

On the Symmetric Algebra of a Module

CRAIG HUNEKE*

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

Communicated by N. Jacobson

Received March 15, 1980

INTRODUCTION

The purpose of this article is to give a simple criterion in order for the symmetric algebra, $S_R(M)$, of an R -module M to be a domain. Of course, we must assume R is a domain. Before we state the main result let us introduce some notation.

If A is a matrix with coefficients in a commutative ring R , by $I_t(A)$ we denote the ideal generated by the $t \times t$ minors of A . If M is an R -module we denote the least number of generators of M by $v(M, R)$. We recall a commutative Noetherian ring R is said to satisfy Serre's property S_k if

$$\text{depth } R_p \geq \min\{\dim R_p, k\}$$

for all primes p in R . If I is an ideal in R , by $\text{grade}(I)$ we denote the length of a maximal R -sequence in I . We may now state the main result.

THEOREM 1.1. *Let R be a universally catenarian Noetherian domain satisfying S_m and let M be an R -module having a finite free resolution,*

$$0 \rightarrow R^m \xrightarrow{A} R^n \rightarrow M \rightarrow 0, \quad A = (a_{ij}).$$

Then the following are equivalent:

- (1) $S_R(M)$ is a domain.
- (2) $\text{grade}(I_t(A)) \geq m + 2 - t$ for $1 \leq t \leq m$.
- (3) $v(M_p, R_p) \leq n - m + \text{grade}(p) - 1$ for all non-zero primes p in R .

If any (and hence all) of the above conditions hold, then $S_R(M)$ is a complete intersection in $R[T_1, \dots, T_n]$. In particular, if R is Cohen–Macaulay (respectively Gorenstein) then $S_R(M)$ is Cohen–Macaulay (respectively Gorenstein).

* Jr. Fellow, Michigan Society of Fellows.

Several authors have investigated the case in which $S_R(I)$ is a domain if I is an ideal. See [6, 8–10]. The present author has used the idea of a d -sequence to provide a wide class of examples such that $S_R(I)$ is a domain. In this case, $gr_I(R)$, the graded algebra of I , will be isomorphic to $S_{R/I}(I/I^2)$, and consequently is considerably easier to deal with. This author used this approach in [7] to prove results about the symmetric powers of prime ideals which are almost complete intersections, generic Pfaffians, or maximal minors of a generic $n \times (n + 1)$ matrix. Also treated in [7] is the following well-known result (see [4] or [11]).

PROPOSITION. *Suppose R is a commutative Noetherian domain and M a finitely generated module, with resolution*

$$0 \rightarrow R \xrightarrow{(a_1, \dots, a_n)} R^n \rightarrow M \rightarrow 0.$$

If $\text{grade}(a_1, \dots, a_n) \geq 2$, then $S_R(M)$ is domain.

We will recover this result as an immediate corollary of Theorem 1.1. We will apply the theorem to several situations (these results may be found in Section 2). For instance, we show the following corollary, which we apply to primes defining varieties of the form $k[t^{n_1}, t^{n_2}, t^{n_3}]$, where k is a field.

COROLLARY. *Let R be Cohen–Macaulay local domain and p a height 2 prime generated by three elements, having projective dimension 1. Then $S(p)$ is a domain.*

We now recall some facts about $S_R(M)$, the symmetric algebra of a module M . It is a non-negatively graded algebra. If $S^n(M)$ denotes its n th graded piece then

$$S^n(M) = M \otimes \dots \otimes M/N,$$

$\underbrace{\hspace{10em}}_{n\text{-times}}$

where N is the submodule of $M \otimes \dots \otimes M$ generated by elements of the form

$$\dots \otimes m \otimes \dots \otimes m' \otimes \dots - \dots \otimes m' \otimes \dots \otimes m \otimes \dots.$$

If M has the presentation

$$R^m \xrightarrow{(a_{ij})} R^n \rightarrow M \rightarrow 0$$

then $S_R(M)$ can be identified with $R[T_1, \dots, T_n]/J$, where J is the ideal generated by the m linear forms $\sum_{i=1}^n a_{ij} T_j$.

$S_R(M)$ has the following universal property: if S is an R -algebra and $f: M \rightarrow S$ is an R -module homomorphism, then there exists a unique $f^*: S_R(M) \rightarrow S$ extending the identifications of M as $S^1(M)$; i.e., the following commutes:

$$\begin{array}{ccc}
 S_R(M) & \xrightarrow{f^*} & S \\
 & \swarrow & \nearrow f \\
 & S^1(M) = M &
 \end{array}$$

1. THE MAIN THEOREM

We now begin the proof of Theorem 1.1. We will show that (2) implies (1), (1) implies (3), and (3) implies (2). So now assume (2).

Suppose $\text{grade}(I_t(A)) \geq m + 2 - t$ for $1 \leq t \leq m$. From the Introduction, we noted that

$$S_R(M) \cong R[T_1, \dots, T_n]/J,$$

where J is the ideal generated by the m linear equations $\sum_{j=1}^n a_{ij}T_j$.

Since $\text{grade}(I_m(A)) \geq m + 2 - m = 2$ it is well known that M must be torsion-free of rank $d = n - m$. By standard arguments we may imbed M in R^d . (Let $K =$ fraction field of R ; M imbeds in $M \otimes_R K \cong K^d$. Take generators and clear denominators.)

Taking the symmetric algebras of this imbedding we obtain a map f ,

$$f: S_R(M) \rightarrow R[X_1, \dots, X_d].$$

Denote by Q the kernel of f lifted back to $R[T_1, \dots, T_n]$. Then Q is a "generic point" for $S_R(M)$ in the following sense: $S_R(M)$ is a domain if and only if $Q = J$. For clearly if $Q = J$ then $S_R(M)$ is a domain. Conversely $S_R(M)$ a domain implies f is an imbedding as then $\text{height } J = \text{height } Q$ (invert a maximal minor) and so $Q = J$. The height (Q) can be computed by the above remarks: $\dim S_R(M)/Q = \dim R[X_1, \dots, X_d] = \dim R + d$. Hence $\text{height } Q = \dim(R[T_1, \dots, T_n]) - (\dim R + d) = n - d = m$.

Now induct on $n =$ (the minimal number of generators of M) to show that under these conditions $S_R(M)$ is a domain. If $n = 1$, then the conditions force M to be torsion-free, $M \cong R$ and $S_R(M) \cong R[X]$ is a domain.

So assume $n > 1$ and we have proved this half of the theorem for all N ,

$$0 \rightarrow S^p \rightarrow S^k \rightarrow N \rightarrow 0, \quad \text{where } k < n,$$

and S a universally catenarian Noetherian domain.

Set $a = a_{ij}$ and consider R_a . Over R_a we may change A to the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{pmatrix}$$

and clearly $(I_t(A'))_a = (I_{t+1}(A))_a$. This gives an exact sequence

$$0 \rightarrow R_a^{m-1} \xrightarrow{A'} R_a^{n-1} \rightarrow M_a \rightarrow 0.$$

Now $\text{grade}(I_t(A'))_a = \text{grade}(I_{t+1}(A))_a \geq m + 2 - (t + 1) = (m - 1) + 2 - t$. Hence the assumptions hold for A' and we may assume $S_{R_a}(M_a) = (S_R(M))_a$ is a domain. By the discussion above this implies $a^{n_1}Q \subseteq J$ for some n_1 and so there is $N \geq 0$ such that $I_1(A)^N Q \subseteq J$.

We claim this implies $\text{height } J = m$. For if $J \subseteq P$, a prime, and $\text{height } P < m$, then since $I_1(A)^N Q \subseteq P$ either $I_1(A) \subseteq P$ or $Q \subseteq P$ and so we see J must have height m . But J is generated by m elements and $R[T_1, \dots, T_n]$ satisfies S_m . Hence every associated prime of J has grade at most m . But $\text{grade}(I_1(A)) \geq m + 1$ and so $I_1(A)$ cannot consist of zero divisors modulo J . But $I_1(A)^N Q \subseteq J$ then forces $Q = J$ and so $S_R(M)$ is a domain.

Assume $S_R(M)$ is a domain. Induct on n to show (3). We first show a special case of (3).

LEMMA 1.1. *Let R be a Noetherian local domain satisfying S_k and let N be an R -module with a minimal resolution,*

$$0 \rightarrow R^k \xrightarrow{B} R^d \rightarrow N \rightarrow 0.$$

If $S_R(N)$ is a domain, then $k \leq \text{depth } R - 1$.

Proof. By the argument above, if $S_R(N)$ is a domain, then $S_R(N) = R[T_1, \dots, T_d]/Q$, where Q is prime and $\text{height } Q = k$. Also Q is generated by $\sum_{j=1}^d b_{ij}T_j$. Hence $Q \subseteq I_1(B)$. But $Q \neq I_1(B)$ since clearly $b_{ij} \notin Q$.

Since the resolution is minimal, $I_1(B)$ is contained in m , the maximal ideal of R . In addition, since R satisfies S_k , so does $R[T_1, \dots, T_d]$ and so as Q is a height k ideal generated by k elements, these elements form an $R[T_1, \dots, T_d]$ -sequence. Since Q is a prime contained in m , the grade of m in $R[T_1, \dots, T_d]$ must be at least $k + 1$. Since the map from R to $R[T_1, \dots, T_d]$ is faithfully flat, the grade of m in R must be at least $k + 1$ which concludes the proof of Lemma 1.1.

We now show (3). Since $S_R(M)$ is a domain, M is torsion-free and $\text{rank}(M) = \text{rank}(M_p) = n - m$ is well defined. Since $S_{R_p}(M_p) = (S_R(M))_W$, where $W = R - p$, to show (3) it is enough to assume R is local with maximal ideal m and to show

$$v(M, R) \leq n - m + \text{depth}(R) - 1.$$

We may choose $n = v(M, R)$, and so it is enough to show

$$m \leq \text{depth}(R) - 1.$$

However, this is precisely the statement of Lemma 1.1.

We now show (3) implies (2). Observe that if p is a prime in R , then $I_t(A)$ is not contained in p if and only if $v(M_p, R_p) \leq n - t$, as in this case we may transform A (over R_p) into

$$t \left\{ \left(\begin{array}{ccc|c} \overbrace{1 \quad \dots \quad 0}^t & & & 0 \\ & \ddots & & \\ 0 & & 1 & \\ \hline & & 0 & A' \end{array} \right) \right\}.$$

Now suppose (3) holds, yet $\text{grade}(I_t(A)) \leq m + 1 - t$. Hence there is a prime p containing $I_t(A)$ whose grade is at most $m + 1 - t$. From the observation above, we deduce $v(M_p, R_p) > n - t$. However, by (3), $v(M_p, R_p) \leq n - m + (m + 1 - t) - 1 = n - t$. This contradiction completes this part of the proof.

To complete the proof we observe that if $S_R(M)$ is a domain then as has been observed, the prime Q defining it has height m and is generated by m elements. Since R satisfies S_m , these elements must form an $R[T_1, \dots, T_n]$ -sequence.

COROLLARY 1.1. *Suppose R is a Cohen-Macaulay domain and M a module as in Theorem 1.1. If $S_R(M)$ is a domain then*

$$v(R, M) \leq \text{rank } M + \dim R - 1.$$

Proof. We use the result of Forster [1].

LEMMA 1.2 [1]. *Let R be a Noetherian ring, and M an R -module. Then*

$$v(R, M) \leq \sup \{v(R_p, M_p) + \dim R/p\}.$$

We apply this together with (3) of Theorem 1.1.

$$v(R_p, M_p) \leq \text{rank } M + \text{height}(p) - 1.$$

Hence,

$$\sup\{\dim R/p + v(R_p, M_p)\} \leq \dim R + \text{rank } M - 1.$$

COROLLARY 1.2. *Let R be a Cohen–Macaulay local domain and p a height 2 prime having projective dimension 1 and having three generators. Then $S_R(p)$ is a domain.*

Proof. From (3) of Theorem 1.1. we need only check that $v(p_q, R_q) \leq \text{height}(q)$. Since p has only three generators we only need to check this at p ; however, R_p is regular so $v(p_p, R_p) = 2$.

2. APPLICATIONS

In [4], Hochster studied rings of the form $R[T_1, \dots, T_n]$ modulo a linear form, $a_1 T_1 + \dots + a_n T_n$. He called these generic grade reductions. We note that one simple result follows immediately from Theorem 1.1.

PROPOSITION 2.1. *Suppose R is a commutative Noetherian domain and M is defined to be a cokernel of the map,*

$$0 \rightarrow R \xrightarrow{(a_1, \dots, a_n)} R^n.$$

If $\text{grade}(a_1, \dots, a_n)$ is at least 2, then $S_R(M) = R[T_1, \dots, T_n]/(a_1 T_1 + \dots + a_n T_n)$ is a domain.

Proof. Since R is a domain, R satisfies S_1 . We now apply (2) of Theorem 1.1.

PROPOSITION 2.2. *Let k be a field and let p be the prime in $k[X, Y, Z]$ defining the variety $k[t^{n_1}, t^{n_2}, t^{n_3}]$. Set $R = k[X, Y, Z]$. Then $S_R(p)$ is a Gorenstein domain.*

Proof. Herzog[2] has shown that p is generated by at most three elements. Since R is regular and R/p is Cohen–Macaulay with $\text{height}(p) = 2$, we see the projective dimension of p is 1. Corollary 1.2 shows $S_R(p)$ is a domain and Theorem 1.1 shows $S_R(p)$ is Gorenstein since R is.

PROPOSITION 2.3 [3]. *Let k be a field, and fix numbers $0 < t_1 < t_2 < \dots < t_n = r$. Let X_i be a generic $t_i \times t_{i+1}$ matrix and set*

$X = X_1 \cdots X_{n-1}$. Let T be a generic $r \times 1$ matrix. Set $R = k[X_1, \dots, X_{n-1}, T]$ by which we mean k with the entries of the matrices adjoined. Let J be the ideal generated by the entries of XT . Then J is prime and is generated by an R -sequence.

Proof. Set $S = k[X_1, \dots, X_{n-1}]$ and $t = t_1$. The matrix X defines a module M by the exact sequence,

$$0 \rightarrow S^t \xrightarrow{X} S^r \rightarrow M \rightarrow 0.$$

Further, $S_S(M)$ is R/J . Hence, provided $\text{height}(I_i(X))$ is at least $t + 2 - i$, Theorem 1.1 will give the required conclusion. However, the nilradical of $I_i(X)$ contains $I_i(X_1) \cdots I_i(X_{n-1})$, and it is well known (see, for example, [3]) that

$$\text{grade}(I_i(X_j)) = (t_j - i + 1)(t_{j+1} - i + 1).$$

Since i is at most t and t is strictly less than t_2 this is evidently at least $t + 2 - i$.

ACKNOWLEDGMENT

The author would like to thank R. Cowsik for many insightful comments and in particular for the observation that $S_n(M)$ a domain implies it is a complete intersection.

REFERENCES

1. O. FORSTER, Über die Anzahl der Erzeugenden eines Ideals in einem Noetherschen Ringe, *Math. Z.* **84** (1964), 80–87.
2. J. HERZOG, Generators and relations of Abelian semigroups and semigroup rings, *Manuscripta Math.* **3** (1970), 175–193.
3. J. EAGON AND M. HOCHSTER, Cohen–Macaulay rings and the generic perfection of determinantal loci, *Amer. J. Math.* **93** (1971), 1020–1058.
4. M. HOCHSTER, Properties of Noetherian rings stable under general grade reduction, *Arch. Math.* **24** (1975).
5. C. HUNEKE, The theory of d -sequences and powers of ideals, *Advances in Math.*, in press.
6. C. HUNEKE, On the symmetric and Rees algebras of an ideal generated by a d -sequence, *J. Algebra* **62** (1980), 268–275.
7. C. HUNEKE, Symbolic powers of prime ideals and special graded algebras, to appear.
8. A. MICALI, Sur les algèbres symétrique et de Rees d'un idéal, *Ann. Inst. Fourier*.
9. A. MICALI, P. SALMON, AND P. SAMUEL, "Integrité et factoriabilité des algèbres symétriques," *Atas do IV Coloquio Brasileiro de Mathematica*, São Paulo, 1965.
10. P. SALMON, "Sulle algebre simmetriche e di Rees di un ideale," Edizioni Scientifiche, Genoa, 1964.
11. P. SAMUEL, Anneaux gradués factoriels et modules réflexifs. *Bull. Soc. Math. France* **92** (1964), 237–249.