there are only 5 non-isomorphic groups.)	The closed orientable flat 3-manifolds with finite cyclic holonomy. There are just 5 such manifolds.
(b) Two structure constants have opposite signs, third is >0.	$\mathbb{R}^2\circ\mathbb{R}^1$ called the inhomogeneous Lorentz group $E(1, 1)$. A matrix form is obtained by replacing trigonometric functions by hyperbolic ones in the matrix form of the universal covering of $E^+(2)$.	$(Z\times Z)\circ Z$ with infinite cyclic monodromy where both eigenvalues must be positive and distinct from 1. All such possibilities do occur.	A torus bundle over the circle; geometric monodromy given by the algebraic monodromy.
4. Simple (2 types) (a) Compact form	SU(2)	The finite subgroups of SU(2)-cyclic, and binary dihedral, binary tetrahedral, binary octahedral, and binary icosahedral groups.	All are manifolds of constant positive curvature, but not all occur. Explicit description will be given.
(b) The non-compact form	The universal covering of PSL(2, $\mathbb{R}$).	Fuchsian groups of the first kind. Γ has signature $\{g; \alpha_1, \ldots, \alpha_n\}$ in the uniform case. The non-uniform case is also easily described.	Seifert manifolds whose explicit invariants will be described. For $\Gamma\backslash$PSL(2, $\mathbb{R}$), we have: $\{2g-2=b;\ g; (\alpha_1, \alpha_1-1), \ldots, (\alpha_n, \alpha_n-1)\}$

In the first section we shall quickly sketch the first 3 cases of the table. The bulk of the paper concentrates on 4(b). In the next section we examine discrete subgroups Γ of a Lie group G with K a compact subgroup. On $\Gamma\backslash G = M$ we describe the invariant tubular neighborhoods of the orbits of the right K-action $(\Gamma\backslash G, K)$ in terms of the Lie algebra $\mathfrak{g}/\mathfrak{k}$ and the action of the isotropy group K_0 on $\mathfrak{g}/\mathfrak{k}$ induced from the adjoint action. When specialized to 3-dimensions this enables us to determine the slice invariants (α_i, β_i) of the resulting SO(2)-action. For the non-compact case this is essentially all the information needed to determine $\Gamma\backslash G$. However, in the compact case another invariant, the *Euler number* of the SO(2)-section, i.e. the *Seifert "b" invariant*, must be determined. This is done in Sections 5 and 6.

For cases 3(a) and 4 a reduction of our problem is made in Section 4 to the special cases of G in its adjoint form, G_1. That is, G_1 can be regarded as a group of orientation preserving isometries of a simply connected surface having constant Gauss curvature. The identification of $\Gamma\backslash G$ is then captured from $\Gamma_1\backslash G_1$ by examining how the Seifert invariants change under nice cyclic coverings, Section 4.

These results were announced in the informal report of the algebraic topology Conference at Oberwolfach in September 1973. The authors apologize for the subsequent delay in publication. Part of the problem has been the many alternative versions and approaches to these results. We have adopted here what we think is the most straightforward approach and one which generalizes somewhat to other interesting situations. Also as a bonus for this particular presentation we are able to sketch a couple of other approaches in Section 7.

For readers conversant with the terminology of Seifert manifolds we shall now formulate our main results (In cases 3(a) and 4). The usual *normalized Seifert* invariants are used. For other readers the necessary definitions, notations, and references are given in the text at the appropriate times. In particular we use the standard notations of [9], [8] and [7].

Theorem 1. *Let G_∞ be the universal covering of* PSL(2, ℝ) *and Γ^∞ be a uniform discrete subgroup of G_∞. Then $M = \Gamma_\infty\backslash G_\infty$, as an* SO(2)*-manifold, has the form of*

$$M = \{g; b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\},$$

if and only if, there exists a divisor r of

$$\alpha_1 \cdots \alpha_n\left(2g - 2 + n - \sum_{i=1}^{n} 1/\alpha_i\right)$$

prime to each α_i so that

$$\beta_i r \equiv -1 \bmod \alpha_i \quad \textit{and} \quad rb = (2g-2) - \sum_{i=1}^{n} k_i,$$

where

$$\beta_i r = \alpha_i - 1 + k_i\alpha_i.$$

Γ_∞ will be a subgroup of index r in the group $\tilde{\Gamma} = p_1^{-1}(\Gamma_1)$, where p_1: $G_\infty \to G_1 =$ PSL(2, $\mathbb{R}$) is the covering and $\Gamma_1 = p_1(\Gamma_\infty)$ is a Fuchsian group with signature $(g; \alpha_1, \ldots, \alpha_n)$. The condition that Γ_1 exists is $2g-2+n-\sum_{i=1}^n 1/\alpha_i > 0$. In particular,

$$\Gamma_1\backslash G_1 = \{g; b = 2g-2; (\alpha_1, \alpha_1 - 1), \ldots, (\alpha_n, \alpha_n - 1)\}.$$

Γ_∞/G_∞ will be a r-fold central covering of $\Gamma_1\backslash G_1$. $\Gamma_\infty = \tilde{\Gamma}$, and hence, $\Gamma_\infty\backslash G_\infty = \Gamma_1\backslash G_1 = \Gamma_1\backslash$PSL(2, $\mathbb{R}$) if and only if $r = 1$.

The reader may wish to compare the present statement with the version in [6; Theorem 3.1] which uses the *unnormalized Seifert invariants*. (The reader is requested to observe that a typographical error, in 3.1 has g appearing in the formulae where $2g$ is what is intended.)

It is also appropriate to point out that if $g = 0$, $n = 3$, $1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 > 1$, then we are in *case* 4(a). The same formulae as above applies since the proofs are essentially the same. However, the list is correspondingly smaller. We shall list all of them below. The triples refer to $(\alpha_1, \alpha_2, \alpha_3)$

$(2, 2, n)$ $\quad \Gamma\backslash SU(2) = \{0; -2, (2, 1), (2, 1), (n, n-1)\}$.

These are the *prism manifolds*. There fundamental groups are the *binary dihedral groups*, $n \geq 2$.

$(2, 3, 3)$ $\quad \Gamma\backslash SU(2) = \{0; -2; (2, 1), (3, 2), (3, 2)\}$.

The fundamental group is the *binary tetrahedral group.*

$(2, 3, 4)$ $\quad \Gamma\backslash SU(2) = \{0; -2, (2, 1), (3, 2), (4, 3)\}$.

The fundamental group is the *binary octahedral* group.

$(2, 3, 5)$ $\quad \Gamma\backslash SU(2) = \{0, -2; (2, 1), (3, 2), (5, 4)\}$.

The space is the *Poincaré homology sphere* with fundamental group the *binary icosahedral group.*

For $n \leq 2$ and $g = 0$, we are in the case of Γ cyclic. Then $\Gamma\backslash SU(2)$ or $\Gamma_1\backslash SO(3)$ occur as lens spaces. They are all of the form $L(p, 1)$ (or $L(p, -1)$ depending upon orientation conventions). As SO(2)-manifolds they have many representations (unlike all the other cases). For $\Gamma \subset SO(3)$, one can choose the right circle subgroup and write $\Gamma\backslash SO(3)$ as $\{g = 0, b = -2, (\alpha, \alpha-1), (\alpha, \alpha-1)\}$. It is of the form $L(2\alpha, 1)$.

Case 3(a) occurs for $2g-2+n-\sum_{i=1}^n 1/\alpha_i = 0$, $g = 0$, (with the exception of T^3 in which case $g = 1$, $n = 0$). We may also, using the same argument, list these possibilities:

(a) $(2, 2, 2, 2,) \leftrightarrow \{0: -2; (2, 1), (2, 1), (2, 1)$ $(2, 1)\}$.
(b) $(2, 4, 4) \leftrightarrow \{0\ -2; (2, 1), (4, 3), (4, 3)\}$.
(c) $(3, 3, 3) \leftrightarrow \{0, -2; (3, 2), (3, 2), (3, 2)\}$.
(d) $(2, 3, 6) \leftrightarrow \{0; -2; (2, 1), (3, 2), (6, 5)\}$.
(e) The 3-torus $\leftrightarrow \{1; 0\}$.

These are the flat orientable 3-manifolds with holonomy $\mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbb{Z}/4$, $\mathbb{Z}/6$, and 1.

Because the "Euler number" [6] is 0, these manifolds cover themselves and so when one looks at any cover of $E^+(2)$ nothing new is found.

Of considerable interst is the situation where $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ and $\Gamma\backslash\mathrm{PSL}(2, \mathbb{R})$ is *not* compact but Γ is finitely presented. More generally, $\Gamma_\infty \subset G_\infty$ so that Γ_∞ is finitely presented. $\Gamma_\infty\backslash G_\infty$ will be an SO(2) manifold and its orbit space, $\Gamma_\infty\backslash G_\infty/\mathrm{SO}(2)$, will be an open 2-manifold which needs $h > 0$ points to compactify it to a surface of genus g. If we again let $p_1: \Gamma_\infty \to \Gamma_1$ be the projection then Γ_1 will have "signature": $\{g; h; \alpha_1, \ldots, \alpha_n\}$

Analagous to Theorem 1 (but obtained with less difficulty) we are already able in Section 4 to prove

Theorem 2. *$M = \Gamma_\infty\backslash G_\infty$, an SO(2) manifold, has orbit invariants*

$$\{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$$

if and only if, there exists a positive integer r so that

$$\beta_i r = -1 \mod \alpha_i.$$

As before Γ_∞ is a subgroup of index r in the group $\tilde{\Gamma} = p_1^{-1}(\Gamma_1)$.

Note that in the closed case the number of distinct r's is finite. In the open case there are an infinite number of r's but only a finite number of manifolds since the β_i's are normalized between $0 < \beta_i < \alpha_i$. As before, if $\tilde{\Gamma} = \Gamma_\infty$, that is $r = 1$, then the orbit invariants of $\Gamma_1\backslash G_1$ are

$$\{(\alpha_1, \alpha_1 - 1), \ldots, (\alpha_n, \alpha_n - 1)\}$$

Finite presentation of $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ turns out also not to be a necessary assumption for a version of Theorem 2; see the Remarks at the end of Section 6.)

As an illustration, it is amusing to consider the complement, $M(a, b)$, of an (a, b) *torus knot*, $K(a, b)$, in the 3-sphere. Here, a and b are relatively prime and $2 \leq b < a$. $M(a, b)$ admits the structure of an SO(2) manifold with exactly 2 exceptional orbits $(a, b^{-1}) = (\alpha_1, \beta_1)$, and $(b, a^{-1}) = (\alpha_2, \beta_2)$ and orbit space the open 2-disk. As *normalized* invariants this means that β_1 is the integer reduced modulo $\alpha_1 = a$, so that $\beta_1 b \equiv 1 \mod a$. Similarly, β_2 is the integer reduced modulo b so that $\beta_2 a \equiv 1 \mod b$. We see, by a simple calculation, that *only the trefoil knot* K (3, 2) can appear in the form $\Gamma\backslash\mathrm{PSL}(2, \mathbb{R})$. In fact, it is readily seen to be $\mathrm{PSL}(2, \mathbb{Z})\backslash\mathrm{PSL}(2, \mathbb{R})$. A more standard form of this knot space is $\mathrm{SL}(2, \mathbb{Z})\backslash\mathrm{SL}(2, \mathbb{R})$. Of course, they are the same space. Again a simple calculation shows that for $K(a, b)$, $(a, b) \neq (3, 2)$, $M(a, b)$ is of the form $\Gamma_r\backslash G_r$, $r \neq 1$. In fact, r satisfies the simultaneous congruences $r\beta_1 \equiv -1 \mod a$ and $r\beta_2 \equiv -1 \mod b$.

One would expect manifolds of the form $\Gamma\backslash G$ to be geometrically interesting. In [6] one can find proofs that all the Seifert manifolds which are integral homology spheres are of the form $\Gamma\backslash G$. Characterizations for complete intersection of Brieskorn varieties to be of the form $\Gamma\backslash G$ are also given there. See also [5] and [3].

1. Abelian, nilpotent, and solvable groups

We make use of the convenient catalogues in [1] and [4] of the 3-dimensional Lie algebras. All of the associated Lie groups contain uniform discrete subgroups except for one family on non-unimodular solvable Lie groups.

1.1. *Abelian case.* $\Gamma\backslash G$ must be a 3-dimensional torus.

1.2. *Nilpotent case.* Let Γ be any torsion free nilpotent finitely generated group. Then according to Malcev, see for example [10], there exists a unique, up to isomorphism, simply connected nilpotent Lie group G and an embedding $\Gamma \to G$ as a lattice in G. All such embeddings are equivalent via an automorphism of G. If C is the center of G, then it is not hard to see that C is connected $\neq 1$ and center $(\Gamma) = C \cap \Gamma$. Moreover, $\Gamma \cap C$ is discrete in C. Therefore, in $\Gamma\backslash G$, we have an action of the torus $K = C \cap \Gamma\backslash C$. This action is free since $\Gamma \cap C\backslash\Gamma$ is a nilpotent subgroup of G/C (We shall see, in general, that the K stability groups are isomorphic to torsion subgroups of $\Gamma \cap C\backslash\Gamma$ which, in the present case, is torsion free). Since G/C is nilpotent and $\Gamma \cap C\backslash\Gamma$ acts freely we see that *$\Gamma\backslash G$ is a principal K*-bundle over the nilmanifold $((\Gamma \cap C)\backslash\Gamma)\backslash(G/C)$.

In the special case of dimension 3 this means that K is a circle and $\Gamma\backslash G$ is a principal circle bundle over the 2-torus. As an SO(2) manifold its Seifert invariants are $\{g = 1, -b = \text{Euler class of the SO(2) bundle}\}$. Explicit matrix representations are found in [1, 4] and in Table 1.

1.3. *Solvable (non-nilpotent) case.* There are two non-isomorphic, 3-dimensional solvable Lie algebras whose corresponding simply connected Lie groups S_1 and S_2 admit lattices, [1] and [4]. The adjoint form of S_2 is the Euclidean group, $E^+(2)$, and so S_2 is $\mathbb{R}^2 \circ \mathbb{R}^1$, where $\mathbb{R}^1$ operates on $\mathbb{R}^2$ by the rotations induced from $SO(2) \subset E^+(2)$. The center of S_2 is infinite cyclic, corresponding to π_1 (SO(2)). Let $p: S_2 \to E^+(2)$ denote this covering projection. For each lattice Γ in S_2 the projection $p(\Gamma) = \Gamma/\Gamma \cap (\text{center } S_2)$ is a uniform lattice in $E^+(2)$. Hence $\Gamma \cap \mathbb{Z}$ is infinite cyclic and $\Gamma\backslash S_2$ is diffeomorphic to $(\Gamma \cap \mathbb{Z}\backslash S_2)/(\Gamma/\Gamma \cap \mathbb{Z})$. This is a finite cyclic covering of $p(\Gamma)\backslash E^+(2)$. It will turn out, as we shall observe later, that the two manifolds in question will be diffeomorphic. Therefore the determination of $p(\Gamma)\backslash E^+(2)$ to be accomplished in Section 6 will determine all the possibilities of $\Gamma\backslash S_2$.

Let us now assume that $\Gamma \subset G$ is uniform and $G = S_1$. Using the fact that $S_1 = \mathbb{R}^2 \circ_\phi \mathbb{R}^1$ with $\Gamma \cap \mathbb{R}^2$ and (projection $\Gamma) \cap \mathbb{R}^1$ is discrete, it is shown in [1] that Γ has no center and $\Gamma\backslash G$ is a torus bundle over a circle. $\Gamma\backslash G$ admits no S^1-action, since Γ has no center. The eigenvalues of the geometric monodromy are also described. Since any torus bundle M over a circle is described completely by a matrix $\Phi \in GL(2, \mathbb{Z})$ we see that to obtain M as $\Gamma\backslash G$ above we need a $\Phi \in SL(2, \mathbb{Z})$, diagonalizable, with both eigenvalues positive. If $'\Phi \in SL(2, \mathbb{Z})$ is diagonalizable but has both eigenvalues negative and not -1, then $'\Phi^2$ will have the desired properties.

If $'M$ corresponds to $'\Phi$, then a double covering M corresponding to $('\Phi)^2$ will yield a manifold of the type $\Gamma\backslash S_1$.

2. $\Gamma\backslash G$ as a K-space

Let Γ be a discrete subgroup of the Lie group G and K a compact subgroup of G. Consider the (right) action of K on $M = \Gamma\backslash G$. Let $m_0 \in M$ and K_0 be the isotropy subgroup of K at m_0, i.e. $\{k \in K | m_0 k = m_0\}$. The proposition below shows that a *K-invariant neighborhood* in M of the orbit *through m_0 depends* (as a K-space) *only* on the triple (G, K, K_0). In particular, Γ and the coset $m_0 = \Gamma g_0 \in M$ are totally irrelevant. More specifically we consider the adjoint action of G on its Lie algebra $\mathfrak{g}$. If we restrict this action to K it leaves invariant K's Lie algebra, $\mathfrak{k}$. Thus we get an induced action of K on $\mathfrak{g}/\mathfrak{k}$. If K_0 is any subgroup of K we can further restrict and obtain an action of K_0 on $\mathfrak{g}/\mathfrak{k}$. We may form the associated vector bundle over $K_0\backslash K$, $(\mathfrak{g}/\mathfrak{k}) \times_{K_0} K$. This is, by definition, the set of K_0-orbits of the diagonal left action, $k_0 \times (v, k) \to (v k_0^{-1}, k_0 k)$, of K_0 on $(\mathfrak{g}/\mathfrak{k}) \times K$. There is an obvious right action of K on $(\mathfrak{g}/\mathfrak{k}) \times_{K_0} K$ induced from right multiplication on the right hand factor in $\mathfrak{g}/\mathfrak{k} \times K$. Thus, $(\mathfrak{g}/\mathfrak{k}) \times_{K_0} K$ is a right K-space. Note that the 0-section is a single K-orbit. The isotropy subgroup of the point $\langle 0, e\rangle$ is K_0. (We denote the image of an element of $\mathfrak{g}/\mathfrak{k} \times K$ in $(\mathfrak{g}/\mathfrak{k}) \times_{K_0} K$ by enclosing it in the bracket "$\langle\ ,\ \rangle$".)

2.1. Proposition. *Let Γ, G, K be as above. There is a K equivariant diffeomorphism between a K-invariant neighborhood of the K orbit through $m_0 = \Gamma g_0 \in \Gamma\backslash G$ and the K-space $(\mathfrak{g}/\mathfrak{k}) \times_{K_0} K$. Furthermore, the map takes m_0 to $\langle o, e\rangle$ and the K-orbit through m_0 to the 0-section of the vector bundle $(\mathfrak{g}/\mathfrak{k}) \times_{K_0} K$.*

We emphasize that K_0 acts on $\mathfrak{g}/\mathfrak{k}$ via the K action on $\mathfrak{g}/\mathfrak{k}$ induced from the adjoint action.

Proof. We shall deduce this from the slice theorem, (see [2; VIII] for a convenient proof), and we shall recall its statement. The orbit $m_0 K$ is in an obvious way diffeomorphic to the coset space $K_0\backslash K$. This is a smooth submanifold of M. If N denotes the fiber of its normal bundle at the point m_0, the differential provides us with a (right) linear action of K_0 on N. We may form the vector bundle $N \times_{K_0} K$ which, by definition, is the space of K_0-orbits of the following left K_0 action on $N \times K$; $k_0(n, k) = (n k_0^{-1}, k_0 k)$. It is possible to identify N with a *K_0-invariant subset of M* transverse to $m_0 K$ *in such a way that the map* $N \times K \to M$: $(n, k) \to nk$ induces a diffeomorphism of $N \times_{K_0} K$ onto a K invariant neighborhood of $m_0 K$. The diffeomorphism is obviously K-equivariant relative to the obvious (right) K-action on $N \times_{K_0} K$. Furthermore, it takes the 0-section of the vector bundle $N \times_{K_0} K$ diffeomorphically onto $m_0 K$.

In view of the slice theorem we see that the problem is to show that the right linear action of K_0 on N is independent of Γ and the point $m_0 = \Gamma g_0$ and, in fact, depends only on the triple (G, K, K_0).

Our task reduces to verifying that we can identify the linear space N with $\mathfrak{g}/\mathfrak{k}$ and the action of K_0 on N with the restriction to K_0 of the adjoint induced action of K on $\mathfrak{g}/\mathfrak{k}$. For this, we consider the map $p: G \to M = \Gamma\backslash G$ defined by $p(g) = \Gamma g_0 g$. It is a K-equivariant map if we let K act by right multiplication on G. It is a local diffeomorphism (since Γ is discrete) taking e (e = the identity of G) to m_0. Thus the differential of p at e identifies $\mathfrak{g} = T(G)_e$ with $T(M)_{m_0}$. Further, it identifies $\mathfrak{k}$ with the tangent space to the orbit m_0K; thus it identifies $\mathfrak{g}/\mathfrak{k}$ with N = the fiber of the normal bundle to the orbit m_0K at the point m_0. It remains to find the action of K_0 on N. It is easily verified that $K_0 = (g_0^{-1}\Gamma g_0) \cap K$. Hence it follows that $p(k_0 g k_0^{-1}) = p(g)\, k_0^{-1}$, for all $k_0 \in K_0$ and all $g \in G$. If we convert the right action, in the familiar way, to a left action (define $k_0 \cdot n$ to be nk_0^{-1}, for all $n \in N$, $k_0 \in K_0$) then the adjoint action of K_0 on G is equivariant, via p, with the new left K_0 action on M. If we now recall that $Ad(k_0): \mathfrak{g} \to \mathfrak{g}$ is defined to be the differential at e of the map $g \to k_0 g k_0^{-1} : G \to G$, the result follows immediately.

2.2. For future reference let Γ be a discrete subgroup of G and K a compact subgroup of G acting on the right. Form the left transitive G-space G/K and the right K-space $\Gamma\backslash G$. If $x = gK \in G/K$, then

$$G_x = gKg^{-1} \quad \text{and} \quad \Gamma x = gKg^{-1} \cap \Gamma.$$

Let $\Gamma g = y \in \Gamma\backslash G$, then $Ky = g^{-1}\Gamma g \cap K$. Obviously, $g^{-1}\Gamma_x g = K_y$. The restriction to Γ of the G action on G/K is properly discontinuous and thus the orbit map $G/K \to \Gamma\backslash(G/K) = \Gamma\backslash G/K$ is a branched covering map. Thus if $b = \Gamma gK$ represents a point in $\Gamma\backslash G/K$, then the K-orbit over b, $(\Gamma g)K$, is isomorphic to the coset space $K_y\backslash K$, and the Γ-orbit over b, $\Gamma(gK)$, is isomorphic to Γ/Γ_x. Hence, branching occurs exactly when $K_y \cong \Gamma_x$ is not trivial. We may also represent a tubular neighborhood to the Γ-orbit of x in G/K as $(\Gamma, \Gamma \times_{\Gamma_x} \mathfrak{g}/\tilde{\mathfrak{k}})$. Here $\tilde{\mathfrak{k}}$ is the Lie algebra of the subgroup gKg^{-1}, conjugate to K. The tangent space to gK is naturally represented by $\mathfrak{g}/\tilde{\mathfrak{k}}$, $(\cong \mathfrak{g}/\mathfrak{k})$, the action of $\Gamma_x = gKg^{-1} \cap \Gamma$ on $\mathfrak{g}/\tilde{\mathfrak{k}}$ is induced by the restriction of the adjoint action of gKg^{-1} on $\mathfrak{g}/\tilde{\mathfrak{k}}$. This we see by considering the Γ-equivariant mapping $q: G \to G/K$, defined by $q(\bar{g}) = \bar{g}\, gK$, and taking the differential at e. Note if $\gamma \in \Gamma \cap gKg^{-1} = \Gamma_x$, then

$$q(\gamma \bar{g} x^{-1}) = \gamma \bar{g} g K = \gamma q(g).$$

Thus, the actions of $\Gamma_x \cong K_y$ are equivalent on the normal disks $\mathfrak{g}/\tilde{\mathfrak{k}}$ and $\mathfrak{g}/\mathfrak{k}$ after we take into account shifting from left to right actions.

3. Dimension $G = 3$

To treat the remaining cases 3(a) and 4 (as well as 2) of our table from the introduction, we specialize to dim $G = 3$, K compact, connected and 1-dimensional, with Γ discrete in G. First note $\Gamma\backslash G$ *is* always *orientable*. Note further that K is isomorphic as a Lie group to $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. We obtain a right action of S^1 on $\Gamma\backslash G$ without fixed points.

These actions are studied and classified in [8] and our purpose is to describe which of the possibilities in [8] arise in this way. These S^1 manifolds all admit the structure of a Seifert fiber space, [9], with the orbit mapping corresponding to the fibering and the exceptional orbits corresponding to the singular (or multiple) fibers.

Choose an isomorphism $\theta : S^1 \to K$ (there are two such and the effect on the invariants of making the other choice is easily understood, see [8, p. 63]), and obtain a *right* action of S^1 on the 3-manifold $\Gamma\backslash G$. In [8], *left* actions are studied but S^1 is abelian and so we may view our right action as a left action; that is if $z \in S^1$, and $m \in \Gamma\backslash G$ then $z \cdot m$ will mean the point previously denoted by $m\theta(z)$. (Needless to say other conventions could be made with at least as much rationale.) As stated above, $\Gamma\backslash G$ is *orientable* but of course, *not* oriented; here again we have a choice and the effect of the choice on the resulting invariants is easily understood [8, p. 63]). Consider

$$S^1 \xrightarrow{\theta} K \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g}/\mathfrak{k}).$$

From the above analysis this composition is non-trivial. It follows therefore that we may choose a real linear isomorphism $\psi : \mathbb{C} \to \mathfrak{g}/\mathfrak{k}$ so that

$$(3.1) \qquad (\mathrm{Ad} \circ \theta)(z)(\psi(\lambda)) = \psi(z^r \lambda)$$

for some positive integer r; (indeed r equals cardinality of the kernel of the composition ($\theta \circ \mathrm{Ad}$)). Any two such isomorphisms differ by a rotation of $\mathbb{C}$; in particular by an orientation preserving isomorphism of $\mathbb{C}$. Thus by using such a ψ to transfer the usual orientation of $\mathbb{C}$ to $\mathfrak{g}/\mathfrak{k}$ we have thus oriented $\mathfrak{g}/\mathfrak{k}$. This, of course, orients $\mathfrak{g}$ for $\mathfrak{k}$ is oriented by $d\theta : \mathbb{R} \xrightarrow{\cong} \mathfrak{k}$ and the usual orientation of $\mathbb{R}$, (here we are thinking of $\mathbb{R}$ as being the Lie algebra of S^1, $t \to e^{2\pi i t} \in S^1$, for $t \in \mathbb{R}$). This orients G and thus $\Gamma\backslash G$. In summary, the special nature of the pair (G, K) is such that once we choose an isomorphism $\theta : S^1 \to K$ we may orient G; if we choose the other isomorphism $S^1 \to K$ we would have arrived at the same orientation on G.

Let us return to the problem of determining the *oriented Seifert invariants* (α, β) of an *exceptional orbit* in these cases, see [8, p. 61]. Suppose $x \in M = \Gamma\backslash G$ is a point on the exceptional orbit. Then we know α is the *cardinality* of the *isotropy subgroup*. Furthermore, β satisfies $0 < \beta < \alpha$ and $\beta\nu \equiv 1 \bmod \alpha$ where ν describes the *slice representation*. Now α describes the isotropy subgroup, $\{z \in S^1_x \mid z^\alpha = 1\}$. We will refer to $\gamma = e^{2\pi i/\alpha}$ as the "generator" of this group despite the obvious objections to this usage. Now if we identify the fiber at x of the normal bundle to the orbits with $\mathbb{C}$, via the isomorphism described above, we see that the action of γ on $\lambda \in \mathbb{C}$ is $\gamma^{-r}\lambda$. Hence ν is determined by $0 < \nu < \alpha$ and $\nu \equiv -r \bmod \alpha$. The minus sign comes from the conversion of a right action to a left action and the remarks about the adjoint action. The general theory of [8] shows that ν is relatively prime to α. Since $-r \equiv \nu \bmod \alpha$, this *limits* (for the pair (G, K) which determined r) the α's that can arise for a particular pair (G, K). Summarizing we have

Proposition 3.2. *Let r be the cardinality of the kernel of the map* $\mathrm{Ad}: K \to \mathrm{Aut}(\mathfrak{g}/\mathfrak{k})$. *Let $x \in \Gamma\backslash G$ be an exceptional orbit. Its oriented Seifert invariant is (α, β) where α = the cardinality of the isotropy subgroup of x, $0 < \beta < \alpha$ and $-r\beta \equiv 1 \bmod \alpha$. (In particular r and β are relatively prime to α.)*

3.3. We shall be especially interested in the following situation: G is the group of *orientation preserving isometries of a complete simply connected surface S of constant (Gauss) curvature κ.* Thus, G acts on S (on the left). Via the differential G acts on $U(S)$, the bundle of unit tangent vectors of S. G acts transitively on S, and if G_x is the isotropy group of point $x \in S$, then $K = G_x$ acts effectively and transitively on $U(S)_x$, the unit tangent vectors to S at x. Thus each G_x is a circle and all the G_x's are conjugate. These subgroups are the only such subgroups of G. Now $r = 1$ is equivalent to the assertion that K acts freely on the unit vectors at the point in question. *Hence,*

Proposition. $\beta = \alpha - 1$.

Although this material is undoubtedly familiar we will be even more explicit.

3.4. *Case I. Curvature κ is negative.* We may choose as the model of the surface, S, the interior of the unit disc, provided with the Poincaré metric. We will choose 0 in S to be a special point. G is equally well described as the set of complex analytic automorphisms of S and is naturally identified with $\mathrm{PSL}(2, \mathbb{R})$. K is then the set $\{f\}$ of the form $f(z) = \gamma z$ for some $\gamma \in S^1$. This, of course, gives a preferred isomorphism $S^1 \cong K$. The orbit space of K-orbits, G/K, is naturally isomorphic to S. Thus, $\mathfrak{g}/\mathfrak{k}$ is identified with the tangent space of S at 0. Via the map $\psi: \mathbb{C} \cong \mathfrak{g}/\mathfrak{k}$, used above in our orientation conventions, we have managed to identify $\mathbb{C}$ with the tangent space to the interior of the unit disk at the origin. Our conventions were chosen so that this is the *usual* isomorphism. (More precisely, since ψ is not completely specified, we shall say that the usual isomorphism is a permissable ψ.)

In this case a discrete group Γ automatically acts properly discontinuously on S and $\Gamma\backslash S$ is compact if Γ is uniform. As is well known $S \to \Gamma\backslash S$ is a *branched* covering space; $\Gamma\backslash S$ inherits a complex structure from S and is hence a (compact, if Γ is uniform) Riemann surface. It does not, in general, inherit the Riemannian manifold structure despite the fact that Γ operates isometrically; at the branch points the differential of the map $S \to \Gamma\backslash S$ is *not* a linear isomorphism. (Indeed, it is 0.) However, in the special case in which there is no branching, (Γ is torsion free), $\Gamma\backslash S$ is naturally a Riemannian manifold of constant negative curvature. By what was said above, $\Gamma\backslash G$ is the *unit tangent bundle* of $\Gamma\backslash S = \Gamma\backslash(G/K) = \Gamma\backslash G/K$ *when Γ is torsion free*. In any case when Γ is an arbitrary discrete subgroup of G, $\Gamma\backslash S$ is identified with $\Gamma\backslash(G/K) = \Gamma\backslash G/K$ = the set of K orbits of the K-action on $M^3 = \Gamma\backslash G$.

The classical analysis of this situation yields a presentation of Γ of the form:

$$(3.5) \qquad \langle a_1, b_1, \ldots, a_g, b_g, q_1, \ldots, q_n, t_1, \ldots, t_s \mid 1 = q_1 q_2 \cdots q_n t_1 t_2 \cdots t_s [a_1, b_1] \cdots [a_g, b_g],\ q_1^{e_1} = \cdots = q_n^{e_n} = 1 \rangle,$$

Here $[x, y] = xyx^{-1}y^{-1}$, and Γ, for convenience is *assumed finitely presented*, g is the genus of the Riemann surface having s-holes ($s \geq 0$) and n is the number of branch points ($n \geq 0$). The e_i's describe the branching and are consequently integers ≥ 2. Indeed, e_i is the α_i associated with the exceptional K-orbit which lies over the branch point, (by 2.2). Γ is uniform if and only if $s = 0$, *ie.* there are no "t"-generators. The algebraic structure, when Γ is *not* uniform, just reduces to

$$Z * \cdots * Z * (Z/e_1) * \cdots * (Z/e_n)$$

where we have $2g + s - 1$ free factors. There is the restriction

$$2 - 2g - \sum_{i=1}^{i=n} (1 - e_i^{-1}) < 0$$

in the uniform case and

$$n + 2g + s - 1 > 1$$

in the non-compact case. As is well known this classical presentation does not determine the embedding of the abstract group Γ in G (not even up to conjugation). In Γ there are n-conjugacy classes of (maximal) finite subgroups; these are represented by the cyclic subgroups of order e_i generated by q_i. Thus, the *abstract* group Γ determines the integer n, the integers $e_i = \alpha_i$, g, and $s = 0$. Since we know $\beta_i = \alpha_i - 1$, these *invariants* of the *abstract group come very close* to describing the action $(S^1, \Gamma\backslash G)$ in the *uniform case*. According to [8] only one further integer the (normalized) "b"-invariant, is needed to complete the description in the *uniform case*. We will later exploit the relation to the unit tangent bundle of $\Gamma\backslash S$ to show that this last integer invariant *is determined by the abstract group as well.*

In the *non-compact* case the "b"-invariant is *not* needed but the abstract finitely presented group only determines the integer $g + s$. However, $\Gamma \subset G$, and so the genus of the compactification of $\Gamma\backslash G/K$ is g, say. Two different embeddings determine the same g if the orbit spaces $\Gamma\backslash G/K$ are homeomorphic. (This information is contained in our presentation of Γ). *Therefore, all* the Seifert invariants are *known* for the K-action $(\Gamma\backslash G, K)$ and is given by:

(3.6) $\{g; s; (\alpha_1, \alpha_1 - 1), \ldots, (\alpha_n, \alpha_n - 1)\}$

This determines the topological type of $\Gamma\backslash \mathrm{PSL}(2, R)$ when Γ is finitely presented and not uniform.

3.7. *Case II. Curvature* $= 0$. $S = \mathbb{R}^2$ with the usual Euclidean metric. K is the isotropy subgroup of the origin again. K is

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

in the usual matrix description of $G \cong E^+(2)$ relative to an oriented orthonormal bases of $\mathbb{R}^2$. The matrix above corresponds to $e^{i\theta} \in S^1$ and we thus get our isomorphism $S^1 \cong K$. Again $r = 1$ for essentially the same reason; thus again $\beta = \alpha - 1$.

The α's are very restricted in the uniform case. The list has been given in the introduction under the discussion of 3(a). (For geometric reasons Γ can be presented as in 3.5 for Γ in PSL(2, R), $E^+(2)$ or SU(2). The expression

$$2-2g-\sum_{i=1}^{n}(1-1/\alpha_i)$$

being <0, 0, or >0 determines, in the uniform case, whether Γ is contained in PSL(2, $\mathbb{R}$), $E^+(2)$ or SU(2), respectively.) The "b"- invariant still needs to be determined.

In the non-uniform case we are also severely restricted, namely:

(i) $g=0$, $n=0$, $s=1$, $\Gamma=e$ and (G, K) is equivalent to

$$(\mathbb{R}^2\times S^1, S^1),$$

(ii) $g=0$, $n=1$, $s=1$, $\Gamma\approx Z_{\alpha_1}$ and $(\Gamma\backslash G, K)$ is equivalent to a linear

$$(\mathbb{R}^2\times_{Z_{\alpha_1}} S^1, S^1)$$

on a solid torus, and

(iii) $g=0$, $n=0$, $s=2$, $\Gamma\cong \mathbf{Z}$ and $(\Gamma\backslash G, K)$ is equivalent to

$$((\mathbb{R}^1\times_Z \mathbb{R}^1)\times S^1, S^1).$$

3.8. *Case III. Curvature is positive.* Here we may take S to be the unit sphere in $\mathbb{R}^3$ with the metric inherited from the usual one in $\mathbb{R}^3$.

$$G=SO(3)=\{A\in GL(3,\mathbb{R})|A^t=A^{-1}\},$$

where A^t is the transpose of A.

$$K=SO(2)=\{A\in SO(3)|A\,e_3=e_3\},$$

where e_3 is the unit vector along z-axis. Again in an obvious way

$$\theta: S^1\to K,\qquad e^{i\theta}\mapsto\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\in GL(3,\mathbb{R}).$$

Again $r=1$ and so $\beta=\alpha-1$. Since G is compact the Γ's involved are finite. In our presentation for Γ above we need only take $g=0$, $s=0$, $n=3$ and $1/\alpha_1+1/\alpha_2+1/\alpha_3>1$, or $g=0$, $s=0$, $n\leq 2$. The orbit invariants are listed under our discussion of (4a) in the Introduction. The "b"-invariant also needs to be determined.

4. Reduction of the problem to the adjoint forms of G

4.1. To describe the general situation in 3(a) and 4 of Table 1 we need to describe all quotients $\Gamma\backslash G$ where G is simply connected. In the previous section we have examined not simply connected G, but the adjoint forms of G, namely PSL(2, $\mathbb{R}$), $E^+(2)$ and SO(3). It is the purpose of this section to show that this, together with a technical fact about central cyclic coverings of Seifert fiber spaces is sufficient to prove the Theorem.

For the following facts about lattices and centers of lattices in semi-simple Lie groups we refer the reader to [10; particularly V, 5.18]. For the special case at hand, PSL(2, $\mathbb{R}$), which is all that we are concerned with, the reader may also consult [1, § 4].

Let G_∞ be the universal covering of PSL(2, $\mathbb{R}$) $= G_1$. The center of G_∞, $\mathfrak{z}(G_\infty)$, is isomorphic to $\mathbb{Z}$. If Γ_∞ is a lattice in G_∞, then $\mathfrak{z}(\Gamma_\infty) = \Gamma_\infty \cap \mathfrak{z}(G_\infty)$ and must be an infinite subgroup, $r\mathbb{Z}$, of $\mathfrak{z}(G_\infty)$. Moreover, if $p_m : G_\infty \to G_m = G_\infty/m\mathbb{Z}$ is the central covering projection, then the lattice Γ_∞ is projected into a lattice

$$p_m(\Gamma_\infty) = \Gamma_m = \Gamma_\infty/(m\mathbb{Z} \cap \Gamma_\infty) \quad \text{in } G_m.$$

Induced is the finite cyclic $(m\mathbb{Z}/\mathrm{lcm}(m, r)\mathbb{Z})$ covering projection: $\Gamma_\infty\backslash G_\infty \to \Gamma_m\backslash G_m$. This may be controlled as follows: Project Γ_∞ to G_r where $\mathfrak{z}(\Gamma_\infty) = r\mathbb{Z}$, then $\Gamma_\infty\backslash G_\infty$ is naturally diffeomorphic to $\Gamma_r\backslash G_r$. Now, observe for any lattice $\Gamma \subset G_m$, $\mathfrak{z}(\Gamma) = \mathfrak{z}(G_m) \cap \Gamma$. We may take $\Gamma' = p_m^{-1}(\Gamma)$, a lattice in $G_\infty = p_m^{-1}(G_m)$, and $\Gamma'\backslash G$ will be diffeomorphic to $\Gamma\backslash G_m$. Therefore, we may as well assume, without any loss of generality, that:

(1) *Γ_r is a lattice in G_r*

(2) *Γ_r is centerless.*

We note that as K_r-space, $\Gamma_r\backslash G_r$, the action of K_r is *effective* since the center of Γ_r is trivial (and $\mathfrak{z}(G_r) \approx \mathbb{Z}/r\mathbb{Z}$ is in each conjugate of K_r). *So to determine $\Gamma_\infty\backslash G_\infty$ it suffices to determine $\Gamma_r\backslash G_r$ as a K_r-space.*

On the other hand, we may make yet another simplification. If we now project $G_r \to G_1 =$ PSL(2, $\mathbb{R}$) then Γ_r is projected *isomorphically* onto the lattice Γ_1 in G_1 since $\mathfrak{z}(\Gamma_r) = 1$, and $\Gamma_r\backslash G_r \to \Gamma_1\backslash G_1$ is an r-fold cyclic covering projection. This is given by dividing out the center $\mathbb{Z}/r\mathbb{Z} \subset K_r \subset G_r$. We obtain the following:

4.2. Reduction proposition. *To determine all $\Gamma_\infty\backslash G_\infty$ we need only examine lattices Γ_1 in G_1 and determine $\Gamma_1\backslash G_1$ as a K_1-space. We then obtain all possibilities by considering the finite number of cyclic $\mathbb{Z}/r$ coverings G_r of G for which Γ_1 lifts to $\Gamma_1 \times \mathbb{Z}/r$ and determining $\Gamma_1\backslash G_r$ as a K_r-space.*

It is exactly this proposition that enables us to carry out the classification for $\Gamma_\infty\backslash G_\infty$: We know all the possible Γ_1's from Section 3 and we have already determined all of their orbit invariants (except the "b"-invariant when Γ_1 is uniform). We now show how one determines the invariants for $\Gamma_r\backslash G_r$ once those of $\Gamma_1\backslash G_1$ are known.

4.3. Proposition ([9, § 14], cf. [6, 1.3]). *Let (S^1, M) be any S^1-3 manifold (acting without fixed points) with orbit invariants $\{g; b; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)$. Let C_r be the set of rth roots of $1 \in S^1$. Let C_r act as a subgroup of S^1 on M. Then,*

(1) *C_r acts freely on M if and only if r is relatively prime to each of the α_i.*

Let M' denote the orbit space M/C_r of this free action. Then, $S^1/C_r = {}'S^1$ acts effectively on M'. The covering map p: $M \to M'$ induces a bijection of the set of exceptional orbits and a diffeomorphism of orbit spaces. The new invariants are

(2) $\{b'; g'; (\alpha'_1, \beta'_1), \ldots, (\alpha'_n, \beta'_n)\}$ *where*

(a) $\alpha_i = \alpha'_i$, $0 \leq i \leq n$,

(b) $r\beta_i \equiv \beta'_i \mod \alpha_i$, $0 < \beta_i < \alpha_i$, $0 < \beta'_i < \alpha'_i$.

Letting $r\beta_i = \beta'_i + k_i\alpha_i$, *then*

(c) $rb + \sum_{i=1}^{n} k_i = b'$,

(d) $g = g'$.

We have given the form of the theorem for a *closed* manifold *M*. For an *open* manifold the "*b*"-invariant will not appear and so the proposition in this case *remains unchanged from above except that* 2(*c*) *does not appear.*

Propositions 3.3, 4.2 *and* 4.3 *complete the proof of Theorem* 2

The argument of 4.2 for the coverings of PSL(2, **R**) can be adapted to the coverings of $E^+(2)$. The co-compact lattices in the universal covering of $E^+(2)$ project to the crystallographic groups in $E^+(2)$. *Outside of* $\mathbb{Z}\oplus\mathbb{Z}$, there are just 4 isomorphisms classes of such groups isomorphic to $(\mathbb{Z}\oplus\mathbb{Z}) \circ_\phi (\mathbb{Z}/k\mathbb{Z})$ where $k = 2, 3, 4$ or 6, as is well known. Now, except for T^3, $g = 0$ and we shall see in § 6 that $b = -2$. The list of allowable α's is given in the Introduction and comes from the discussion in Section 3. So applying 4.3, we see that *the allowable r-fold coverings all yield exactly the same orbit invariants as* $\Gamma \subset E^+(2)$. This involves only a small calculation but if one switches to unoriented orbit invariants and replaces the "*b*"-invariant with the equivalent Euler number which in this case turns out to be 0, then all the allowable coverings must also have 0 Euler number (cf. [6]) and *unchanged* "β"-invariants.

For SO(3), the finite subgroups are well known. In the *non-cyclic* case, the group $\Gamma_1 \subset SO(3)$ always lifts to a non-trivial central extension Γ of $\mathbb{Z}/2$ by Γ_1. Therefore $\Gamma_1\backslash SO(3)$ as a K_1 space is exactly the same as $\Gamma\backslash SU(2)$. Therefore, in both 3(a), and 4(a), *the result for the adjoint form is all that is needed.*

5. The "*b*"-invariant

Let $E \xrightarrow{\pi} B$ be a principal G-bundle, with G connected. Let B be triangulated as a cell complex. Let us assume s: $B^n \to E$ is a section over the n-skeleton. We try to extend it to the $(n+1)$-skeleton. On each $(n+1)$-cell e choose a section $\sigma_e : e \to E$. Then if $x \in \pi^{-1}(e)$, we can write x uniquely as $\sigma_e(\pi(x)) \cdot g = x$. For $y \in \partial e$, we have $s(y) = \sigma_e(y) \cdot f_e(y)$, where $f_e : \partial e \to G$. This defines a map $S^{n-1} \to G$, and the homotopy class of this map is independent of the choice of the section σ_e. The assignment $e \to [f_e]$, where "[ ]" is the homotopy class, defines an obstruction cocycle. The cohomology class of this cocycle is the characteristic class sought.

We now explain the "b"-invariant for an S^1-action without fixed points. For background see [9, pp. 18–185], [8; p. 68] or [7]. Triangulate M^* as a cell complex and surround each exceptional orbit with 2 cells $e_i = N_i^*$ $(i = 1, \ldots, n)$, and with boundary components q_i^*. Choose x_0^*, an ordinary orbit, in interior N_0^*, where $N_0^* \subset M^* - \bigcup_{i=1}^n N_i^*$. Denote $M^* - \bigcup_{i=1}^n$ interior N_i by M'^*. The orbit mapping $M' \to M'^*$ is a principal SO(2) bundle projection, (this bundle is trivial unless $n = 0$, and M^* is compact.) M'^* is an oriented 2 manifold with boundary components $q_1^*, \ldots, q_n^*$. Choose cross-sections $q_1, \ldots, q_n$ over $q_1^*, \ldots, q_n^*$. (The N_i must be sewn-in in a prescribed equivariant way to extend the SO(2)-action to obtain M as an SO(2)-manifold.) Now this partial cross-section can be extended to a section s on all of M'^*-interior N_0^*. If M' is not compact, it can be extended to all of M'^*). We can therefore choose σ_{e_0} and $\sigma_{e_i} = s|_{e_i}$ and so f_e is constant for all e except possibly e_0. We orient M^* and the cells coherently and compatibly with the orientation of M and so the homotopy class of $f_{e_0}: \partial e_0 \to S^1$ is a well-defined integer. Since

$$\pi^{-1}(e_0) = N_0 \cong D^2 \times S^1,$$

the cross-sectional curve $s(q_0^*) = q_0$, where $s: q_0^* \to \pi^{-1}(\partial e_0^2)$, is a homology class in $H_1(\partial D^2 \times S^1)$. With the obvious notation for homology classes in $H_1(\partial D^2 \times S^1)$, we put $[m_0] = (1, 0)$ and $[h] = (0, 1)$, where "[ ]" denotes the homology class of an oriented closed curve. That is, we think of m_0 as the oriented image $\sigma|_{\partial e_0}: \partial e_0 \to \partial D^2 \times 1$ and h as $0 \times S^1$ in $D^2 \times S^1 = \pi^{-1}(e_0)$. Now $[q_0] = [m_0] - b[h]$ for some integer b. That is, $-b = [f_{e_0}]$. This "characteristic class" is represented by a cocycle which has value 0 on all the 2 cells but one, (e_0), and on that 2-cell it has value $-b$. Thus, via the obvious isomorphism,

$$H^2(M'^*, q_1^* \cup \cdots \cup q_n^*; \mathbb{Z}) \approx \mathbb{Z},$$

the characteristic class is $-b$. (Observe that if M'^* is not compact, then this obstruction class vanishes which means that s extends over e_0. This is why the "b"-invariant is not needed for open manifolds). Of course, "b" is an invariant of the S^1-action as explained in [9] and [8]. Moreover, $\{b; g; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$ forms a complete set of invariants for the oriented S^1-action. Actually, (with the exceptions of $n \leq 2$ when $g = 0$), they are, up to change of orientation, a complete set of invariants for the topological type as well as the fundamental group of M, see [7].

6. Calculation of "b" for $G = I^+(S(\kappa))$

We return to the notation of 3.3–3.8 where Γ is a discrete uniform subgroup of G, the group ($\cong$PSL(2, $\mathbb{R}$), $\mathcal{E}^+(2)$, SO(3)) of orientation preserving isometries of a complete simply connected 2-dimensional surface S of constant curvature ($\kappa <$ 0, $=0$, >0, respectively).

The proof of Theorem 1 will be complete when we obtain the

6.1. Proposition. *The "b"-invariant for $\Gamma \backslash G$ is precisely* $2g - 2$.

Proof. Let us choose $x_0 \in S$ and v_0 among the unit tangent vectors to S at x_0. Via the differential, G acts simply transitively on the entire unit tangent bundle $U(S)$. We may, therefore, identify G with $U(S)$ via: $g \to gv_0 \in U(S)_{gx_0}$. The right action of K on G becomes a right action on $U(S)$. This is, of course a free action, the orbits are just the circles $\{U(S)_x\}$, $x \in S$.

Denote the surface $\Gamma\backslash S$ by $\bar{S}$; we assume it given a C^∞-structure in the usual way. Let $d\pi: T(S) \to T(\bar{S})$ denote the differential. If $x \in S$ and $\pi(x)$ is *not* a branch point $d\pi_x: T(S)_x \to (T(\bar{S}))_{\pi(x)}$ is an isomorphism; if $\pi(x)$ is a branch point $d\pi_x = 0$.

Let $\bar{V} \subset \bar{S}$ be the complement of the branch points and $V = \pi^{-1}(\bar{V})$. Γ operates on $T(V)$ and we clearly have $\Gamma\backslash T(V) \simeq T(\bar{V})$. Since Γ operates isometrically, we may introduce a Riemannian metric in $T(V)$ so that $\bar{V} = \Gamma\backslash V$ "inherits" the geometry of constant curvature of S. Thus, $U(V)$ makes sense and $\Gamma\backslash U(V) \simeq U(\bar{V})$.

On the other hand $U(V) \subset U(S) = G$; indeed $\Gamma\backslash U(V)$ is precisely the complement in $\Gamma\backslash G$ of the exceptional fibers.

Thus we have *an explicit identification* of the *complement* of the *exceptional orbits* and the *unit tangent bundle of* $\bar{V}$. ($\bar{V}$ = the complement of the branch points in $\bar{S} = \Gamma\backslash S$.)

If there are no exceptional fibers, the proof is easy. The "b"-invariant is the negative (Section 5) of the obstruction to a section of $(\Gamma\backslash G \to \Gamma\backslash S)$. By the identification just described and Hopf's Theorem this obstruction is the Euler characteristic of $\Gamma\backslash S$ and equals $2-2g$, (g = the genus of $\Gamma\backslash S$).

However, in general, there are finitely many branch points, $y_1, y_2, \ldots, y_n$ on $\Gamma\backslash S$. As in Section 5, choose arbitrarily small disk shaped neighborhoods. N_i^* of the y_i's and specific sections, q_i defined over $N_i^* = q_i^*$ of the y_i's of the bundle P. ($P = \Gamma\backslash G$-(exceptional orbits); P "=" $U(\bar{V}) \subset (T(\bar{V}))' \subset (T(\Gamma\backslash S))'$, where "prime" denotes the *non-zero tangent vectors.*) We can assume that N_i^* are 2-cells in a regular cell decomposition of $\Gamma\backslash S$. Let χ be a partial section of $(T(\Gamma\backslash S))'$ defined over the 1 skeleton and agreeing with the q_i on ∂N_i^*. Let F be the 2-cocycle which is the obstruction to extending χ over all of $\Gamma\backslash S$. By Hopf's theorem:

$$\text{Euler characteristic of } (\Gamma\backslash S) = 2-2g = \Sigma F(e^2),$$

where the sum extends over *all* the 2-cells in $\Gamma\backslash S$. In view of P "=" $U(\bar{V})$ the "b"-invariant equals $-\Sigma F(e^2)$, where the sum now extends only to the cells *other* than the $\{N_i^*\}_{i=1}^n$. *Therefore,*

$$2-2g = -b + \sum_{i=1}^{n} F(N_i^*).$$

To complete the proof we establish

6.2. Lemma. *For* $i = 1, 2, \ldots, n$, $F(N_i^*) = 0$.

Geometrically, this means that q_i, viewed as a non-vanishing vector field defined on ∂N_i^*, has an extension to N_i^* which never vanishes.

We describe $F(e^2)$ homologically:

$$(T(e^2))' = e^2 \times (\mathbb{R}^2 - 0),$$

$$(T(\partial e^2))' = \partial e^2 \times (\mathbb{R}^2 - 0).$$

The image χ of ∂e^2 determines an element of

$$H_1((T(\partial e_2))') \approx H_1((\partial e_2) \times \mathbb{R}^2 - 0)) \approx \mathbb{Z} \oplus \mathbb{Z}$$

of the form (l, k). Since χ is a section $l = 1$, and $F(e_2)$ will be l_k. For $e^2 = N_i^*$, $\chi(\partial e_2) = q_i$, $[q_i] \in H_1((T(\partial N_i^*))')$. So, via the isomorphism $[q_i] = (1, k)$. We wish to show that k must be zero. (We drop the "i" for convenience.)

Let $x \in S$ be a point lying over $y \in \Gamma\backslash S$ with α the branching index at y. Relative to suitable coordinate patches the map $\pi: S \to \Gamma\backslash S$ is described by $z \to z^\alpha$. We can assume N^* is a small enough neighborhood so that y corresponds to 0 under the mapping. Put $X = U(N^*)$, the part of $\Gamma\backslash G$ lying over N^*. Let N be the component of the part of S lying over N^* and containing x. Now $U(N)$, the part of G lying over N has image X under the map $G \to \Gamma\backslash G$.

Since X is a solid torus and represents an invariant tubular neighborhood of the exceptional orbit y, we have oriented curves q, m, h on X and satisfying the following homological relationship [8]:

$$[m] = \alpha[q] + \beta[h].$$

h is a principal orbit on ∂X, q is our cross-sectional curve and m is the boundary of a normal disk to the exceptional orbit and its homology class is characterized by:

(i) $[m]$ is an element of the kernel $H_1(\partial X) \to H_1(X)$. (In fact, it is the generator of this infinite cyclic group.)

(ii) Via $H_1(\partial X) \to H_1(\partial N^*)$, the class $[m]$ goes to $\alpha[\partial N^*]$.

Suppose $v: N \to U(N)$ is any section. Let m_1 be the oriented closed curve $v(\partial N)$. Then $[m_1] \in H_1(U(\partial N))$. We claim that $[m_1]$ goes to $[m]$ in $H_1(U(\partial N)) \to H_1(\partial X)$.

(i) $[\partial N] \in$ kernel $H_1(\partial N) \to H_1(N)$ and so $[v_*(\partial N)] = [m_1] \in$ kernel $H_1(U(\partial N) \to H_1(U(N))$. So $\tau_*[m_1] \in H_1(\partial X)$ is in the kernel $H_1(\partial X) \to H_1(X)$, where $\tau: U(N) \to X$ is the α-fold branched covering.

(ii) As $v(\partial N) \to \partial N \to \partial(N^*)$ commutes with $v(\partial N) \to \tau(V(\partial N)) \to \partial(N^*)$ and since $\partial N \to \partial(N^*)$ is an α-covering, we have that $\tau_*[m_1] = [m]$.

Let us make use of our special coordinate systems:

$$U(\partial N) \subset (T(D - x))' = (D - x) \times (\mathbb{C} - 0),$$

$$U(\partial N^*) \subset (T(D^* - y))' = (D^* - y) \times (\mathbb{C} - 0).$$

The inclusions are homotopy equivalences and so we may compute either in $(T(D - x))'$ or $U(\partial N)$ and $U(\partial N^*)$ or $(T(D^* - y))'$ as we wish.

Since $\pi: D \to D^*$ is given by $z \to z^\alpha$; $d\pi: (T(D - x))' \to (T(D^* - y))'$ is given by $(z, v) \to (z^\alpha, \alpha z^{\alpha-1} v)$. The coordinate system for $v \to U(N)$ is described by $v(z) =$

$(z, f(z))$, where $f: N \to (\mathbb{C}-0)$. The curve m_1 gets mapped to the curve

$$z \to (z^{\alpha}, \alpha z^{\alpha-1} f(z)) \in (T(D^* - y))', \quad z \in (\partial N).$$

Now ∂N is of degree 1 in $D-x$ so the curve $z \to z^{\alpha}$ has degree α in D^*-y. Similarly, $z \to \alpha z^{-1} f(z)$ has degree $\alpha - 1$. (Recall $f: \partial N \to \mathbb{C}-0$ has an extension to N and so has degree 0). Hence,

$$[m] \in H_1(\partial X) = H_1((T(D^*-y))') \approx \mathbf{Z} \oplus \mathbf{Z} \quad \text{"is"} \quad (\alpha, \alpha - 1),$$

in terms of ∂D^* and the fiber $\mathbb{C}-0$.

Since h is a principal orbit over a point in ∂N^*, we see that

$$[h] \in H_1((T(D^*-y))') \cong \mathbf{Z} \oplus \mathbf{Z} \quad \text{"is"} \quad (0, 1).$$

So $[m] = \alpha[\partial D^*] + \alpha - 1[h]$. Now, $H_1(\partial X) \xrightarrow{\cong} H_1((T(D^*-y))')$ is generated by [q] and $[h]$ while m also satisfies $[m] = \alpha[q] + \beta[h]$. We know q is related to ∂D^* and h by $[q] = a[\partial D^*] + c[h]$. But the orientations must be compatible, so $a = 1$ and we easily deduce $c = 0$ from $0 < \beta < \alpha$, and $(\alpha, \beta) = 1$. Consequently, $\beta = \alpha - 1$, and $[q] = [\partial D^*]$ "=" $(1, 0)$. This completes the proof.

6.3. Remark. Observe that $\beta = \alpha - 1$ was deduced in the course of the proof of Lemma 6.2. In fact, the deduction of $\beta = \alpha - 1$ does not require either the compactness of $\Gamma \backslash G$ nor the finiteness of the number of branch points. This means, in effect, that we could have dispensed with some of Sections 2 and 3 only retaining what is needed to establish notations, orientations and the reduction proposition of Section 4. However, the point of Section 2 is that Proposition 2.1 is a general fact which has implications for all Lie groups. Moreover, in the next section we shall give a more sophisticated proof of our theorem which utilizes Sections 2, 3 and 4 but dispenses with parts of Sections 5 and 6.

7. Neumann's method of finding the "b" invariant

As mentioned earlier, the problem that has been considered is so rich that one expects to find many alternative methods for obtaining proofs of Theorems 1 and 2. Our alternative proofs are based upon either the Gauss–Bonnet Theorem, the Hurwitz formula for branched coverings of surfaces, or injective actions of Conner–Raymond and the cohomology of Fuchsian and crystallographic groups. The proof we have given seems to us to be the most attractive of our arguments.

W.D. Neumann has shown us an elegant method for obtaining the "b"-invariant also utilizing the Hurwitz formula, his *unnormalized* Seifert invariants, and the *Euler number* associated to a Seifert fibering, (see [6] for definitions). We sketch Neumann's argument here.

Let Γ be a *uniform* discrete subgroup of $G \cong \mathrm{PSL}(2, \mathbb{R})$ or $E^+(2)$. Let Γ_1 be a normal, torsion free subgroup of index $k < \infty$. The projection $S \to \Gamma_1 \backslash S$ is an unbranched covering, while $\Gamma_1\backslash S \to \Gamma\backslash S$ is a branched covering of index k. The "b"-invariant for $\Gamma_1\backslash S$ is easily $2g_1-2$, by the remark made in 6.1. Therefore the Euler number, $e(M_1)$, of $\Gamma_1\backslash G$ is $2-2g_1$, where g_1 is the genus of $\Gamma_1\backslash S$. The k-fold covering projection $\Gamma_1\backslash G \to \Gamma\backslash G$ is clearly an S^1-equivalent map; and so using the formula for the resulting Euler number [6, Theorem 1.2] we have:

$$\begin{aligned} 2-2g_1 &= k(e(M)) \\ &= k\left(-\left(b+\frac{\beta_1}{\alpha_1}+\cdots+\frac{\beta_n}{\alpha_n}\right)\right) \\ &= k\left(-\left(b+\frac{\alpha_1-1}{\alpha_1}+\cdots+\frac{\alpha_n-1}{\alpha_n}\right)\right), \quad \text{(by 3.3)} \\ &= -k\left(b+n-\sum_{i=1}^{n}\frac{1}{\alpha_i}\right). \end{aligned}$$

Now using the Hurwitz formula for $\Gamma_1\backslash S \to \Gamma\backslash S$ we have

$$2g_1-2 = k\left(2g-2+n-\sum_{i=1}^{n}\frac{1}{\alpha_i}\right),$$

where α_i is the multiplicity of branching. Now substituting for $2g_i-2$ we have $b=2g-2$, where g is the genus of $\Gamma\backslash S$.

This argument also can be adjusted for $\Gamma \subset \mathrm{SO}(3)$. We just take M_1 to be SO(3) itself and Γ_1 and the identity element of Γ. Then $k=|\Gamma|$, and SO(3) as the unit sphere bundle of the tangent bundle of the 2 sphere has Euler class 2 (as a principal SO(2) bundle). It follows that $b=-2$.

References

[1] L. Auslander, L. Green and F. Hahn, Flows on some 3-dimensional homogeneous spaces, Chapter III of Annals of Math. Study 53 (Princeton, 1963).

[2] A. Borel et al., Seminar on Transformation Groups, Annals of Math. Study 46 (Princeton, 1960).

[3] A.M. Macbeath, The fundamental groups of the 3-dimensional Brieskorn manifolds (preprint).

[4] J. Milnor, Curvatures of left invariant metrics on Lie groups., Adv. in Math. 21, 293–329 (1976).

[5] J. Milnor, On the 3-dimensional Brieskorn manifold $M(p, q, r)$, Papers Dedicated to the Memory of R.H. Fox, Annals of Math. Study 48 (Princeton, 1975) 175–225.

[6] W.D. Neumann and F. Raymond, Seifert manifolds, Plumbing, μ-invariant, and orientation reversing maps, Algebraic and Geometric Topology, Proceedings 1977, Springer Lecture Notes, Vol. 664 (Springer-Verlag, New York, 1977) 162–195.

[7] P. Orlik, Seifert Manifolds, Springer Lecture Notes, Vol. 291 (Springer-Verlag, New York).

[8] F. Raymond, Classification of the actions of the circle on 3-manifolds, Trans. Amer. Math. Soc. 131 (1968) 51–78.

[9] H. Seifert, Topologie dreidimensioneler Gefaserter Räume, Acta Math. 60 (1933) 147–238.

[10] M.S. Raghunathan, Discrete Subgroups of Lie Groups, Band 68, Ergebnisse Series (Springer-Verlag, Berlin).