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Global Bifurcation of Steady-State Solutions

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INTRODUCTION

In this paper, we study the bifurcation of steady-state solutions of a reaction-diffusion equation in one space variable. The steady-state solutions satisfy the equation $u'' + f(u) = 0$ on the interval $-L < x < L$, where we take $f(u)$ to be a cubic polynomial. The solution is assumed to satisfy either homogeneous Dirichlet, or homogeneous Neumann, or periodic boundary conditions. We take as bifurcation parameter the number L, and we obtain global bifurcation diagrams; that is, we count the exact number of solutions.

These solutions can be viewed as the "rest points" of the equation $u_t = u_{xx} + f(u)$. In order to determine the global flow of this latter equation, the precise knowledge of the number of rest points is a necessary first step.

Our technique is a careful analysis of the so-called "time-map" $S(\alpha)$, a function defined by an elliptic integral, which measures the "time" an orbit takes to get from one boundary line to another. The relevant point is that we are able to count the exact number of critical points of S. This is done for both positive and negative solutions of the Dirichlet problem,[1] by proving estimates of the form $S'' + cS' > 0$, or <0, for some (nonzero) function $c = c(\alpha)$. For the Neumann problem, we use entirely different techniques to prove that S is never critical, for any cubic polynomial f. This implies at once that the Neumann problem can have at most one nonconstant solution (having a given number of maxima or minima). This solution is necessarily strongly nondegenerate, in the sense that zero is not contained in the spectrum of the linearized operator (see [2]).

For the Dirichlet problem, the situation is far more complicated, and the bifurcation diagrams undergo qualitative changes, depending on the positions of the roots of f. For example, Fig. 1 shows the bifurcation diagram (for the positive solutions), for two different cubic functions of the form $f(u) = (a-u)(u-b)(u-c)$. It is interesting to note, however, that in every case we study, there are at most three solutions to the Dirichlet problem, for each

[1] For essentially all cubic polynomials f; see the discussion at the end of Section 4.

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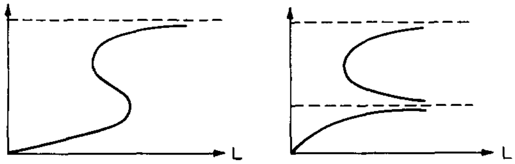

FIG. 1. (A) $-1 \ll a < b < 0 < c$; (B) $0 < a < b < c$

value of L. Furthermore, there can be (at most) two values of L for which the corresponding solutions are degenerate; for all other values, the solutions are strongly nondegenerate.

We remark that the stability of these steady-state solutions is discussed in [1].

1. Notation and Conventions

We consider equations of the form

$$u'' + f(u) = 0, \qquad -L < x < L, \tag{1.1}$$

with homogeneous Neumann or Dirichlet or periodic boundary conditions. Here $f(u)$ is a cubic polynomial of the form

$$f(u) = -(u-a)(u-b)(u-c), \tag{1.2}$$

with $a < b < c$; i.e., we assume that f has three distinct[2] real roots. We shall denote by σ, π and τ, respectively, the sum of the roots, the product of the roots, and the sum of the products of all pairs of the roots. Thus we can write f in the alternate form

$$f(u) = -u^3 + \sigma u^2 - \tau u + \pi. \tag{1.3}$$

We shall always assume that $\sigma \geqslant 0$; this involves no loss in generality since the transformation $u \to -u$ takes Eq. (1.1) into $u'' - f(-u) = 0$, and $-f(-u) = -u^3 - \sigma u^2 - \tau u - \pi$; thus if we can do the analysis when $\sigma \geqslant 0$, we can also do it when $\sigma < 0$.

We shall also assume that $\int_a^c f > 0$ when we draw the phase portraits; the reader should note, however, that this assumption is never used in our analysis.

[2] The reader will see that our techniques can handle the other cases: we have omitted them for the sake of brevity.

We now list some functions (see [2]), which are repeatedly used in the following sections: $F(u) = \int_0^u f(s)\,ds$,

$$\theta(u) = 2F(u) - uf(u) = u^4/2 - (\sigma/3)u^3 + \pi u, \tag{1.4}$$

$$\theta'(u) = 2u^3 - \sigma u^2 + \pi = f(u) - uf'(u). \tag{1.5}$$

Note that θ does not depend on τ; this remark will be useful later on.

We shall partition the set of solutions into three classes, $\mathscr{P}$, $\mathscr{N}$ and $\mathscr{C}$. These are, respectively, the nonnegative solutions, the nonpositive solutions, and the solutions which change sign.

2. Dirichlet Boundary Conditions—Positive Solutions

We consider Eq. (1.1), with homogeneous Dirichlet boundary conditions

$$u(L) = u(-L) = 0, \tag{2.1}$$

and seek positive solutions; in Section 4 we shall consider other solutions. In order to study bifurcation of solutions, we write (2.1) in the usual way as the system $\dot{u} = v$, $\dot{v} = -f(u)$. We observe that a solution of (1.1), (2.1) is an integral curve which "begins" and "ends" on the line $u = 0$, and makes "time" $2L$ to make the journey; see Fig. 2, where we have indicated two typical types of solutions. Now as in [2], we introduce the "time" map[3]

$$T(p) = \int_0^{\alpha(p)} 2^{-1/2}(F(\alpha(p)) - F(u))^{-1/2}\,du, \qquad 0 < p < A. \tag{2.2}$$

The number A is defined by $A^2 = 2F(c)$, and $\alpha(p)$ is the first point on the line $v = 0$ which meets the orbit through p. In order to study the number of

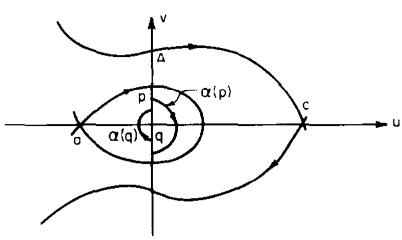

Figure 2

[3] If we are considering solutions in $u \leqslant 0$, then we replace (2.2) by $T(q) = \int_{\alpha(q)}^0 (F(\alpha(q)) - F(u))^{-1/2}\,du$.

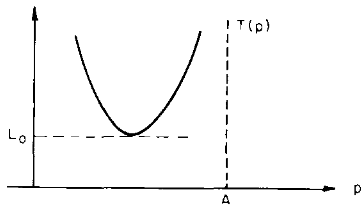

FIGURE 3

solutions of (1.1) and (2.1), and their bifurcations, we must study the qualitative shape of the time map. Observe that $\alpha'(p) > 0$, so that we may actually consider T as a function of α, defined by $S(\alpha) = 2^{1/2}T(p)$, whenever this is convenient to us.

We shall show how the function T changes with the position of the roots of f. We consider first the case where there are no negative roots.

THEOREM 2.1. *If $0 \leqslant a < b < c$, then T has exactly one critical point, a minimum.*

Remark. It follows froms this that if $a = 0$ the time map has the form given in Fig. 3. Thus for short intervals, $L < L_0$, the only solution is $u \equiv 0$; then for $L = L_0$, a nonconstant solution appears, while for $L > L_0$, this solution bifurcates into two nonconstant solutions. Therefore, for $L > L_0$ there are exactly three solutions. (We shall discuss below the case $a > 0$.)

Proof of Theorem 2.1. We shall first assume that $a = 0$.

In (2.2), if we make the change of vaiable $u = \alpha t$, we get

$$S(\alpha) = \int_0^1 (F(\alpha) - F(\alpha t))^{-1/2} \alpha \, dt,$$

where we are using the notation, $S(\alpha(p)) = 2^{1/2}T(p)$. Thus

$$\begin{aligned} S'(\alpha) &= \int_0^1 \frac{(\Delta F)^{1/2} - \frac{1}{2}\alpha(\Delta F)^{-1/2}(f(\alpha) - tf(\alpha t))}{\Delta F} \, dt \\ &= \int_0^\alpha \frac{\Delta F - \frac{1}{2}(\alpha f(\alpha) - uf(u))}{(\Delta F)^{3/2}} \frac{du}{\alpha} \\ &= \int_0^\alpha \frac{\theta(\alpha) - \theta(u)}{2(\Delta F)^{3/2}} \frac{du}{\alpha}, \end{aligned} \tag{2.3}$$

where we are using the obvious notation $\Delta F = F(\alpha) - F(u)$, and[4] $\theta(x) =$

[4] Note that if $\theta'(u) \neq 0$, $\alpha > u > 0$, then $S' \neq 0$, so T is monotone.

$2F(x) - xf(x)$. Now from (1.4), we have $\theta(x) = (x^3/2)(x - (2\sigma/3))$ so that $\theta(\alpha) - \theta(u) > 0$ if $\alpha > 2\sigma/3$, while $\theta(\alpha) - \theta(u) < 0$ if $\alpha < \sigma/2$. It follows that $S'(\alpha) < 0$ for $\alpha \leqslant \sigma/2$, and $S'(\alpha) > 0$ for $\alpha \geqslant 2\sigma/3$, and therefore, $S'(\alpha)$ has at least one zero. Thus, we need only consider $\alpha \geqslant \sigma/2$. We have

$$2S'(\alpha) = \int_0^1 \frac{\theta(\alpha) - \theta(\alpha t)}{(F(\alpha) - F(\alpha t))^{3/2}}\, dt,$$

so that

$$2S''(\alpha) = \int_0^1 \frac{\left[\begin{array}{l}(F(\alpha) - F(\alpha t))^{3/2}(\theta'(\alpha) - t\theta'(\alpha t)) \\ \qquad - \frac{3}{2}(\theta(\alpha) - \theta(\alpha t))(F(\alpha) - F(\alpha t))^{1/2}(f(\alpha) - tf(\alpha t))\end{array}\right]}{(F(\alpha) - F(\alpha t))^3}\, dt$$

$$= \int_0^\alpha \frac{(\Delta F)(\alpha\theta'(\alpha) - u\theta'(u)) - \frac{3}{2}\Delta\theta(\Delta\tilde{f})}{\alpha^2(F(\alpha) - F(u))^{5/2}}\, du,$$

where $\Delta\tilde{f} = \alpha f(\alpha) - uf(u)$. Hence, for any K, we have

$$S''(\alpha) + KS'(\alpha) = \int_0^\alpha \frac{K\alpha(\Delta F)(\Delta\theta) - \frac{3}{2}\Delta\theta(\Delta\tilde{f}) + \Delta F(\Delta\tilde{\theta}')}{2\alpha^2(\Delta F)^{5/2}}\, du.$$

It follows that if we put $K = 3/\alpha$, we have

$$S''(\alpha) + \frac{3}{\alpha} S'(\alpha) = \int_0^\alpha \frac{\frac{3}{2}(\Delta\theta)^2 + (\Delta F)(\Delta\tilde{\theta}')}{2\alpha^2(\Delta F)^{5/2}}\, du.$$

Now $x\theta'(x) = 2x^3(x - \sigma/2)$, so that if $x \geqslant \sigma/2$, $x\theta'(x) \geqslant 0$. But $(x\theta'(x))' = 8x^2(x - 3\sigma/8)$ and thus for $\alpha \geqslant \sigma/2$, $(\Delta F)(\Delta\tilde{\theta}') \geqslant 0$, and

$$S''(\alpha) + (3/\alpha)S'(\alpha) > 0.$$

Thus if $\alpha < \sigma/2$, $S'(\alpha) < 0$; if $\alpha \geqslant 2\sigma/3$, $S'(\alpha) > 0$, and for $\sigma/2 < \alpha < 2\sigma/3$, if $S'(\alpha) = 0$, then $S''(\alpha) > 0$. This proves that S' has exactly one zero. But now $T'(p) = S'(\alpha)\, d\alpha/dp$, and since $d\alpha/dp > 0$, we see that T has exactly one critical point, a minimum. This completes the proof in the case $a = 0$.

We turn now to the case where $a > 0$; note that $\alpha(p) > a$. We shall first show that $\alpha < \sigma/2$ implies that $T'(\alpha) < 0$ so that there cannot be any bifurcation in this case. To do this, we write $T(p) = T_1(p) + T_2(p)$, where

$$T_1(p) = \int_0^a (p^2 - 2F(u))^{-1/2}\, du.$$

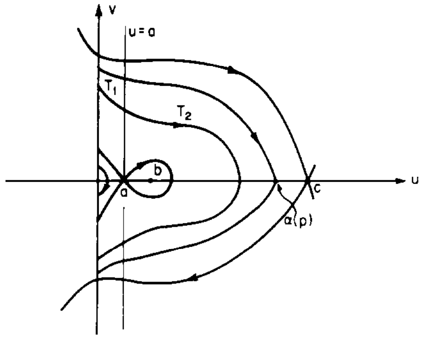

FIGURE 4

and

$$T_2(p) = \int_a^{\alpha(p)} 2^{-1/2}(F(\alpha(p)) - F(u))^{-1/2}\,du;$$

see Fig. 4.

Now if we set $S_2(\alpha) = 2^{1/2}T_2(p)$, and make the change of variables $w = u - a$, $\tilde{F}(x) = F(x + a)$, then we can write

$$S_2(\alpha) = \int_0^{\alpha - a} (\tilde{F}(\alpha - a) - \tilde{F}(w))^{-1/2}\,dw.$$

Thus, as before

$$S_2'(\alpha) = \int_0^{\alpha - a} \frac{\tilde{\theta}(\alpha - a) - \tilde{\theta}(w)}{(\tilde{F}(\alpha - a) - \tilde{F}(w))^{3/2}}\,dw,$$

where $\tilde{\theta}(x) = 2\tilde{F}(x) - (x + a)f(x + a)$. Now $\tilde{\theta}'$ vanishes at $(c + b)/2$, so that $\alpha < \sigma/2$ implies $\alpha - a < (c + b)/2$; hence $S_2' < 0$ if $\alpha < \sigma/2$. On the other hand, by direct calculation

$$\frac{dT_1}{dp} = -p\int_0^a (p^2 - 2F(u))^{-3/2}\,du < 0.$$

It then follows that $T'(\alpha) < 0$ if $\alpha < \sigma/2$.

We now consider the case where $\alpha \geqslant \sigma/2$. Here we define, as above, $S(\alpha) = 2^{1/2}T(p)$. Note that, as in the previous case, T has at least one critical point. We shall show here that for $\sigma \geqslant \sigma/2$, we have the estimate

$$S''(\alpha) + \frac{2}{\alpha}S'(\alpha) > 0. \qquad (2.4)$$

As before, this implies that T has exactly one critical point.

We shall in fact show that $\alpha \geqslant 4\sigma/9$ implies (2.4) holds. Now as before

$$S'' + \frac{3}{\alpha} S' > \frac{1}{2\alpha^2} \int_0^\alpha (\Delta F)^{-5/2} (\Delta F)(\Delta \tilde{\theta}')\, du, \tag{2.5}$$

and thus for any c

$$S'' + \frac{3}{\alpha} S' + cS' > \frac{1}{2\alpha^2} \int_0^\alpha (\Delta F)^{-3/2} [\alpha c(\Delta\theta) + \Delta \tilde{\theta}']\, du. \tag{2.6}$$

Now consider the function $\phi(x) = x\theta'(x) - \theta(x) = \frac{3}{2}x^3(x - 4\sigma/9)$. Thus, $\alpha > 4\sigma/9$, $u < \alpha$ imply $\phi(\alpha) - \phi(u) > 0$. Hence if we put $\alpha c = -1$ in (2.6), we obtain (2.4). This completes the proof of the theorem.

We remark that in the case where $a > 0$, there are two different classes of solutions of the Dirichlet problem; namely, those solutions having $\alpha(p) > a$, and those with $\alpha(p) \leqslant a$ (see Fig. 4). Our theorem applies to the former solutions, but from the results in [2], it follows that the latter solutions cannot undergo bifurcations. It is easy to see that the bifurcation diagram takes the form of Fig. 1B, in this case.

We now consider the remaining types of cubic functions $f(u)$. In order to classify the bifurcations, we shall use the notation (1.3). In this notation, our Theorem 2.1 implies that we have completely classified the bifurcations in the case when $\sigma > 0$, $\tau > 0$ and $\pi \geqslant 0$.

THEOREM 2.2. *If $a < 0 \leqslant b < c$, then no solution in $\mathcal{N}$ can bifurcate, and for solutions in $\mathcal{P}$, there is exactly one bifurcation; i.e., T has exactly one critical point.*

Remark. Observe that for solutions in $\mathcal{P}$, this theorem implies that we have classified the complete bifurcation diagram in the cases when $\pi \leqslant 0$, for any τ (where, of course, we are always considering $\sigma > 0$, in view of our previous convention).

Proof of Theorem 2.2. We consider first solutions in $\mathcal{N}$. Since $\sigma > 0$, it follows from (1.5) that θ' has no negative roots. As we have observed earlier (see footnote 4), this implies T is monotone, and so there cannot be any bifurcations.

For solutions in $\mathcal{P}$, we can use an argument similar to that in the proof of Theorem 2.1. Thus, from (1.5), we see that θ' can have at most one positive root. Also, since $\theta'(0) = \pi \leqslant 0$, $\theta'''(0) = -2\sigma < 0$, and $\theta'(u) \to +\infty$, as $u \to +\infty$ it follows that θ' has exactly one zero in $u \geqslant 0$. But $\theta'(4\sigma/9) = 2(4\sigma/9)^2(4\sigma/9 - \sigma/2) + \pi$. Hence $\alpha \leqslant 4\sigma/9$ implies that $\theta' < 0$, so $S' < 0$ and there cannot be any bifurcations in this case. On the other hand, if $\alpha > 4\sigma/9$, then as in Theorem 2.1, $S'' + (2/\alpha)S' > 0$, so there can be at most one bifur-

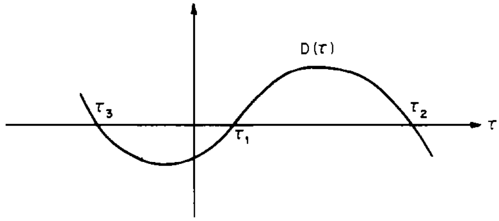

FIGURE 5

cation. Since $\theta(c) = 2F(c) > 0$, we see that for α near c, $S'(\alpha) > 0$. But we have already seen that $S'(\alpha) < 0$ if $\alpha < 4\sigma/9$; this implies that S has exactly one critical point, and the proof is complete.

For the convenience of the reader, we remark that Theorems 2.1 and 2.2 imply that we have classified the bifurcation diagrams for solutions in $\mathscr{P}$, in the cases (i) $\pi \geqslant 0$, $\tau \geqslant 0$, and (ii) $\pi \leqslant 0$, all τ, when f has three real roots. We shall not give a complete classification of all of the remaining cases, but shall content ourselves in demonstrating the existance of some new bifurcation diagrams. The omitted cases will be discussed at the end of this section.

For the cubic polynomial defined by Eq. (1.3) the discriminant D is defined by

$$D = -4\tau^3 + \sigma^2\tau^2 + 18\sigma\pi\tau - (27\pi^2 + 4\sigma^3\pi). \tag{2.7}$$

Here we are thinking of $D = D(\tau)$ as a function of τ, with $\sigma > 0$ and π fixed. We first consider the case where $\pi \geqslant 0$. Now since $D'(0) > 0$ and $D(0) < 0$, we see that $D(\tau)$ has one negative, (τ_3), and two positive, $(\tau_1, \tau_2, \tau_1 < \tau_2)$, roots (see Fig. 5).

We consider several cases; first suppose that $t \geqslant \tau_2$. Then $D(\tau) < 0$ so f has one real root, which must be nonnegative since $f(0) \geqslant 0$. If $f(0) = 0$, then the only solution to the Dirichlet problem is $u \equiv 0$. Hence, we can assume $f(0) > 0$. Next, suppose that $\tau = \tau_2$; then f has a double root,[5] so that the graph of f takes one of the two forms in Fig. 6. In the case where f has

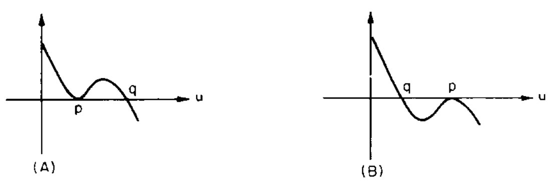

FIGURE 6

[5] If $\sigma^3 = 27\pi$, f has a triple root and conversely, since $f = (u - (\sigma/3))^3$; the analysis of this case is similar.

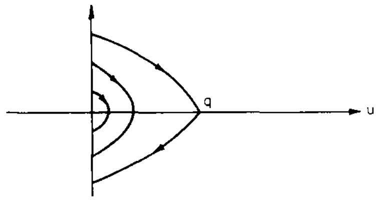

FIGURE 7

the form in Figure 6A we see that the flow has a rest point at p, and the phase portrait is that of Fig. 7.

We observe that this case is handled in a manner similar to that of Theorem 2.1., i.e., T has exactly one critical point, and remarks analogous to those made after the proof of that theorem are valid here. In the case where f is as in Fig. 6B, the time map is monotone; this follows from the results of [2] since $\theta' = f(u) - uf'(u)$ is nonzero.

Now suppose that $\tau > \tau_2$. First note that for $\tau = \tau_2$, the graph of f takes the form of Figure 6A. (This follows from consideration of τ in the open interval (τ_1, τ_2), where f has three positive roots, and allowing τ to increase; i.e., as τ increases, $-\tau u$ decreases, so f gets "pulled down.") At $\tau = \tau_2, f' < 0$ before the nondouble root r, so $\theta' = f - uf' > 0$ on $0 \leqslant u \leqslant r$. Since $df'/d\tau = -1$, we see that $\theta' > 0$ for $\tau \geqslant \tau_2$. Hence T is monotone, if $\tau \geqslant \tau_2$.

Next, we consider the case $\tau_1 \leqslant \tau < \tau_2$. Again, if $\tau = \tau_1$, then f is as in Fig. 6A, and we have done this case. Also, if $\tau_1 < \tau < \tau_2$ then f has three positive roots (since $f(-u)$ has positive coefficients), and this case again was considered in Theorem 2.1.

We consider now the case where $\tau \leqslant 0$. This case is rather difficult, and requires some new estimates, which we shall formulate in the lemmas below. We shall see too, that the time map has a distinctly different qualitative feature than those which we previously studied; namely, in this case T has exactly two critical points.

Now if $\tau < \tau_3$, then f has three real roots. Using Descarte's rule of signs, we see that f has exactly two negative roots, and one positive root; thus $a < b \leqslant 0 < c$. Now if we let $p(u) = \theta(u)/u$, then the discriminant of $p(u)$ is $(64\pi/27)[\sigma^3 - (27/4)^2\pi]$. Thus, for

$$\sigma^3 > (27/4)^2\pi \tag{2.8}$$

we see that $p(u)$ has two positive roots, and one negative root. Thus $\theta(u)$ has the form of Fig. 8, if (2.8) holds, independent of τ. Now if we show that $c > D$, and $\theta(c) > \theta(A)$, then we see from the above figure, that for $u < \alpha \leqslant c$, if $\Delta\theta = \theta(\alpha) - \theta(u)$, that $\Delta\theta > 0$ on $0 \leqslant \alpha \leqslant A$, $\Delta\theta < 0$ on $B \leqslant \alpha \leqslant C$, and $\Delta\theta > 0$ on $E \leqslant \alpha \leqslant b$. Thus since

$$S'(\alpha) = \int_0^\alpha \Delta\theta/2\alpha(\Delta F)^{3/2}\, du,$$

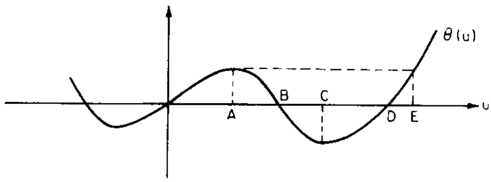

FIGURE 8

(see, e.g., (2.3)), we see that S', and hence T', has at least two zeros. We make this precise in the following lemma.

LEMMA 2.3. *If* (2.8) *holds, then T has at least two critical points, for any τ.*

Proof. As we have observed above, it suffices to show that $c > D$ and $\theta(c) > \theta(A)$. (Recall that the roots of f satisfy $a < b \leqslant 0 < c$.)

Now $\theta(c) = 2F(c) > 0$; if we show $c > A$ and $\theta(c) > \theta(A)$, then it follows that $c > D$ and the proof will be complete. We begin by showing that $c > A$. To see this, note that $0 = \theta'(A) = Ap'(A) + p(A)$ so $p'(A) < 0$. Hence $0 > 6A^2 - 2\sigma A = 6A(A - \sigma/3)$ and thus $A < \sigma/3$. But also $c \geqslant \sigma/3$; for if $c < \sigma/3$, then $\sigma/3 > \sigma/3 + (a + b) > c + a + b = \sigma > \sigma/3$, an impossibility. Thus $A < c$. Now $\theta(A) = 2F(A) - Af(A) < 2F(A) < 2F(c) = \theta(c)$. This completes the proof.

Our goal in the remainder of this section is to prove that T has exactly two critical points. This will follow from a series of lemmas. We first state the main theorem.

THEOREM 2.4. *If* (2.8) *holds, together with $\sigma > 0$, $\pi > 0$, and $\tau < 0$, then T has exactly two critical points.*[6]

LEMMA 2.5. *If $c \leqslant \alpha \leqslant E$ (se Fig. 7), then $S''(\alpha) + (2/\alpha) S'(\alpha) > 0$; in fact, this estimate holds, if $\alpha \geqslant 4\sigma/9$.*

COROLLARY 2.6. *If $B \leqslant \alpha(p) \leqslant E$, then $T(p)$ has exactly one critical point, a maximum.*

It is clear that the proof of the corollary follows at once from the lemma.

Proof of Lemma 2.5. As in the proof of Theorem 2.1, we have

$$S''(\alpha) + KS'(\alpha) = \int_0^\alpha \frac{K\alpha(\Delta F)(\Delta\theta) - \frac{3}{2}\Delta\theta(\Delta\tilde{f}) + \Delta F(\Delta\tilde{\theta}')}{2\alpha^2(\Delta F)^{5/2}}\, du.$$

[6] We remark that we do not consider the case $0 \leqslant \tau < \tau_1$. This case is rather delicate, and requires some new estimates. The hypothesis $\tau < 0$ is only used in Lemma 2.9.

In the numerator of the integrand, we put $K\alpha = -1$, and let $\phi(x) = x\theta'(x) - \theta(x)$. Note that $\phi(c) = -\theta(c) > 0$. Also, $\phi(x) = \frac{3}{2}x^3(x - 4\sigma/9)$. Hence, if $\alpha \geqslant 4\sigma/9$, $\phi'(x) = 6x^2(x - \sigma/3)$, so $\Delta\sigma \geqslant 0$ if $u < 4\sigma/9$ since $\phi(u) < 0$ and $\phi(\alpha) > 0$. If $u > 4\sigma/9$, then $\phi'(\alpha) > 0$, so $\Delta\phi > 0$. Thus

$$S''(\alpha) - \frac{1}{\alpha} S'(\alpha) \geqslant \frac{1}{2\alpha^2} \int_0^{\alpha} \frac{-\frac{3}{2}\Delta\theta(\Delta \tilde{f})}{(\Delta F)^{5/2}} du.$$

Now

$$S''(\alpha) - \frac{1}{\alpha} S'(\alpha) - dS'(\alpha) \geqslant \frac{1}{2\alpha^2} \int_0^{\alpha} \frac{(d\alpha \Delta F - \frac{3}{2}\Delta \tilde{f})\Delta\theta}{(\Delta F)^{5/2}} du$$

and if we put $d\alpha = 3$, we have

$$\begin{aligned} S''(\alpha) + \frac{2}{\alpha} S'(\alpha) &\geqslant \frac{3}{4\alpha^2} \int_0^{\alpha} \frac{(2\Delta F - \Delta \tilde{f})(\Delta\theta)}{(\Delta F)^{5/2}} du \\ &= \frac{3}{4\alpha^2} \int_0^{\alpha} \frac{(\Delta\theta)^2}{(\Delta F)^{5/2}} du > 0. \end{aligned}$$

This proves the lemma.

We now consider the case where $A \leqslant \alpha < B$ (see Fig. 8); our goal is to show that for some M, $S''(\alpha) + MS'(\alpha) < 0$.

In view of Lemma 2.5, we may assume that $\alpha < 4\sigma/9$, whenever this is convenient.

Before stating the next lemma, we need some notation. Let the function $H(u)$ be defined by

$$H(u) = f'(u)\,\theta'(u) - f(u)\,\theta''(u) \tag{2.9}$$

where, as usual $\theta(u) = 2F(u) - uf(u)$.

LEMMA 2.7. *If*

$$\sigma^3 > 3^5 2^{-3}(2 - \sqrt{3})^{-1} 3^{-1/2}\pi, \tag{2.10}$$

then $H(B) > 0$, *when* $\tau = 0$ *(recall* B *from Fig. 7). In particular, the conclusion holds if* (2.8) *is valid.*

Proof. A computation shows that $H(u) = -\sigma u^4 - 9\pi u^2 + 4\sigma\pi u$, at $\tau = 0$. Hence $H(u)/u = -\sigma u^3 - 9\pi u + 4\sigma\pi$. Now $\theta(B) = 0$, so that $\pi = -B^3/2 + \sigma B^2/3$. Substituting gives

$$\frac{H(B)}{B} = (4\sigma - 9B)\left(\frac{-B^3}{2} + \frac{\sigma B^2}{3}\right) - \sigma B^3$$
$$= B^2\left[(4\sigma - 9B)\left(\frac{\sigma}{3} - \frac{B}{2}\right) - \sigma B\right]$$
$$= B^2\left[\frac{9}{2}B^2 - 6\sigma B + \frac{4}{3}\sigma^2\right].$$

The quadratic in parenthesis has roots at $(2\sigma/9)(3 \pm \sqrt{3})$. Thus $H(B) > 0$ if $B < (2\sigma/9)(3 - \sqrt{3})$; i.e., if $\theta((2\sigma/9)(3 - \sqrt{3})) < 0$. This yields condition (2.10), and the proof of the lemma is complete.

LEMMA 2.8. *$\partial H(B)/\partial\tau < 0$ at $\tau = 0$, if (2.10) holds.*

Proof. Since $H(B)/B > 0$ at $\tau > 0$ (Lemma 2.7), we have $(4\sigma - 9B)\pi > \sigma B^3$, where the coefficient of π is positive. This is true since $\theta(4\sigma/9) < 0$ (2.8), so $B < 4\sigma/9$. Thus

$$\left.\frac{\partial H(B)}{\partial\tau}\right|_{\tau=0} = 4B^3 - \sigma B^2 - \pi$$
$$< 4B^3 - \sigma B^2 - \sigma B^3/(4\sigma - 9B)$$
$$= -4B^2(3B - \sigma)^2/(4\sigma - 9B) < 0.$$

LEMMA 2.9. *$H(u) > 0$ for all $\tau < 0$, $0 \leqslant u \leqslant B$, if (2.10) holds.*

Proof. We have

$$\left.\frac{H(u)}{u}\right|_{\tau=0} = -\sigma u^3 - 9\pi u + 4\sigma\pi,$$

so that

$$\left.\frac{d}{du}\left(\frac{H(u)}{u}\right)\right|_{\tau=0} = -3\sigma u^2 - 9\pi < 0.$$

Thus $H(B) > 0$ at $\tau = 0$ (Lemma 2.7), implies $H(u) > 0$ at $\tau = 0$, $0 \leqslant u \leqslant B$. Now $\partial H(u)/\partial\tau = -\theta'(u) + u\theta''(u) = 4u^3 - \sigma u^2 - \pi$. But $\theta(B) = 0$ implies $\pi = \sigma B^2/3 - B^3/2$. Hence

$$\frac{\partial H(B)}{\partial\tau} = \frac{9}{2}B^2\left(B - \frac{8}{27}\sigma\right) < 0$$

since $B < 2\sigma/9 < 8\sigma/27$. Now $\partial H(u)/\partial\tau$ has at most one positive root, it is negative at $u = 0$ and at B; thus it is negative on $0 \leqslant u \leqslant B$. This implies the result.

LEMMA 2.10. *If* $H(\alpha) > 0$, *then*

$$\max_{0 \leqslant u \leqslant \alpha} \Delta\tilde{f}/\Delta F - \min_{0 \leqslant u \leqslant \alpha} \Delta\tilde{\tilde{f}}'/\Delta\tilde{f} = 1, \tag{2.11}$$

where $\Delta\tilde{f} = \alpha f(\alpha) - uf(u)$, *and* $\Delta\tilde{\tilde{f}}' = \alpha^2 f'(\alpha) - u^2 f'(u)$.

Proof. We let $X = \Delta\tilde{f}$, $Y = \Delta F$ and $Z = \Delta\tilde{\tilde{f}}'$. We shall show that the maximum of X/Y occurs at $u = \alpha$ and that the minimum of Z/X occurs at $u = 0$. Then

$$\min_{0 \leqslant u \leqslant \alpha} Z/X = \alpha f'(\alpha)/f(\alpha),$$

and using L'Hospital's rule

$$\max_{0 \leqslant u \leqslant \alpha} X/Y = (\alpha f'(\alpha) + f(\alpha))/f(\alpha),$$

so subtracting, gives the desired result.

We shall show first that the maximum of X/Y occurs at $u = \alpha$. Now

$$\left(\frac{X}{Y}\right)' = \frac{-Y(uf'(u) + f(u)) + Xf(u)}{Y^2},$$

so

$$\left(\frac{X}{Y}\right)'\Bigg|_{u=0} = \frac{f(0)(X - Y)}{Y^2}\Bigg|_{u=0} = \frac{f(0)}{Y^2}(\alpha f(\alpha) - F(\alpha) + F(0))$$

$$= \frac{f(0)}{Y^2}\left(-\frac{3}{4}\right)\alpha^3(\alpha - 8\sigma/9) - \frac{\tau}{2}\alpha^2,$$

and this latter quantity is positive, since $\alpha < 4\sigma/9$, and $\tau < 0$. Thus max X/Y occurs at α or at some internal point. Now set $G(u) = (f(u) + uf'(u))/f(u)$, and note that when $u = \alpha$, $G(u) = X/Y$, and that $G(0) = 1$, and that $G'(u) = H(u)/f(u)^2 > 0$, by Lemma 2.9. Furthermore at an internal critical point u_0 of X/Y, we have $X/Y|_{u=u_0} = G(u_0) < G(\alpha) = X/Y|_{u=\alpha}$. Hence the maximum of X/Y occurs at $u = \alpha$.

Next, we show that the minimum of Z/X occurs at $u = 0$. Now $Z/X|_{u=0} = \alpha f'(\alpha)/f(\alpha)$. We have

$$\frac{Z}{X} - \frac{\alpha f'(\alpha)}{\alpha f(\alpha)} = \frac{\alpha^2 f'(\alpha) - u^2 f'(u)}{\alpha f(\alpha) - uf(u)} - \frac{\alpha f'(\alpha)}{f(\alpha)}$$

$$= \frac{uf(u)(\alpha(f'(\alpha)/f(\alpha)) - u(f'(u)/f(u)))}{\Delta\tilde{f}}$$

$$= \frac{uf(u)}{\Delta \tilde{f}} \left[1 - \frac{\theta'(\alpha)}{f(\alpha)} - \left(1 - \frac{\theta'(u)}{f(u)} \right) \right]$$

$$= \frac{uf(u)}{\Delta \tilde{f}} \left(\frac{\theta'(u)}{f(u)} - \frac{\theta'(\alpha)}{f(\alpha)} \right)$$

$$= \frac{uf(u)}{\Delta \tilde{f}} \frac{(u-\alpha)}{f(\xi)^2} (-H(\xi))$$

for some ξ, $u < \xi < \alpha$. But Lemma 2.9 implies that this last quantity is positive. This completes the proof of the lemma.

Proof of Theorem 2.4. Let $M = G(\alpha) = \max_{0 \leqslant u \leqslant \alpha} X/Y$. We shall show that

$$S''(\alpha) + \frac{M}{2\alpha} S'(\alpha) < 0; \tag{2.12}$$

this will prove that T has at most one critical point in $0 \leqslant \alpha \leqslant B$, which together with Corollary 2.6 and Lemma 2.3, proves the theorem. We have

$$S''(\alpha) + \frac{M}{2\alpha} S'(\alpha)$$
$$= \int_0^\alpha \frac{(M/2)[2(\Delta F)^2 - (\Delta \tilde{f})(\Delta F)] + \frac{3}{2}(\Delta \tilde{f})^2 - 2(\Delta \tilde{f})(\Delta F) - (\Delta \tilde{\tilde{f}}')(\Delta F)}{2\alpha^2 (\Delta F)^{5/2}} du.$$

Now from Lemma 2.10, the numerator, call it N, of the integrand is less than $\frac{3}{2}(\Delta \tilde{f})^2 - (2 + m + (M/2))(\Delta \tilde{f})(\Delta F) + M(\Delta F)^2$, where $m = \min_{0 \leqslant u \leqslant \alpha} Z/X$. If we let $\lambda = \Delta \tilde{f}/\Delta F$, then $1 \leqslant \lambda \leqslant M$, and

$$N \leqslant (\Delta F)^2 \left[\frac{3}{2} \lambda^2 - \lambda \left(2 + m + \frac{M}{2} \right) + M \right].$$

Denoting the quadratic in λ by $p(\lambda)$, and noting that $M - 2 = m - 1$, (from Lemma 2.10), we have

$$p(\lambda) = \tfrac{3}{2}\lambda^2 - \lambda(\tfrac{3}{2}M + 1) + M = (\lambda - M)(\lambda - \tfrac{2}{3}).$$

It then follows that $p(\lambda) \leqslant 0$ if $1 \leqslant \lambda \leqslant M$. Hence $N \leqslant 0$ and $S''(\alpha) + (M/2\alpha)S'(\alpha) < 0$. This completes the proof of Theorem 2.4.

For the convenience of the reader, we shall now make explicit just which cubic functions f we have not considered for solutions of the Dirichlet problem in the class $\mathscr{F}$. First note that the discriminant of θ' is $\pi/4(\sigma^3 - 27\pi)$; thus if $\sigma^3 < 27\pi$, θ' has no positive roots, and hence T' can never be zero. Now if $(27\pi) \leqslant \sigma^3 \leqslant 3^5 2^{-3} 3^{-1/2} (2 - \sqrt{3})^{-1} \pi \equiv \rho\pi$, then we have not considered such cubics. As σ increases from below 27π, a single cubic singularity in T develops, and we conjecture that T' has exactly two zeros henceforth; this is true for $\sigma^3 > \rho\pi$ (Theorem 2.4). But our techniques fail here. To analyze this case, would require a detailed analysis of the third derivative of T. Finally, if $\sigma^3 > \rho\pi$ and $0 < \tau < \tau_1$, we have not considered this case either. Our techniques seem to be applicable here, but the technical effort invoved, doesn't seem to be woth the result.

3. Neumann and Periodic Boundary Conditions

In this section we shall consider the problem (1.1) with periodic boundary conditions, or homogeneous Neumann boundary conditions,

$$u'(L) = u'(-L) = 0. \tag{3.1}$$

We shall show that in both cases, the "time" map is monotone, so that there is never any bifurcation. It is clear that if we prove this result for the Neumann problem, then the result for the case of periodic boundary conditions follows almost immediately.

We begin our study of the Neumann problem by noting that it suffices to consider those functions f of the form

$$f(u) = -u(u-a)(u-c), \qquad a < 0 < c. \tag{3.2}$$

To see this, observe that solutions of the Neumann problem can be written as solutions of the system $\dot{u} = v$; $\dot{v} = -f(u)$, $v(\pm L) = 0$, where in general, $f(u) = -(u-A)(u-B)(u-C)$, $A < B < C$. If we set $w = u - B$, then the system becomes $\dot{w} = v$, $\dot{v} = f(w+B) = -(w-(A-B))w(w-(C-B))$, with the same boundary conditions $v(\pm L) = 0$. Thus the Neumann problem is invariant under translations of u, and we need only consider f's of the form (3.2). Observe that in this notation, we have $\sigma = a + c > 0$, and $\tau = ac < 0$. Note too that our solutions lie in the "tear-shaped" region of Fig. 9.

We denote by $T_1(\alpha)$, the (minimal) "time" that the orbit through α takes to get to the line $u = 0$, and by $T_2(\beta)$, the time that this orbit takes to get from the line $u = 0$ to the point β on the line $v = 0$; see Fig. 9. Of course, α is a function of β, so that if we let $T(\beta)$ denote the "time" that the orbit takes in going from α to β, then we have

$$T(\beta) = T_1(\alpha(\beta)) + T_2(\beta).$$

Our main result in this section is the following theorem.

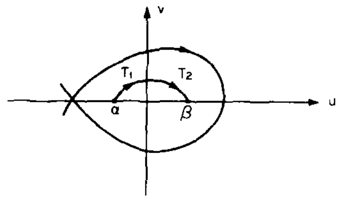

FIGURE 9

THEOREM 3.1. $dT/d\beta > 0$.

We are unable to prove this theorem directly, and so we shall first prove a few lemmas concerning the above "time" functions.

LEMMA 3.2. *$T_1' < 0$, and $T_1'' > 0$; here the primes denote differentiation of T_1 with respect to its argument.*

Proof. If in the expression for $T_1(\alpha)$, we make the change of variable $u = \alpha t$, we have

$$T_1(\alpha) = \int_\alpha^0 \frac{du}{(F(\alpha) - F(u))^{1/2}} = \int_0^1 \frac{-\alpha\, dt}{(F(\alpha) - F(\alpha t))^{1/2}}.$$

Thus using our usual notation $\Delta F = (F(\alpha) - F(\alpha t))$, we have

$$T_1'(\alpha) = \int_0^1 \frac{-(\Delta F)^{1/2} + \frac{1}{2}\alpha(\Delta F)^{-1/2}(f(\alpha) - tf(\alpha t))}{(\Delta F)}\, dt$$

$$= \int_0^\alpha \frac{-\Delta F + \frac{1}{2}(\alpha f(\alpha) - uf(u))}{(\Delta F)^{3/2}} \frac{du}{\alpha}$$

$$= \int_\alpha^0 \frac{\theta(\alpha) - \theta(u)}{2(\Delta F)^{3/2}} \frac{du}{\alpha}.$$

Now observe that since $\pi = 0$, $\theta(x) = (x^3/2)(x - 2\sigma/3)$. It follows that θ is a monotone decreasing function if $x < 0$. Thus $u > \alpha$ implies $\theta(\alpha) > \theta(u)$, so that the integrand in the expression for T_1', is negative.

To calculate T_1'', we have

$$2T_1' = \int_0^1 \frac{-\Delta\theta}{(\Delta F)^{3/2}}\, dt,$$

where $\Delta\theta = \theta(\alpha) - \theta(\alpha t)$. Thus,

$$2T_1'' = -\int_0^1 \frac{(\Delta F)^{3/2}(\theta'(\alpha) - t\theta'(\alpha t)) - \frac{3}{2}(\Delta\theta)(\Delta F)^{1/2}(f(\alpha) - tf(\alpha t))}{(\Delta F)^3}\,dt$$

$$= \int_0^\alpha \frac{\frac{3}{2}\Delta\theta(\Delta\tilde{f}) - (\Delta F)(\Delta\tilde{\theta}')}{(\Delta F)^{5/2}}\,du,$$

where $\Delta\tilde{\theta}' = \alpha\theta'(\alpha) - u\theta'(u)$, and $\Delta\tilde{f} = \alpha f(\alpha) - uf(u)$. Hence

$$T_1'' = \int_0^\alpha \frac{\frac{3}{4}(\Delta\theta)(\Delta\tilde{f}) - 3\Delta\theta\Delta F + 3\Delta\theta\Delta F - \Delta F(\Delta\tilde{\theta}')}{(\Delta F)^{5/2}}\,du$$

$$= \int_0^\alpha \frac{-\frac{3}{2}(\Delta\theta)^2 + \Delta F(3\Delta\theta - \Delta\tilde{\theta}')}{(\Delta F)^{5/2}}\,du.$$

Since $\alpha < 0$, we see that $T_1'' > 0$ since $3\theta(x) - x\theta'(x) = -\frac{1}{2}x^4 \leqslant 0$. But also, $(3\theta(x) - x\theta'(x))' = -2x^3 > 0$. This completes the proof of the lemma.

LEMMA 3.3. *If* $0 < \beta \leqslant 2\sigma/3$, *then* $f'(\beta) > 0$, $f(\alpha) + f(\beta) \geqslant 0$, *and* $\alpha + \beta \leqslant 0$.

Proof. A calculation shows that $f'(2\sigma/3) = -ab > 0$, so that $f'(\beta) > 0$. Hence if we think of β as a function of α via $F(\beta) = F(\alpha)$, and set $G(\alpha) = f(\alpha) + f(\beta)$, we see that $G(0) = 0$. Also,

$$\begin{aligned}(d/d\alpha)\, G(\alpha) &= f'(\alpha) + f'(\beta)\, d\beta/d\alpha \\ &= f'(\alpha) + f'(\beta)(f(\alpha)/f(\beta)) \\ &= (f'(\alpha)f(\beta) + f'(\beta)f(\alpha))/f(\beta).\end{aligned}$$

Now consider $G(\alpha)$ at a point where $G'(\alpha) = 0$. At such a point, $f(\alpha) = f'(\alpha)f(\beta)/f'(\beta)$ so

$$\begin{aligned}G(\alpha) &= f(\beta)(f'(\beta) - f'(\alpha))/f'(\beta) \\ &= f(\beta)(3(\alpha - \beta)(\alpha + \beta) + 2\sigma(\beta - \alpha))/f'(\beta)\end{aligned}$$

by direct computation. Thus, if $\alpha + \beta \leqslant 0$, then we see that $G(\alpha) > 0$, if $G'(\alpha) = 0$. Since $G(a) > 0$, we see that G cannot be negative on $(a, 0)$; i.e., $G \geqslant 0$. This proves the first two statements provided that the last one holds.

To prove the last statement, we consider β fixed and consider α as a function of σ via the equation $F(\alpha) = F(\beta)$. Now, let us define $H(\sigma) = \alpha + \beta$. Note that if $\sigma = 0$, then f is an odd function and $\alpha = -\beta$ so $H(0) = 0$. Also, $H'(\sigma) = d\alpha/d\sigma$. Now differentiating the relation $F(\alpha) = F(\beta)$ with respect to σ

gives $\beta^3/3 = \alpha^3/3 + f(\alpha)\, d\alpha/d\sigma$ so $d\alpha/d\sigma = (\beta^3 - \alpha^3)/3f(\alpha) < 0$. Thus $H(\sigma) \leqslant 0$ and the proof is complete.

LEMMA 3.4. $d/d\beta(T_1(\alpha) - T_1(-\beta) > 0$.

Proof. We will show, equivalently, that $d/d\alpha(T_1(\alpha) - T_1(-\beta)) < 0$. Now, if "prime" again denotes differentiation with respect to its argument, we have

$$\begin{aligned}\frac{d}{d\alpha}(T_1(\alpha) - T_1(-\beta)) &= T_1'(\alpha) - T_1'(-\beta)(-d\beta/d\alpha)\\ &= T_1'(\alpha) + T_1'(-\beta)(f(\alpha)/f(\beta))\\ &= (T_1'(\alpha) - T_1'(-\beta)) + T_1'(-\beta)\left(1 + \frac{f(\alpha)}{f(\beta)}\right)\\ &= T_1''(\xi)(\alpha + \beta) + T_1'(-\beta)\left(1 + \frac{f(\alpha)}{f(\beta)}\right),\end{aligned}$$

for some ξ between α and $-\beta$. Thus the result follows from Lemmas 3.2 and 3.3.

LEMMA 3.5. $(d/d\beta)(T_1(-\beta) + T_2(\beta)) > 0$.

Observe that Lemmas 3.4 and 3.5 prove Theorem 3.1.

Proof. We have

$$T_1(-\beta) = \int_{-\beta}^{0} \frac{du}{(F(-\beta) - F(u))^{1/2}} = \int_{\beta}^{0} \frac{-du}{(F(-\beta) - F(-u))^{1/2}}$$

and

$$T_2(\beta) = \int_0^{\beta} \frac{du}{(F(\beta) - F(u))^{1/2}}.$$

Let $T(\beta) = T_1(-\beta) + T_2(\beta)$; then

$$\begin{aligned}T(\beta) &= \int_0^{\beta} [(F(\beta) - F(u))^{-1/2} + (F(-\beta) - F(-u))^{1/2}]\, du\\ &= \int_0^{1} [\beta(F(\beta) - F(\beta t))^{-1/2} + \beta(F(-\beta) - F(-\beta t))^{1/2}]\, dt.\end{aligned}$$

If we let $\Delta F = F(\beta) - F(u)$, $\Delta F_- = F(-\beta) - F(-u)$, $\Delta \tilde{f} = \beta f(\beta) - \beta t f(\beta t)$, and $\Delta \tilde{f}_- = -\beta f(-\beta) + \beta t f(-\beta t)$, then

$$T'(\beta) = \int_0^1 \left\{ \frac{2\Delta f - \Delta \tilde{f}}{2(\Delta F)^{3/2}} + \frac{2\Delta F_- - \Delta \tilde{f}_-}{2(\Delta F_-)^{3/2}} \right\} dt$$

$$= \int_0^1 \left\{ \frac{\theta(\beta) - \theta(\beta t)}{2(\Delta F)^{3/2}} + \frac{\theta(-\beta) - \theta(-\beta t)}{2(\Delta F_-)^{3/2}} \right\} dt$$

$$= \int_0^\beta \left\{ \frac{\theta(\beta) - \theta(u)}{2(F(\beta) - F(u))^{3/2}} + \frac{\theta(-\beta) - \theta(-u)}{2(F(-\beta) - F(-u))^{3/2}} \right\} \frac{du}{\beta}.$$

Thus, it suffices to show that for $0 \leqslant u \leqslant \beta$, that

$$(F(\beta) - F(u))^{3/2}(\theta(-\beta) - \theta(-u)) + (F(-\beta) - F(-u))^{3/2}(\theta(\beta) - \theta(u)) \geqslant 0. \tag{3.3}$$

Now $\theta(u) + \theta(-u) = u^4 \leqslant \beta^4$ so that

$$\theta(-\beta) - \theta(-u) \geqslant -(\theta(\beta) - \theta(u)). \tag{3.4}$$

But also, as we have observed in the proof of Lemma 3.2, $\theta'(x) < 0$ if $x < 0$; hence

$$\theta(-\beta) - \theta(-u) \geqslant 0. \tag{3.5}$$

Again, $F(-u) - F(u) = -2\sigma u^3/3 \geqslant -2\sigma\beta^3/3$, so $F(-u) - F(u) \geqslant F(-\beta) - F(\beta)$, and thus

$$F(\beta) - F(u) \geqslant F(-\beta) - F(-u). \tag{3.6}$$

Now (3.5) and (3.6) yield (3.3) and hence the proof of the lemma is complete.

We next consider the case of periodic boundary conditions. To do this, we merely note that the time map here is also an increasing function, as follows from Theorem 3.1, and the fact that orbits in phase space are symmetric about the line $v = 0$. We thus have

THEOREM 3.6. *For the problem* (1.1), *with periodic boundary conditions, the derivative of the time map is positive.*

COROLLARY 3.1. *There is at most one nonconstant periodic solution of* (1.1) *with homogeneous Neumann boundary conditions, having a prescribed number of maxima on* $-L \leqslant x \leqslant L$.

4. Dirichlet Boundary Conditions, Part 2

We shall begin this section by considering solutions which are never positive. We remind the reader that the conventions of Section 1 are still in effect; in particular, we assume that σ, the sum of the roots of f, is nonnegative. Thus, there are only two cases to consider; namely, when (i) $a < 0 \leqslant b < c$, or (ii), $a < b \leqslant 0 < c$. However, case (i), was considered in Theorem 2.2. Case (ii) is handled in the following theorem.

THEOREM 4.1. *Considering solutions in* $\mathscr{N}$, *the following are true:* (i) *If* $a < 0 \leqslant b < c$, *then* T *is monotone;* (ii) *if* $a < b \leqslant 0 < c$, *then* T *has exactly one critical point, a minimum.*

Proof. Suppose that the hypotheses (ii) hold. From (2.6), we have, with $c = -1/\alpha$,

$$S'' + \frac{2}{\alpha} S' > \frac{1}{2\alpha^2} \int_{\alpha}^{0} (\Delta F)^{-3/2}\, \Delta\phi \, du,$$

where $\phi(x) = x\theta'(x) - \theta(x) = \frac{3}{2}x^3(x - 4\sigma/9)$. Thus, if $u < 0$, then $\phi' < 0$ so that $\Delta\phi > 0$ and hence $S'' + (2/\alpha)S' > 0$. Thus T can have at most one critical point.

Now from (1.5), $\theta'(0) = \pi > 0$, so $S' > 0$ for α near 0, $\alpha < 0$. It follows that $T'(p) > 0$ for p near zero. But for p near a, $T(p)$ is very large; thus T' has exactly one critical point. This completes the proof.

We note that in case (ii) of the last theorem, the time map takes the form of Fig. 10.

Finally, we make some remarks about solutions which oscillate; i.e., solutions of the Dirichlet problem which lie neither in $\mathscr{P}$ nor in $\mathscr{N}$. These solutions correspond to orbits which begin on the line $u = 0$, encircle the origin one or more times, and then end on $u = 0$. Since T is always monotone for periodic orbits (Theorem 3.2), we see that if the orbit encircles the origin sufficiently-many times, then there cannot be any bifurcation.

That is, solutions with sufficiently many maxima, do not bifurcate. The case of solutions with a small number of maxima, requires further investigation. We shall not pursue this here.

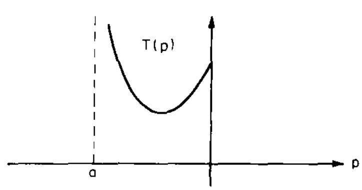

FIGURE 10

5. Concluding Remarks

We end this paper by making a few observations. First, we note that our results are immediately applicable to certain systems. For example, consider the Fitz–Hugh–Nagumo equations, $v_t = v_{xx} + f(v) - u$, $u_t = \sigma v - \gamma u$, where σ and γ are positive constants, and $f(v) = -v(v-b)(v-c)$, $0 < b < c$. If σ and γ are chosen so that the polynomial $\phi(v) = f(v) - (\sigma/\gamma)v$ has three roots, then our results apply at once to the steady-state solutions of these equations. Similar remarks apply to the Hodgkin–Huxley equations.

Next, we wish to point out that we can view the results of Section 3 in a somewhat different manner. Namely, we can think of these results as examples of bifurcations with two parameters, namely, the length of the interval L, and the middle root b. Thus, abstractly, we have an equation of the form $F(u, L, b) = 0$, corresponding to the problem (1.1) and (2.1), where f is given by (1.2), with both a and c considered as fixed. We now can study the "bifurcation diagram" for this equation as *both* L and b vary. For example, if $\sigma^3 > \rho\pi$, (recall that ρ was defined at the end of Section 3), and $\tau < 0$, then for $a < b < 0 < c$, Theorem 3.4 implies that the bifurcation diagram for the nonnegative solutions, has the form of Fig. 11.

Now let $b \to 0$, holding a and c fixed. When $b = 0$, Theorem 3.2 applies, and it is easy to see that the bifurcation diagram takes the form of Fig. 12, together with the positive L-axis. Thus the bifurcation diagrams do not vary continuously with b. One can view this as a "higher order" bifurcation; i.e., as a "bifurcation" in the bifurcation diagrams.

Next, we wish to point out that in [2], it was proved that for solutions of Eq. (1.1), with either homogeneous Neumann or Dirichlet boundary conditions, if $T'(p) \neq 0$, then the corresponding solution is strongly

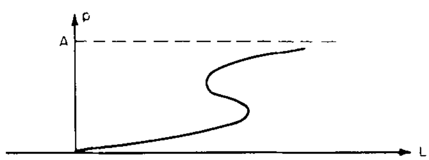

Figure 11

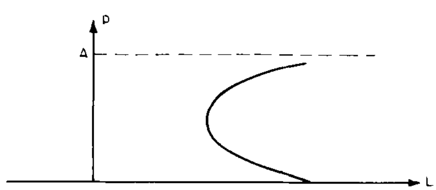

Figure 12

nondegenerate, in the sense that zero is not in the spectrum of the linearized operator. This implies at once the validity of our remarks in the introduction concerning strongly nondegenerate solutions.

Finally, we end this paper by noting that our methods are applicable to other linear homogeneous boundary value problems for equation (1.1). Again in the interest of brevity, we shall not discuss these here.

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