THE β FUNCTION OF THE TWO-DIMENSIONAL NON-LINEAR σ MODEL

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Improved MCRG methods are used to determine the β function of the two-dimensional O(3) standard action σ model. We argue that the previously observed anomalous behavior of the mass gap can be understood as a general scaling behavior according to the full β function.

1. Introduction

The two-dimensional non-linear O(3) σ model is one of the simplest asymptotically free theories. However, Monte Carlo simulations showed that the standard action model

$$Z=\int \mathrm{D}\bar{s}\prod_{\bar{n}}\delta(\bar{s}_{\bar{n}}^2-1)\exp\left(\beta\sum_{n}\sum_{\mu=1}^{2}\bar{s}_{\bar{\mu}}\bar{s}_{\bar{n}+\mu}\right)$$

does not show asymptotic scaling behavior even at a relatively large correlation length [1]. By asymptotic scaling we mean the continuum scaling behavior which can be described by the first two universal terms of the β function.

For earlier scaling behavior, different improvement methods were proposed. The Symanzik improvement program is based on a systematic perturbative expansion to cancel the lattice artifact corrections to scaling [2]. The idea behind the real space renormalization group improvement method is to find the renormalized trajectory of a given block transformation and to perform a Monte Carlo simulation along it [3–5]. These improved actions showed earlier asymptotic scaling behavior, however, the question arises: Is the behavior of the standard action model at correlation length $\xi \simeq 2$–10 a real non-scaling behavior or is it a scaling behavior in the general sense: scaling according to the full β function?

In this paper, using Monte Carlo renormalization group methods (MCRG), we determine the β function of the standard model. We argue, that the standard action O(3) model shows scaling behavior even at correlation length of about 2 lattice units. Therefore the value of the mass gap in terms of the lattice Λ parameter can be given using the high statistics MC data for the mass gap at small lattice correlation length.

2. The MCRG methods

The aim of an MCRG method is to determine the β function of the theory. We can directly determine a discrete version of the β function, $\Delta\beta(\beta)$ which gives the change in the coupling $\beta \to \beta' = \beta - \Delta\beta(\beta)$ when the lattice correlation length is *decreased* by a factor of 2. The function $\Delta\beta(\beta)$ contains the same information as the full β function itself.

In ref. [6] two different MCRG methods were proposed [‡1]. Both methods worked extremely well for

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[‡1] One of the methods discussed in ref. [6] (optimization of the block transformation for a given fixed action) has also been suggested in ref. [7].

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the exactly solvable two-dimensional O(n), $n \to \infty$ model and were promising for the NL σ model as well. We applied both methods in a high statistics Monte Carlo calculation. We give here only a brief summary of the methods. For details the interested reader is referred to refs. [6,8].

2.1. The probability blocking method. The block spin MCRG is a powerful technique for the study of critical phenomena. However, it can be effective only if the fixed point and the renormalized trajectory (RT) of the given block transformation lie close to the action we are investigating. Earlier MCRG investigations revealed that the RT of the simplest block transformation, where the block spin is just the normalized average of the four spins in a 2 × 2 block,

$$\bar{t}_{\bar{n}} = \frac{\bar{S}_{\bar{n}_1} + \bar{S}_{\bar{n}_2} + \bar{S}_{\bar{n}_3} + \bar{S}_{\bar{n}_4}}{\|\bar{S}_{n_1} + \bar{S}_{\bar{n}_2} + \bar{S}_{\bar{n}_3} + \bar{S}_{\bar{n}_4}\|},$$

$$\bar{S}_{\bar{n}_1}, \bar{S}_{\bar{n}_2}, \bar{S}_{\bar{n}_3}, \bar{S}_{\bar{n}_4} \in \text{block } \bar{n},$$

is relatively far from the standard action [5]. However, with a simple modification one can find a block transformation which works much better for the standard action [6]. Let us choose the block spin with a probability distribution

$$\bar{t}_{\bar{n}} = \text{Prob} \exp\left(c\beta \bar{t}_{\bar{n}} \sum_{i \in \text{block}} \bar{S}_{\bar{n}_i}\right).$$

The parameter c is free. It can be tuned in order to find an optimized block transformation. Tree level perturbation theory suggests

$$c_{\text{opt}} = 2.3 \quad \text{for O(3)}.$$

We would like to emphasize that in principle every value of c is appropriate, only the number of blocking steps which is required to find convergence can depend on the specific choice of c.

2.2. The ratio method. The ratio of two spin–spin correlation functions

$$f(\bar{n}_1, \bar{n}_2) = \langle \bar{S}_0 \bar{S}_{\bar{n}_1} \rangle / \langle \bar{S}_0 \bar{S}_{\bar{n}_2} \rangle, \quad n_1, n_2 \gg 1$$

satisfies the homogeneous RG equation in the continuum limit, therefore, it can be used to determine the β function of the theory. The matching condition is

$$f(2n_1, 2n_2, L, \beta) = f(n_1, n_2, L/2, \beta'),$$

where the left- and right-hand side of this equation should be calculated on a lattice of $L \times L$ and $(L/2) \times (L/2)$ in order to minimize the finite size effects. If n_1 and n_2 are not large enough, the ratio f suffers from lattice artifacts. There is a systematic way to minimize the distortion effect of the lattice. In the improved ratio method linear combinations of the ratios are considered and the mixing coefficients are determined so that the lattice artifacts are cancelled systematically order by order in perturbation theory [8]. Combining two basic ratios one can define ratios which are correct at tree level, combining three basic ratios one can define ratios which are correct at tree level and at one-loop level as well, and so on. This improvement requires the calculation of the tree level, one-loop, ... coefficients of the correlation functions [9].

3. The calculation

We used the probability improved blocking method with $c = 2.3$ parameter, and the tree-level and one-loop improved ratio test as well for matching from 16^2 lattice size to 8^2 lattice size at 9 different coupling values: $\beta = 1.9, 1.8, 1.7, 1.6, 1.5, 1.4, 1.3, 1.2, 1.1$. The lattice correlation length at $\beta = 1.9$ is about 100 while at $\beta = 1.1$ it is about 2.5. On the 16^2 lattice at $\beta = 1.9$ we did measurements on 30 000 configurations, at $\beta = 1.8, 1.7$ and 1.4 on 20 000 configurations while at the other β values on 10 000 configurations, each separated by 10 heat bath steps. On the 8^2 lattice we measured at $\beta = 1.79, 1.77; 1.69, 1.67; 1.58, 1.56; 1.48, 1.46; 1.36, 1.34; 1.24, 1.22; 1.11, 1.09; 0.96, 0.94$; and 0.83, 0.81; using linear interpolation between the adjacent β values for matching. For the blocking matching at every β value we measured the nearest neighbor $\langle S_0 S_{\hat{\mu}} \rangle$, the diagonal $\langle S_0 S_{\hat{\mu}+\hat{\nu}} \rangle$, $(\hat{\mu} \neq \hat{\nu})$, and the next-to-nearest neighbor $\langle S_0 S_{2\hat{\mu}} \rangle$ block expectation values doing 2 and 3 blocking steps on the 8^2 and 16^2 lattice, respectively. For the ratio test we measured all the possible spin–spin correlation functions with distance less than 5 on the 8^2 lattice and the corresponding even correlation functions on the 16^2 lattice. The mixing coefficients of the tree level and one-loop improved ratios can be determined by perturbation theory. On the finite lattice one should handle the zero modes carefully [9]. We considered only ratios with positive mixing coefficients (to assure monotonic behavior in β) and restricted the relative magnitude of the mixing coefficients to be of order 1 (to suppress the dominance of certain basic ratios). Altogether we analyzed 264 tree-level and 384 one-loop improved ratios.

In the error analysis we used both the time relaxation method and the "bunching" method.

4. Results

We analyzed the improved ratios by introducing different cuts. First we considered only ratios which gave the matching value with an error smaller than a given error cut. Then we considered those ratios which did not contain the smallest correlation functions and analyzed them separately. Table 1 shows the matching values at different β values, introducing different cuts for the tree level and one-loop improved ratios. The first error is the usual statistical error, while the second one corresponds to the average fluctuation of the matching values from the different ratios. The stability of the results with respect to the different cuts and improvements is remarkable, only the average fluctuation depends on the details of the analysis.

Similar stability is found in the blocking matching when we considered different block expectation values.

Our results at the different coupling values are summarized in table 2. We give the $\Delta\beta$ value from the blocking matching obtained by comparing the first and second block expectation values, and the second and third block expectation values. The results from the tree-level and one-loop improved ratio test and the two-loop asymptotic values are also given.

5. Conclusion

Table 2 shows that at $\beta = 1.7-1.9$ we are already in the asymptotic scaling region, while at $\beta < 1.6$ there are important perturbative and/or non-perturbative contributions to the β functions. However, we are still in the scaling region as we could do a consistent blocking and ratio matching as well.

In fig. 1 the full $a\Lambda$ parameter obtained from the data of table 2 is fitted to the mass gap MC data. It shows that the mass gap scales according to the full $a\Lambda$

Table 1
Matching values at different coupling values introducing $c_1 = 0.025$ and $c_2 = 0.030$ error cuts in the analysis.

β	Tree level improved ratios		One-loop improved ratios		
	all ratios	long distance ratios	all ratios	long distance ratios	
1.9	0.1259(0.0140)(0.0006)	0.1206(0.0154)(0.0010)	0.1223(0.0193)(0.0003)	0.1211(0.0204)(0.0004)	
1.7	0.1166(0.0133)(0.0006)	0.1116(0.0154)(0.0008)	0.1123(0.0172)(0.0002)	0.1102(0.0185)(0.0002)	c_1
1.4	0.1823(0.0127)(0.0006)	0.1821(0.0150)(0.0011)	0.1737(0.0151)(0.0002)	0.1730(0.0178)(0.0002)	
1.2	0.2537(0.0123)(0.0006)	0.2546(0.0148)(0.0007)	0.2468(0.0145)(0.0005)	0.2512(0.0153)(0.0003)	
1.9	0.1248(0.0162)(0.0006)	0.1202(0.0189)(0.0009)	0.1229(0.0218)(0.0003)	0.1214(0.0228)(0.0004)	
1.7	0.1158(0.0156)(0.0006)	0.1100(0.0198)(0.0007)	0.1120(0.0188)(0.0002)	0.1100(0.0204)(0.0002)	c_2
1.4	0.1824(0.0138)(0.0006)	0.1820(0.0168)(0.0011)	0.1737(0.0167)(0.0002)	0.1731(0.0178)(0.0002)	
1.2	0.2537(0.0131)(0.0006)	0.2549(0.0164)(0.0007)	0.2467(0.0155)(0.0005)	0.2512(0.0164)(0.0003)	

Table 2
$\Delta\beta$ matching values at the different coupling values obtained by using 2/1 and 3/2 block transformation, tree level and one-loop improved ratio test. The two-loop asymptotic values of $\Delta\beta$ are also given.

β	2/1 Blocking	3/2 Blocking	Tree level ratio	One-loop ratio	Two-loop pert.
1.9	0.125|6|	0.116|7|	0.126|14|	0.122|19|	0.120
1.8	0.122|9|	0.119|9|	0.123|13|	0.118|18|	0.121
1.7	0.120|7|	0.115|7|	0.117|13|	0.112|17|	0.122
1.6	0.132|7|	0.124|7|	0.128|14|	0.125|17|	0.122
1.5	0.149|10|	0.142|13|	0.155|14|	0.150|15|	0.123
1.4	0.177|5|	0.176|6|	0.182|13|	0.174|15|	0.124
1.3	0.212|7|	0.208|10|	0.219|14|	0.204|19|	0.126
1.2	0.246|7|	0.249|6|	0.254|12|	0.247|14|	0.127
1.1	0.297|7|	0.304|7|	0.310|13|	0.281|18|	0.129

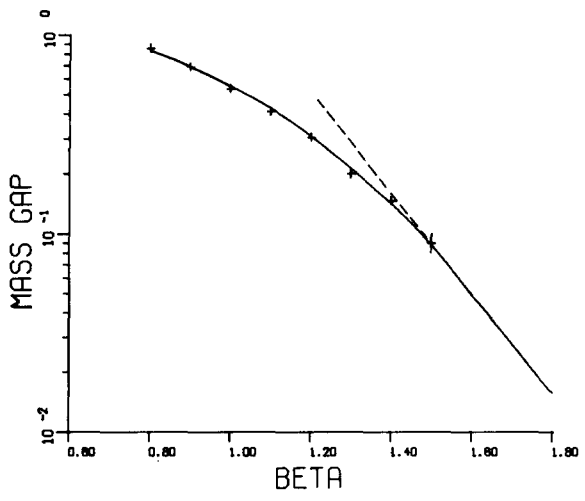

Fig. 1. The full $a\Lambda$ parameter obtained from table 2 is fitted to the mass gap Monte Carlo data [2]. The dashed line corresponds to the two-loop asymptotic formula.

parameter up to $\beta \approx 0.9$. The value of the mass gap can be obtained using the high precision measurement at $\beta = 1.0$–1.2

$$m = (113 \pm 10)\Lambda_{\text{lattice, standard act.}}$$

We think that this prediction for the mass gap and the excellent scaling property we found illustrates the usefulness and the power of an MCRG analysis.

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