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GORMAN AND MUSGRAVE ARE DUAL
An Antipodean Theorem On Public Goods

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This paper examines the conditions under which the 'allocation branch' of the government can determine the optimum provision of public goods independently of the distribution of private goods. We draw attention to the duality with the literature on aggregation over consumers private goods economies.

The theory of public goods would be greatly simplified if it were possible to determine the optimal amount of public goods independently of the distribution of private goods among individuals. Richard Musgrave (1958, 1969) has vigorously argued for treating the allocation and distribution functions of government as distinct. But as Paul Samuelson (1955, 1969) has demonstrated, in general equlibrium different Pareto optima corresponding to different distributions of utility will typically require different quantities of public goods. Therefore an 'allocation branch' of the government cannot independently determine a Pareto optimal amount of public goods unless it is informed in advance by the 'distribution branch' about the distribution of private goods that would simultaneously be instituted.

There is a well-known special case in which allocation and distribution can for practical purposes be treated separately. This is the case of 'quasi-linear utility' where preferences of each individual, i, can be represented by a utility function of the form:

$$U_i(X_i,Y)=X_i+f_i(Y), \tag{1}$$

where X_i is i's consumption of private good and Y is the amount of public good available. As Samuelson (1969) points out, in this case the partial equilibrium analyses of Bowen (1943) and Lindahl (1910) extend without complication to the case of general equilibrium. The trouble with this special case is that it is too special to serve as even a reasonable approximation to a realistic model. One of the implications of quasi-linear utility is that if the wealth of all individuals in the community were increased, the Pareto optimal amount of public good for the community would be unchanged. Another implication is that if tax shares are an increasing function of private wealth, then within a community one's preferred amount of public good would be a decreasing function of his wealth. Several recent empirical studies of the demand for local public goods strongly suggest that this hypothesis is untenable.

It turns out that separation between allocation and distribution is possible over a much broader and more interesting class of preferences. Let there be K public good and one private good. Let Y denote the vector of public goods supplied and X_i the amount of the private good consumed by i. The family of preferences for which the desired separation is possible turns out to be essentially [1] those preferences which are representable by utility functions of the form

$$U_i(X_i,Y)=A(Y)X_i+B_i(Y) \tag{2}$$

for each i. This class of utility functions includes both quasi-linear utility and the case of identical Cobb–Douglas utilities. But members of this class need be neither homothetic nor separable and can be chosen in such a way as to make income elasticities of demand for public goods as large or small as one wishes. Different B_i's for different consumers allow for variation in preferences.

In this paper, for simplicity of exposition we confine our attention to utility functions that are differentiable, monotone increasing [2] in all

[1] This 'essentially' glosses a number of subtleties and qualifications which are treated in Bergstrom and Corner (1981).

[2] This assumption excludes Cobb–Douglas utility functions since they imply that private goods are useless when $Y=0$. We could expand our theory to include this and other cases where indifference curves are asympotic to the X_i axis by assuming only that preferences for private goods are monotone increasing in private goods for strictly positive Y. The method of proof used here could be adapted to these assumptions by choosing a strictly positive 'origin vector' Y_0 to play the role that the vector $Y=0$ plays in this proof. The representation theorem thus obtained would apply only over the domain $Y \geqslant Y_0$. But Y_0

commodities, and strictly quasi-concave.[3] We assume that for any allocation (X_i,Y), there exists an amount of private goods, X_i', large enough so that $U_i(X_i',0) > U_i(X_i,Y)$.[4] Production possibilities are assumed to include all allocations $(X_1,\ldots, X_n,Y)$ such that $\Sigma_i X_i + C(Y) = W$, where $C(Y)$ is a smooth convex function.
Let

$$\Pi_i^k(X_i,Y) \equiv \frac{\partial U_i(X_i,Y)}{\partial Y_k} \Big/ \frac{\partial U_i(X_i,Y)}{\partial X_i}$$

be i's marginal rate of substitution between public good k and private goods. The Samuelson first-order conditions for Pareto efficiency are

$$\sum_i \Pi_i^k(X_i,Y) = \frac{\partial C(Y)}{\partial Y_k} \quad \text{for each } k, \tag{3}$$

$$\sum_i X_i + C(Y) = W. \tag{4}$$

Under the strong regularity and convexity assumptions assumed here, these conditions are sufficient for a Pareto optimum as well as necessary for an interior Pareto optimum.[5]

If utility takes the form $A(Y)X_i + B_i(Y)$ for each i, then it is immediate that

$$\Pi_i^k(X_i,Y) = \alpha^k(Y)X_i + \beta_i^k(Y), \tag{5}$$

where $\alpha^k(Y) \equiv (1/A(Y))(\partial A(Y)/\partial Y_k)$ and $\beta_i^k(Y) \equiv (1/A(Y)) \times$

could be chosen arbitrarily close to the origin so that a limiting argument could be used to extend the theorem to the entire positive orthant.

[3] In Bergstrom and Cornes (1981) we use a different method of proof which shows that none of these assumptions are essential.

[4] This assumption also excludes indifference curves asymptotic to the X_i axis. As remarked in footnote 2 we could extend our results to these cases by substituting an origin vector $Y_0 \gg 0$ for 0 in the statement of this assumption.

[5] An interior Pareto optimum is a Pareto optimum in which Y is strictly positive and $X_i > 0$ for all i.

$(\partial B_i^k/\partial Y_k)$. Therefore condition (3) can be written as

$$\alpha^k(Y)\sum_i X_i + \sum_i \beta_i^k(Y) = \frac{\partial C(Y)}{\partial Y_k} \quad \textit{for each } k. \tag{6}$$

Suppose that $(\bar{X}_1,\ldots,\bar{X}_n,\bar{Y})$ is an interior Pareto optimum and that utility functions are of the form (2). Since (4) and (6) are necessary for an interior Pareto optimum, the allocation $(\bar{X}_1,\ldots,\bar{X}_n,\bar{Y})$ must satisfy both equations. Let $(X_1',\ldots,X_n',\bar{Y})$ be an allocation such that $\Sigma_i X_i' = \Sigma_i \bar{X}_i$. Since $(\bar{X}_1,\ldots,\bar{X}_n,\bar{Y})$ satisfies (4) and (6) it must also be that $(X_1',\ldots,X_n',\bar{Y})$ satisfies (4) and (6). But (4) and (6) are sufficient for Pareto optimality. Therefore $(X_1',\ldots,X_n',\bar{Y})$ must also be Pareto optimal. Thus we have shown that if utility is of the form (2), then a Pareto efficient amount of public goods is determined independently of the distribution of private goods.

A deeper theorem is the converse result that independence between distribution and allocation implies that preferences be representable by utility functions of the form (2). If a Pareto efficient vector of public goods is to remain Pareto efficient after any redistribution of private goods, it is clear that the left-hand side of (3) must remain constant if Y is constant and $\Sigma_i X_i$ is constant. Therefore for each k, it must be that

$$\sum_i \Pi_i^k(X_i,Y) = f^k\Big(\sum_i X_i, Y\Big) \tag{7}$$

for some function f^k. Now an equation of the form

$$\sum_i G_i(X_i) = H\Big(\sum_i X_i\Big) \tag{8}$$

is known as a Pexider functional equation [see Aczel (1966) or Eichhorn (1978)]. A standard result (with an easy proof) is that if the functions $G_i(\cdot)$ are continuous, then they must all be of the form $aX_i + b_i$. Applying this result to eq. (3) (where the Π_i^k are viewed as functions of X_i, holding Y constant) we have

$$\Pi_i^k(X_i,Y) = \alpha^k(Y)X_i + \beta_i^k(Y) \tag{9}$$

for some functions $\alpha^k(Y)$ and $\beta_i^k(Y)$. Recalling the definition of $\Pi_i^k(X_i,Y)$, we can find the functional form required for independence by solving for the family of solutions to the partial differential equations

$$\frac{\partial U_i}{\partial Y_k} \bigg/ \frac{\partial U_i}{\partial X_i} = \alpha^k(Y)X_i + \beta_i^k(Y). \tag{10}$$

Now let us consider a dual problem. Gorman (1953) explores the question of when aggregate demand for private goods is independent of the distribution of income. Let $h_i^k(M_i,P)$ be consumer i's demand for good k as a function of his income, M_i and the price vector $P=(P_i,\ldots,P_K)$. If demand is independent of income distribution, then it must be that

$$\sum_i h_i^k(M_i,P) = f^k\left(\sum_i M_i,P\right) \tag{11}$$

for some function f^k. Eq. (11), like (7) is a Pexider functional equation. Therefore there must be functions $\alpha^k(P)$ and $\beta_i^k(P)$ such that

$$h_i^k(M_i,P) = \alpha^k(P)M_i + \beta_i^k(P). \tag{12}$$

Let $V_i(M_i,P)$ be consumer i's indirect utility function. According to Roy's law:

$$h_i^k(M_i,P) = \frac{\partial V_i(M_i,P)}{\partial P_k} \bigg/ \frac{\partial V_i(M_i,P)}{\partial M_i}. \tag{13}$$

Substituting (13) into (12) yields

$$\frac{\partial V_i(M_i,P)}{\partial P_k} \bigg/ \frac{\partial V_i(M_i,P)}{\partial M_i} = \alpha^k(P)M_i + \beta_i^k(P). \tag{14}$$

Notice that eqs. (7), (9) and (10) of our theory are formally identical to eqs. (11), (12) and (14) of Gorman's where we identify respectively the objects $X_i, Y, \Pi_i^k(X_i,Y)$ and $U_i(X_i,Y)$ from the former theory with the objects $M_i, P, h_i^k(M_i,P)$ and $V_i(M_i,P)$ from the latter. Thus the answer to our question 'What kind of utility functions allow one to solve for efficient amounts of public goods independently of the distribution of private goods?' must be just the dual of the answer to Gorman's question 'What kind of indirect utility function allows one to solve for aggregate demand for private goods independently of income distribution.' If you know that the answer to Gorman's question is 'Indirect utility must be representable in the form

$$V_i(M_i,P)=A(P)M_i+B_i(P)\text{'}, \tag{15}$$

then you also know that the answer to our question is: 'Direct utility must be representable in the form (2).'

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We could stop here, but we have a rather neat demonstration that the partial differential eq. (10) imply that utility is representable in the form (2). We haven't seen this proof elsewhere in the economic literature. Because of the duality between (10) and (15) this proof also offers a new and quite simple proof of Gorman's result.

We assumed that for each i, and all (X_i,Y) there exists X_i' such that $U_i(X_i',0)=U_i(X_i,Y)$. Define $U_i^*(X_i,Y)$ so that $U_i(U_i^*(X_i,Y),0)=U_i(X_i,Y)$.[6] Since preferences are assumed to be monotone increasing in $X_i, U_i^*(X_i,Y)$ is a well-defined function and furthermore, $U_i^*(X_i,Y)$ represents preferences of consumer i. Geometrically, $U_i^*(X_i,Y)$ is the point on the X_i axis that meets the indifference curve through (X_i,Y). Consider a point $(\bar{X}_i,\bar{Y})$ and for any scalar λ such that $0\leqslant\lambda\leqslant 1$, define $X_i(\lambda)$ so that $U_i(X_i(\lambda),\lambda\bar{Y})=U_i(\bar{X}_i,\bar{Y})$.[7] Then from the definitions of $U_i^*(X_i,Y)$, and of $X_i(\lambda)$ it must be that

$$X_i(0)=U_i^*(\bar{X}_i,\bar{Y}) \quad \text{and} \quad X_i(1)=X_i. \tag{16}$$

From the defintion of $X_i(\lambda)$:

$$0=\frac{\mathrm{d}}{\mathrm{d}\lambda}U_i(X_i(\lambda),\lambda\bar{Y})=\frac{\partial U_i}{\partial X_i}\frac{\mathrm{d}X_i(\lambda)}{\mathrm{d}\lambda}+\sum_k \bar{Y}_k\frac{\partial U_i(X_i(\lambda),\lambda\bar{Y})}{\partial Y_k}. \tag{17}$$

6 If we were proving the dual theorem, we would pick a reference price vector $P_0 \gg 0$ and define $V_i^*(M_i,P)$ so that $V_i(V_i^*(M_i,P)P_0)=V_i(M_i,P)$. It is easily seen that the structure of indirect utility theory ensures that $V_i^*(M_i,P)$ is well-defined and represents indirect utility.

7 To see that $X_i(\lambda)$ is well-defined observe the following. Monotonicity in Y implies that $U_i(\bar{X}_i,\lambda\bar{Y})\leqslant U_i(\bar{X}_i,\bar{Y})$ for $\lambda\leqslant 1$. By assumption, there exists X_i' such that $U_i(\bar{X}_i,\bar{Y})=U_i(X_i',0)\leqslant U_i(X_i',\lambda\bar{Y})$. Therefore $U_i(\bar{X}_i,\lambda\bar{Y})\leqslant U_i(\bar{X}_i,\bar{Y})\leqslant U_i(X_i',\lambda\bar{Y})$. Monotonicity and continuity of U_i imply, therefore, that for some unique $X_i(\lambda)$ between $\bar{X}_i$ and X_i', we have $U_i(X_i(\lambda),\lambda\bar{Y})=U_i(\bar{X}_i,\bar{Y})$.

From (10) and (17) it follows that

$$\frac{\mathrm{d}X_i(\lambda)}{\mathrm{d}\lambda} = -\left[\sum_k \bar{Y}_k\alpha^k(\lambda\bar{Y})X_i(\lambda) + \sum_k \beta_i^k(\lambda\bar{Y})\right]. \tag{18}$$

Thus eq. (18) is of the form

$$\frac{\mathrm{d}X_i(\lambda)}{\mathrm{d}\lambda} = \Psi_1(\lambda,\bar{Y})X_i(\lambda) + \Psi_2(\lambda,\bar{Y}). \tag{19}$$

This is a well-known type of ordinary differential equation. Its solution can be found in any differential equations text to be of the form

$$X_i(\lambda) = F_i(\lambda,\bar{Y})\left[X_i(0) + G_i(\lambda,\bar{Y})\right] \tag{20}$$

for some functions F_i and G_i. Recalling (16), (20) implies that

$$U_i^*(\bar{X}_i,\bar{Y}) = A_i(\bar{Y})\bar{X}_i + \hat{B}_i(\bar{Y}) \quad \text{where} \tag{21}$$

$$A_i(\bar{Y}) = \frac{1}{F_i(1,\bar{Y})} \quad \text{and} \quad \hat{B}_i(\bar{Y}) = -\frac{G_i(1,\bar{Y})}{F_i(1,\bar{Y})}.$$

It remains only to be shown that the A_i's in (27) can be chosen to be identical. To see this, notice that (10) and (27) imply that

$$\alpha^k(Y) = \frac{1}{A_i(Y)}\frac{\partial A_i(Y)}{\partial Y_k} = \frac{\partial \ln A_i(Y)}{\partial Y_k} \tag{22}$$

for all k. Integrating (22), we see that there must exist positive scalars k_i such that $A_i(Y) = k_i A(y)$ for some function $A(Y)$. Since preferences can be represented by the utility function (21) for each i, they could also be represented by the utility function obtained by dividing (21) by k_i for each i. This yields the utility representation

$$A(Y)X_i + B_i(Y), \tag{23}$$

where $B_i(Y) + (1/k_i)\hat{B}_i(Y)$. We have demonstrated that if the utility functions U_i satisfy the partial differential eq. (10), they must all be monotone transformations of utility functions of the form (23).

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