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OPTIMAL SWITCHING CONDITIONS FOR MINIMUM FUEL FIXED TIME TRANSFER BETWEEN NON COPLANAR ELLIPTICAL ORBITS

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Abstract—The ordinary theory of maxima and minima is applied to develop the optimal switching conditions for the problem of two-impulse transfer with time constraint between non-coplanar elliptical orbits. It is shown that all the elements of the transfer orbit, the total characteristic velocity for the transfer and the time of flight between impulses can be expressed explicitly in terms of three variables namely the semi-latus rectum of the transfer orbit, and the true anomalies defining the locations of the impulses on the initial and the final orbit, respectively. In terms of these variables, three necessary optimal conditions are derived for solving either the problem of minimum-fuel, fixed-time transfer or the problem of minimum-time transfer for a prescribed fuel consumption. In the special cases of fixed-time coplanar transfer and free-time non-coplanar transfer, the general optimal conditions are reduced to the classical optimal conditions in the published literature. In particular, it is shown that for the minimum-fuel, fixed-time transfer between coplanar circular orbits, the optimal conditions can be expressed in terms of Lambert's invariant parameters.

1. INTRODUCTION

During the past two decades, the problem of minimum-fuel transfer between orbits has been thoroughly investigated for both the high-thrust and the low-thrust propulsion systems. The cumulative results have been masterfully presented in the exhaustive treatise by Marec[1]. With increased manned space flight in this decade, the transfer time becomes an important parameter which should be considered in the optimization process. In this paper, we propose to develop the optimal equations for solving the following two related problems.

There are given two terminal orbits O_1 and O_2 about a Newtonian center of attraction F (Fig. 1). For a high-thrust propulsion system, it is proposed to find the minimum fuel transfer trajectory for a prescribed transfer time, or the minimum-time transfer trajectory for a given fuel consumption.

An orbit is defined by the classical elements a, e, Ω, i and ω. If the plane of the initial orbit is taken as the reference plane and the direction to the pericenter as the reference direction, there are given 7 parameters a_1, e_1, a_2, e_2, Ω_2, i_2 and ω_2 (Fig. 1). We shall assume that the thrust is high enough such that the velocity change at the time of the application of the thrust can be considered as instantaneous. Furthermore, we restrict the problem to the case of two impulsive changes in the velocity with magnitude V_1 and V_2. The total characteristic velocity is then

$$V = V_1 + V_2 = F(\mathrm{X}) \tag{1}$$

where X is the arbitrary variable defining the transfer. The time of flight is

$$\tau = G(\mathrm{X}) \tag{2}$$

The mathematical problem is simply, for a given τ, to minimize the function V, or for a given V, to minimize the function τ. Since this is a parametric optimization problem subject to constraint, the theory of maxima and minima is adequate for its solution provided that the variables selected lead to managable resulting equations.

2. TRANSFER GEOMETRY

We shall define the shape and the size of an orbit by its eccentricity e and semi-latus rectum $p = a|1 - e^2|$ where a is the semi-major axis. Hence, we first have the given parameters p_1, e_1, p_2 and e_2. The elements without subscript correspond to the unknown transfer orbit O. To preserve certain element of symmetry in the equations, we shall use the

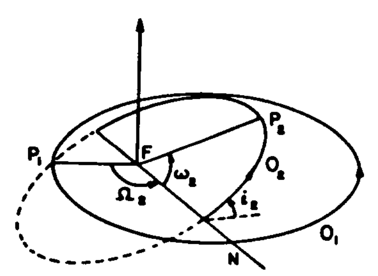

Fig. 1. Geometry of the terminal orbits.

†Deceased.

plane of the transfer orbit as the reference plane. Let ϕ be the angle between the planes on the given orbits which intersect each other along the line of nodes FN. Let β_1 and β_2 be the true anomalies of this line of node in the corresponding planes. Hence, as compared to the classical elements of the orbit in Fig. 1, we have $\phi = i_2$, $\beta_1 = \Omega_2$ and $\beta_2 = 2\pi - \omega_2$.

Let I_1 and I_2 be the locations of the impulses. These are defined in the planes of the terminal orbits by the true anomalies α_1 and α_2, respectively, with corresponding radial distances r_1 and r_2. Hence, in the planes of the terminal orbits

$$r_1 = \frac{p_1}{1 + e_1 \cos \alpha_1}, \quad r_2 = \frac{p_2}{1 + e_2 \cos \alpha_2}. \tag{3}$$

In the plane of the transfer orbit, let f_1 and f_2 be the true anomalies of the points I_1 and I_2 respectively. Then (Fig. 2)

$$p = r_1(1 + e \cos f_1) = r_2(1 + e \cos f_2) \tag{4}$$

Solving for p and e, we have

$$p = \frac{r_1 r_2(\cos f_1 - \cos f_2)}{r_1 \cos f_1 - r_2 \cos f_2} \tag{5}$$

and

$$e = \frac{(r_2 - r_1)}{r_1 \cos f_1 - r_2 \cos f_2}. \tag{6}$$

Let Δ be the transfer angle measured in the plane of the transfer orbit, that is

$$\Delta = f_2 - f_1. \tag{7}$$

We have the following pertinent relations

$$\tan f_1 = \cot \Delta - \frac{r_1(p - r_2)}{r_2(p - r_1) \sin \Delta}$$
$$\tan f_2 = -\cot \Delta + \frac{r_2(p - r_1)}{r_1(p - r_2) \sin \Delta}. \tag{8}$$

Finally, let γ_1 and γ_2 be respectively the plane changes upon the applications of the impulses. More specifically, as shown in Fig. 3, with projection on the unit sphere, the initial plane O_1 is rotated by an angle γ_1, about the vector $\mathbf{FI}_1$, into the plane O which is in turn rotated by the angle γ_2, about the vector $\mathbf{FI}_2$, into the final plane O_2 making an angle ϕ with the plane O_1 along the line of node $\mathbf{FN}$. The sides and the angles for the oblique spherical triangle I_1NI_2 are clearly labelled on the figure. From this, we have the following useful relations

$$\cos \Delta = \cos(\beta_1 - \alpha_1)\cos(\beta_2 - \alpha_2) + \sin(\beta_1 - \alpha_1)\sin(\beta_2 - \alpha_2)\cos\phi \tag{9}$$

$$\cos \phi = \cos \gamma_1 \cos \gamma_2 - \sin \gamma_1 \sin \gamma_2 \cos \Delta \tag{10}$$

$$\sin \gamma_1 = -\frac{\sin(\beta_2 - \alpha_2)}{\sin \Delta} \sin \phi \tag{11}$$

$$\sin \gamma_2 = \frac{\sin(\beta_1 - \alpha_1)}{\sin \Delta} \sin \phi \tag{12}$$

$$\cos \gamma_1 = \cos \gamma_2 \cos \phi + \sin \gamma_2 \cos(\beta_2 - \alpha_2) \sin \phi \tag{13}$$

$$\cos \gamma_2 = \cos \gamma_1 \cos \phi + \sin \gamma_1 \cos(\beta_1 - \alpha_1) \sin \phi. \tag{14}$$

In the figure, the two rotations γ_1 and γ_2 are in the same direction. This is a plane change of the rotating type. In the case where γ_2 is in the opposite direction, we have a plane change of the reflecting type. The same equations apply if we take a negative value for γ_2.

From the geometry of the transfer, it is clear that, for the components of the vector $\mathbf{X}$ in eqns (1) and (2), we can take the true anomalies α_1 and α_2 defining the locations of the impulses, and the semi-latus rectum p of the transfer orbit. They are the unknown independent variables considered in the present formulation. The other elements are obtained explicitly in terms of these variables. First, the radial distances r_1 and r_2 are given by eqns (3), in terms of α_1 and α_2 respectively. Then, the transfer angle Δ is obtained from eqn (9). The plane change angles γ_1 and γ_2 are determined by eqns (11) and (12). This completely fixes the plane of the transfer orbit. In this plane, if p is specified, the true

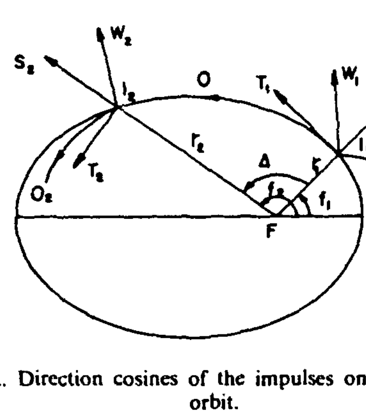

Fig. 2. Direction cosines of the impulses on the transfer orbit.

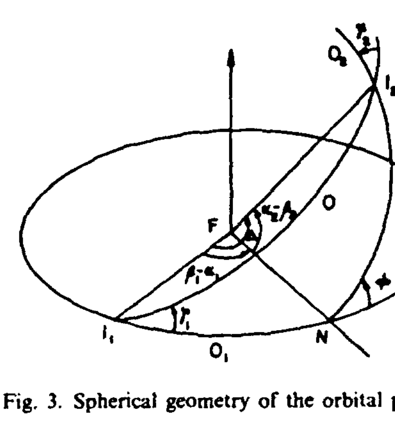

Fig. 3. Spherical geometry of the orbital planes.

Table 1. Dependency of the intermediary variables

Variables	Function of
r_1	α_1
r_2	α_2
Δ	α_1, α_2
γ_1, γ_2	α_1, α_2
f_1, f_2	α_1, α_2, p
e	α_1, α_2, p

anomalies f_1 and f_2 of the impulses are given by eqns (8). We then have the line of the apses of the transfer orbit. Its eccentricity is of course given by eqn (6). The functions V and τ can then be easily computed.

These functions will be given in the next section, in the mixed forms, containing the intermediary variables e, r_i, γ_i and f_i. For the purpose of evaluating the partial derivatives of these variables with respect to the independent variables a_1, a_2 and p, it is convenient to summarize the dependency in Table 1.

Based on this table, and using the appropriate equations for taking the partial derivatives and simplifying, we have the following equations for use in further development

$$\frac{\partial r_1}{\partial \alpha_1}=r_1^2\frac{e_1}{p_1}\sin\alpha_1,\quad \frac{\partial r_2}{\partial \alpha_2}=r_2^2\frac{e_2}{p_2}\sin\alpha_2 \tag{15}$$

$$\frac{\partial \Delta}{\partial \alpha_1}=-\cos\gamma_1,\quad \frac{\partial \Delta}{\partial \alpha_2}=\cos\gamma_2 \tag{16}$$

$$\frac{\partial \gamma_1}{\partial \alpha_1}=\sin\gamma_1\cot\Delta,\quad \frac{\partial \gamma_1}{\partial \alpha_2}=\frac{\sin\gamma_2}{\sin\Delta} \tag{17}$$

$$\frac{\partial \gamma_2}{\partial \alpha_1}=-\frac{\sin\gamma_1}{\sin\Delta},\quad \frac{\partial \gamma_2}{\partial \alpha_2}=-\sin\gamma_2\cot\Delta \tag{18}$$

$$\frac{\partial f_1}{\partial p}=\frac{\partial f_2}{\partial p}=\frac{r_1-r_2}{e^2 r_1 r_2\sin\Delta}=\frac{\cos f_2-\cos f_1}{ep\sin\Delta} \tag{19}$$

$$\frac{\partial f_1}{\partial \alpha_1}=\frac{1}{\sin\Delta}\left(\cos\gamma_1\cos f_1\sin f_2-\frac{e_1 p}{e p_1}\sin\alpha_1\cos f_2\right)$$

$$\frac{\partial f_1}{\partial \alpha_2}=-\frac{\cos f_1}{\sin\Delta}\left(\cos\gamma_2\sin f_2-\frac{e_2 p}{e p_2}\sin\alpha_2\right) \tag{20}$$

$$\frac{\partial f_2}{\partial \alpha_1}=\frac{\cos f_2}{\sin\Delta}\left(\cos\gamma_1\sin f_1-\frac{e_1 p}{e p_1}\sin\alpha_1\right),\quad \frac{\partial f_2}{\partial \alpha_2}=-\frac{1}{\sin\Delta}$$

$$\times\left(\cos\gamma_2\sin f_1\cos f_2-\frac{e_2 p}{e p_2}\sin\alpha_2\cos f_1\right) \tag{21}$$

$$\frac{\partial e}{\partial p}=\frac{1}{\sin\Delta}\left(\frac{\sin f_2}{r_1}-\frac{\sin f_1}{r_2}\right)=\frac{e}{p}+\frac{\sin f_2-\sin f_1}{p\sin\Delta} \tag{22}$$

$$\frac{\partial e}{\partial \alpha_1}=\frac{\sin f_2}{\sin\Delta}\left(e\sin f_1\cos\gamma_1-\frac{e_1 p}{p_1}\sin\alpha_1\right)$$

$$\frac{\partial e}{\partial \alpha_2}=-\frac{\sin f_1}{\sin\Delta}\left(e\sin f_2\cos\gamma_2-\frac{e_2 p}{p_2}\sin\alpha_2\right). \tag{23}$$

3. THE CHARACTERISTIC FUNCTIONS

We now derive the characteristic functions V and τ. In the plane of the motion, let u be the radial component and v the transverse component of the velocity vector. We have the classical relations

$$u=\sqrt{\frac{\kappa}{p}}\,e\sin f,\quad v=\frac{\sqrt{\kappa p}}{r} \tag{24}$$

where κ is the gravitational constant. Let superscripts $(-)$ and $(+)$ denote respectively the elements of the velocity before and after the application of the impulse. Then, for the true anomaly f, using either $f=\alpha_i$ or $f=f_i$, we have

$$u_1^-=\sqrt{\frac{\kappa}{p_1}}\,e_1\sin\alpha_1,\quad u_1^+=\sqrt{\frac{\kappa}{p}}\,e\sin f_1$$
$$u_2^-=\sqrt{\frac{\kappa}{p}}\,e\sin f_2,\quad u_2^+=\sqrt{\frac{\kappa}{p_2}}\,e_2\sin\alpha_2 \tag{25}$$

and

$$v_1^-=\frac{\sqrt{\kappa p_1}}{r_1},\quad v_1^+=\frac{\sqrt{\kappa p}}{r_1}$$
$$v_2^-=\frac{\sqrt{\kappa p}}{r_2},\quad v_2^+=\frac{\sqrt{\kappa p_2}}{r_2}. \tag{26}$$

For the velocity changes $\mathbf{V}_i$, in the plane of the transfer orbit, we use a right-handed coordinate system $Ixyz$ such that the x-axis is along the position vector, the y-axis is orthogonal to it and in the direction of the motion and the z-axis is orthogonal to the orbital plane. For the x_i component, it is simply

$$x_i=u_i^+-u_i^-. \tag{27}$$

For the y_i and z_i components we refer to Fig. 4 which shows the velocity diagram in the local horizontal plane with $\mathbf{V}_{i_h}$ representing the projection of the velocity change into this plane.

In the rotated plane

$$y_i^+=v_i^+-v_i^-\cos\gamma_i$$
$$z_i^+=v_i^-\sin\gamma_i \tag{28}$$

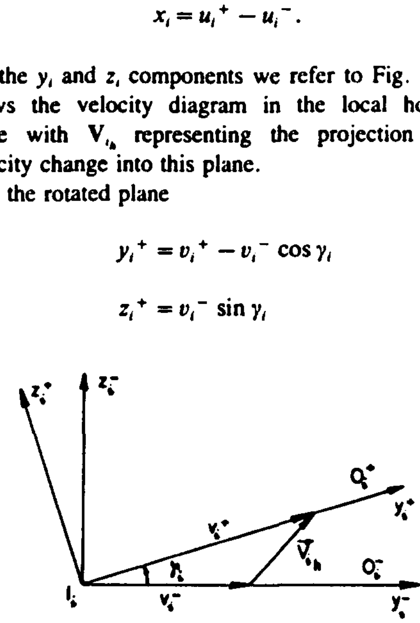

Fig. 4. Velocity components in the horizontal plane at the impulse.

while in the original plane, we have

$$y_i^- = v_i^+ \cos\gamma_i - v_i^- \qquad z_i^- = v_i^+ \sin\gamma_i. \tag{29}$$

Since we refer the velocity changes to the plane of the transfer orbit, for the first impulse $\mathbf{V}_1$ we use eqns (28) and for the second impulse $\mathbf{V}_2$ we use eqns (29) to have the components of the impulses $\mathbf{V}_i$

$$x_1 = \sqrt{\kappa}\left(\frac{e}{\sqrt{p}}\sin f_1 - \frac{e_1}{\sqrt{p_1}}\sin\alpha_1\right)$$
$$y_1 = \frac{\sqrt{\kappa}}{r_1}(\sqrt{p} - \sqrt{p_1}\cos\gamma_1) \tag{30}$$
$$z_1 = \frac{\sqrt{\kappa p_1}}{r_1}\sin\gamma_1$$

and

$$x_2 = \sqrt{\kappa}\left(\frac{e_2}{\sqrt{p_2}}\sin\alpha_2 - \frac{e}{\sqrt{p}}\sin f_2\right)$$
$$y_2 = \frac{\sqrt{\kappa}}{r_2}(\sqrt{p_2}\cos\gamma_2 - \sqrt{p}) \tag{31}$$
$$z_2 = \frac{\sqrt{\kappa p_2}}{r_2}\sin\gamma_2.$$

The total characteristic velocity for the transfer is

$$V = \Sigma(x_i^2 + y_i^2 + z_i^2)^{1/2} \tag{32}$$

It is dependent of the variables α_1, α_2 and p through the intermediary variables e, r_i, γ_i and f_i. For taking the derivatives with respect to the independent variables, it is convenient to express the characteristic velocities in the form

$$V_i = (x_i^2 + h_i^2)^{1/2} \tag{33}$$

where h_i is the horizontal component of $\mathbf{V}_i$ with

$$h_i^2 = y_i^2 + z_i^2 = \frac{\kappa}{r_i^2}(p + p_i - 2\sqrt{pp_i}\cos\gamma_i). \tag{34}$$

Along an elliptic orbit, the time of flight from the pericenter to a point with eccentric anomaly E is

$$t = \sqrt{\frac{a^3}{\kappa}}(E - e\sin E) \tag{35}$$

where a is the semi-major axis

$$a = p/(1-e^2). \tag{36}$$

The eccentric anomaly is related to the true anomaly by the equations

$$\cos E = \frac{e + \cos f}{1 + e\cos f}, \quad \sin E = \frac{\sqrt{1-e^2}\sin f}{1 + e\cos f}. \tag{37}$$

Using the mean anomaly M such that

$$M = E - e\sin E \tag{38}$$

we have the time of flight between the impulses on the transfer orbit, with the provision for N complete revolutions

$$\tau = \sqrt{\frac{a^3}{\kappa}}(M_2 - M_1 + 2\pi N). \tag{39}$$

Since τ is a function of p, e and f_1 and f_2, it is an implicit function of the chosen independent variables p, α_1 and α_2.

If the transfer orbit is hyperbolic, we have the same characteristic function (39) with "a" being the semi-transverse axis such that

$$a = p/(e^2 - 1). \tag{40}$$

The mean anomaly is now

$$M = e\sinh H - H \tag{41}$$

where the hyperbolic anomaly H is related to the true anomaly f by the equations

$$\cosh H = \frac{e + \cos f}{1 + e\cos f}, \quad \sinh H = \frac{\sqrt{e^2-1}\sin f}{1 + e\cos f}. \tag{42}$$

4. THE OPTIMAL CONDITIONS

If the characteristic velocity for the transfer is prescribed, being equal to a value V_0, we have the constraining relation

$$V - V_0 = 0. \tag{43}$$

On the other hand, if the transfer time is prescribed being equal to a value τ_0, we have the constraining relation

$$\tau - \tau_0 = 0. \tag{44}$$

We consider the augmented performance index

$$J = k_1(V - V_0) + k_2(\tau - \tau_0) \tag{45}$$

where k_1 and k_2 are two constants. If ξ is any one of the three independent variables, p, α_1 and α_2, the minimization of J leads to the three necessary conditions

$$\frac{\partial J}{\partial \xi} = k_1\frac{\partial V}{\partial \xi} + k_2\frac{\partial \tau}{\partial \xi} = 0. \tag{46}$$

From eqn (45), it is clear that for a minimum-fuel, free-time problem, we take $k_2 = 0$, while for a fixed-time problem, in addition to eqns (46) we consider eqn (44). On the other hand, if the fuel consumption is prescribed, for the minimum-time problem, the three eqns (46) and the constraining relation (43) constitute the set of optimal conditions.

In all these cases, we can use the ratio $k = k_2/k_1$ and write the necessary conditions

$$\frac{\partial V}{\partial p} + k\frac{\partial \tau}{\partial p} = 0$$

$$\frac{\partial V}{\partial \alpha_1} + k\frac{\partial \tau}{\partial \alpha_1} = 0 \tag{47}$$

$$\frac{\partial V}{\partial \alpha_2} + k\frac{\partial \tau}{\partial \alpha_2} = 0.$$

The problem now is to evaluate the partial derivatives in these equations.

For the function V, from the definition (33), a typical derivative is

$$\frac{\partial V}{\partial \xi} = \Sigma\frac{1}{V_i}\left(x_i\frac{\partial x_i}{\partial \xi} + \frac{1}{2}\frac{\partial h_i^2}{\partial \xi}\right). \tag{48}$$

It is convenient to use the direction cosines of the impulses $\mathbf{V}_i$ associated to the rotating axes along the transfer orbit as seen in Fig. 2. We have by definition

$$S_i = \frac{x_i}{V_i},\ T_i = \frac{y_i}{V_i},\ W_i = \frac{z_i}{V_i}. \tag{49}$$

Then, we have

$$\frac{\partial V}{\partial \xi} = \Sigma\, S_i\frac{\partial x_i}{\partial \xi} + \frac{1}{2V_i}\frac{\partial h_i^2}{\partial \xi}. \tag{50}$$

Based on the eqns (30), (31) and (34) and with the help of the equations in section 2, these derivatives are easy to evaluate. First, with respect to p, we have

$$\frac{\partial x_1}{\partial p} = \frac{1}{2}\sqrt{\frac{\kappa}{p^3}}\left(e\sin f_1 - 2\tan\frac{\Delta}{2}\right)$$
$$\frac{\partial x_2}{\partial p} = -\frac{1}{2}\sqrt{\frac{\kappa}{p^3}}\left(e\sin f_2 + 2\tan\frac{\Delta}{2}\right) \tag{51}$$

and

$$\frac{\partial h_1^2}{\partial p} = \frac{y_1}{r_1}\sqrt{\frac{\kappa}{p}}$$
$$\frac{\partial h_2^2}{\partial p} = -\frac{y_2}{r_2}\sqrt{\frac{\kappa}{p}}. \tag{52}$$

Upon using in eqn (50) we obtain

$$\frac{\partial V}{\partial p} = \frac{1}{2}\sqrt{\frac{\kappa}{p^3}}\left[q_1\left(\frac{S_1\cos\Delta - S_2}{\sin\Delta} + T_1\right) - q_2\left(\frac{S_1 - S_2\cos\Delta}{\sin\Delta} + T_2\right) - (S_1 + S_2)\tan\frac{\Delta}{2}\right] \tag{53}$$

where by definition

$$q_i = \frac{p}{r_i} = (1 + e\cos f_i). \tag{54}$$

Besides these polar equations of the transfer orbit, in the algebraic manipulation we have used the identities

$$e\sin f_1 = \frac{q_1\cos\Delta - q_2}{\sin\Delta} + \tan\frac{\Delta}{2}$$
$$e\sin f_2 = \frac{q_1 - q_2\cos\Delta}{\sin\Delta} - \tan\frac{\Delta}{2}. \tag{55}$$

For the derivatives with respect to α_1, we have

$$\frac{\partial x_1}{\partial \alpha_1} = \sqrt{\frac{\kappa}{p}}\left[\frac{e\sin f_2\cos\gamma_1}{\sin\Delta} - \frac{e_1}{p_1}p\sin\alpha_1\cot\Delta - e_1\sqrt{\frac{p}{p_1}}\cos\alpha_1\right] \tag{56}$$
$$\frac{\partial x_2}{\partial \alpha_1} = -\frac{1}{\sin\Delta}\sqrt{\frac{\kappa}{p}}\left[e\sin f_1\cos\gamma_1 - \frac{e_1}{p_1}p\sin\alpha_1\right]$$

and

$$\frac{\partial h_1^2}{\partial \alpha_1} = 2\sqrt{\frac{\kappa}{p}}\left[z_1q_1\sin\gamma_1\cot\Delta - \frac{e_1}{p_1}y_1p\sin\alpha_1 - \sqrt{\frac{\kappa}{p}}q_1\left(1 - \sqrt{\frac{p}{p_1}}\cos\gamma_1\right) \times e_1\sin\alpha_1\right] \tag{57}$$
$$\frac{\partial h_2^2}{\partial \alpha_1} = -2\sqrt{\frac{\kappa}{p}}\frac{\sin\gamma_1}{\sin\Delta}q_2z_2.$$

Upon using in eqn (50) there comes

$$\frac{\partial V}{\partial \alpha_1} = \sqrt{\frac{\kappa}{p}}\left[\left(\frac{S_1\cos\Delta - S_2}{\sin\Delta} + T_1\right) \times \left(e\sin f_1\cos\gamma_1 - \frac{e_1}{p_1}p\sin\alpha_1\right) + S_1\left(\sqrt{\frac{p}{p_1}} - \cos\gamma_1\right) + \sin\gamma_1\left(\frac{(W_1 - W_2)}{\sin\Delta}q_2 - W_1\tan\frac{\Delta}{2}\right)\right]. \tag{58}$$

By a similar development we have the derivative of V with respect to α_2

$$\frac{\partial V}{\partial \alpha_2} = -\sqrt{\frac{\kappa}{p}}\left[\left(\frac{S_1 - S_2\cos\Delta}{\sin\Delta} + T_2\right) \times \left(e\sin f_2\cos\gamma_2 - \frac{e_2}{p_2}p\sin\alpha_2\right) + S_2\left(\sqrt{\frac{p}{p_2}} - \cos\gamma_2\right) + \sin\gamma_2\left(\frac{(W_2 - W_1)}{\sin\Delta}q_1 - W_2\tan\frac{\Delta}{2}\right)\right]. \quad (59)$$

To express these derivatives in terms of the direction cosines such as in eqn (53) we shall use the following relations for simplification. First, from eqns (30) and (31) we have

$$\frac{T_1}{W_1}\sin\gamma_1 = \sqrt{\frac{p}{p_1}} - \cos\gamma_1$$
$$\frac{T_2}{W_2}\sin\gamma_2 = -\sqrt{\frac{p}{p_2}} + \cos\gamma_2. \quad (60)$$

Next, we have

$$\frac{S_1}{W_1}q_1\sin\gamma_1 = \sqrt{\frac{p}{p_1}}e\sin f_1 - \frac{e_1}{p_1}p\sin\alpha_1$$
$$\frac{S_2}{W_2}q_2\sin\gamma_2 = -\sqrt{\frac{p}{p_2}}e\sin f_2 + \frac{e_2}{p_2}p\sin\alpha_2. \quad (61)$$

Combining these equations, we obtain

$$e\sin f_1\cos\gamma_1 - \frac{e_1}{p_1}p\sin\alpha_1 = \frac{\sin\gamma_1}{W_1}(S_1q_1 - T_1e\sin f_1)$$
$$e\sin f_2\cos\gamma_2 - \frac{e_2}{p_2}p\sin\alpha_2 = -\frac{\sin\gamma_2}{W_2}(S_2q_2 - T_2e\sin f_2). \quad (62)$$

Finally, we have

$$\frac{\sin\gamma_1}{W_1} = \frac{r_1V_1}{\sqrt{\kappa p_1}}, \quad \frac{\sin\gamma_2}{W_2} = \frac{r_2V_2}{\sqrt{\kappa p_2}}. \quad (63)$$

Using eqns (60), (62) and (63) in eqns (58) and (59) and next eliminating $e\sin f_1$ and $e\sin f_2$ by means of eqns (55), we have the final results

$$\frac{\partial V}{\partial \alpha_1} = \frac{r_1V_1}{\sqrt{pp_1}}\left\{X_1\left[\frac{(S_1\sin\Delta - T_1\cos\Delta)}{\sin\Delta}q_1 + \frac{T_1}{\sin\Delta}q_2 - T_1\tan\frac{\Delta}{2}\right] + S_1T_1 + W_1\left[\frac{(W_1 - W_2)}{\sin\Delta}q_2 - W_1\tan\frac{\Delta}{2}\right]\right\} \quad (64)$$

$$\frac{\partial V}{\partial \alpha_2} = \frac{r_2V_2}{\sqrt{pp_2}}\left\{X_2\left[\frac{(S_2\sin\Delta + T_2\cos\Delta)}{\sin\Delta}q_2 - \frac{T_2}{\sin\Delta}q_1 + T_2\tan\frac{\Delta}{2}\right] + S_2T_2 - W_2\left[\frac{(W_2 - W_1)}{\sin\Delta}q_1 - W_2\tan\frac{\Delta}{2}\right]\right\}. \quad (65)$$

In these expressions, by definition

$$X_1 = \frac{S_1\cos\Delta - S_2}{\sin\Delta} + T_1, \quad X_2 = \frac{S_1 - S_2\cos\Delta}{\sin\Delta} + T_2. \quad (66)$$

Then, we can write eqn (53)

$$\frac{\partial V}{\partial p} = \frac{1}{2}\sqrt{\frac{\kappa}{p_i}}\left[q_1X_1 - q_2X_2 - (S_1 + S_2)\tan\frac{\Delta}{2}\right]. \quad (67)$$

For the function τ, from the definition (39), a typical derivative is

$$\frac{\partial \tau}{\partial \xi} = \frac{1}{\sqrt{\kappa}}\left[\frac{\partial}{\partial \xi}(a^{3/2}M_2) - \frac{\partial}{\partial \xi}(a^{3/2}M_1) + 2\pi N\frac{\partial}{\partial \xi}(a^{3/2})\right]. \quad (68)$$

Hence, we shall evaluate the derivatives of $a^{3/2}$ and M. By Kepler's equation (38), M is a function of e and f and hence, based on Table 1, is a function of all three independent variables α_1, α_2 and p. First, we have from eqns (37) and (38)

$$\frac{\partial M}{\partial \xi} = (1 - e\cos E)\frac{\partial E}{\partial \xi} - \sin E\frac{\partial e}{\partial \xi} = \frac{(1 - e^2)}{q}\frac{\partial E}{\partial \xi} - \frac{\sqrt{1 - e^2}}{q}\sin f\frac{\partial e}{\partial \xi}. \quad (69)$$

Next by taking the derivative of the second eqn (37) using the first equation, we have

$$\frac{\partial E}{\partial \xi} = \frac{\sqrt{1 - e^2}}{q}\frac{\partial f}{\partial \xi} - \frac{\sin f}{\sqrt{1 - e^2}\,q}\frac{\partial e}{\partial \xi}. \quad (70)$$

By substituting into eqn (69), we have the derivative of M

$$\frac{\partial M}{\partial \xi}=\frac{(1-e^2)^{3/2}}{q^2}\frac{\partial f}{\partial \xi}-\frac{\sqrt{1-e^2}\,(q+1)}{q^2}\sin f\,\frac{\partial e}{\partial \xi} \quad (71)$$

where ξ is any one of the three independent variables.

For the semi-major axis "a" as given in eqn (36), we first evaluate its derivative with respect to the variables α_i, through the eccentricity e. In general

$$\frac{\partial}{\partial \alpha}a^{3/2}=\frac{3\,ep^{3/2}}{(1-e^2)^{5/2}}\frac{\partial e}{\partial \alpha}. \quad (72)$$

Hence

$$\begin{aligned}\frac{\partial}{\partial \alpha}(a^{3/2}M)&=a^{3/2}\frac{\partial M}{\partial \alpha}+M\frac{\partial a^{3/2}}{\partial \alpha}\\&=\frac{p^{3/2}}{q^2}\left[\frac{\partial f}{\partial \alpha}-\frac{(q+1)}{(1-e^2)}\sin f\,\frac{\partial e}{\partial \alpha}\right]\\&\quad+\frac{3e\,Mp^{3/2}}{(1-e^2)^{5/2}}\frac{\partial e}{\partial \alpha}.\end{aligned} \quad (73)$$

If the derivatives (72) and (73) are substituted into eqn (68) we obtain the derivative of the time function τ with respect to α in the form

$$\begin{aligned}\frac{\partial \tau}{\partial \alpha}=\sqrt{\frac{p^3}{\kappa}}&\left\{\frac{1}{q_2^2}\left[\frac{\partial f_2}{\partial \alpha}-\frac{1}{e}\cot f_2\,\frac{\partial e}{\partial \alpha}\right]-\frac{1}{q_1^2}\right.\\&\times\left[\frac{\partial f_1}{\partial \alpha}-\frac{1}{e}\cot f_1\,\frac{\partial e}{\partial \alpha}\right]\\&\left.+\frac{1}{e}Y\sin \Delta\,\frac{\partial e}{\partial \alpha}\right\}\end{aligned} \quad (74)$$

where, by definition

$$\begin{aligned}Y=\frac{1}{(1-e^2)\sin \Delta}&\left[3e^2\tau\sqrt{\frac{\kappa}{p^3}}\right.\\&-2e\left(\frac{1}{q_2\sin f_2}-\frac{1}{q_1\sin f_1}\right)\\&\left.+\cot f_2-\cot f_1\right].\end{aligned} \quad (75)$$

When using $\alpha=\alpha_1$ in eqn (74), with the derivatives already calculated in Section 2, the first term vanishes and there is a simplification in the second term which leads to

$$\frac{\partial \tau}{\partial \alpha_1}=\frac{1}{e}\sqrt{\frac{p^3}{\kappa}}\left[-\frac{r_1^2e_1\sin \alpha_1}{pp_1\sin f_1}+Y\sin \Delta\,\frac{\partial e}{\partial \alpha_1}\right]. \quad (76)$$

With $\alpha=\alpha_2$, the second term in eqn (74) vanishes and there is a simplification in the first term which leads to

$$\frac{\partial \tau}{\partial \alpha_2}=\frac{1}{e}\sqrt{\frac{p^3}{\kappa}}\left[\frac{r_2^2e_2\sin \alpha_2}{pp_2\sin f_2}+Y\sin \Delta\,\frac{\partial e}{\partial \alpha_2}\right]. \quad (77)$$

Finally, using eqns (23) for the derivatives $\partial e/\partial \alpha_1$ and $\partial e/\partial \alpha_2$, we obtain with the aid of eqns (62) and (63)

$$\begin{aligned}\frac{\partial \tau}{\partial \alpha_1}=\frac{r_1V_1}{e\kappa}\sqrt{\frac{p^3}{p_1}}&\left[-\frac{W_1r_1e_1\sin \alpha_1}{q_1p_1\sin f_1\sin \gamma_1}\right.\\&\left.+Y\sin f_2(S_1q_1-T_1e\sin f_1)\right]\end{aligned} \quad (78)$$

and

$$\begin{aligned}\frac{\partial \tau}{\partial \alpha_2}=\frac{r_2V_2}{e\kappa}\sqrt{\frac{p^3}{p_2}}&\left[\frac{W_2r_2e_2\sin \alpha_2}{q_2p_2\sin f_2\sin \gamma_2}\right.\\&\left.+Y\sin f_1(S_2q_2-T_2e\sin f_2)\right].\end{aligned} \quad (79)$$

For the derivative of $a^{3/2}$ with respect to p, we now have

$$\frac{\partial a^{3/2}}{\partial p}=\frac{3\sqrt{p}\,A}{2(1-e^2)^{5/2}} \quad (80)$$

where, by definition

$$\begin{aligned}A&=(1-e^2)+2ep\,\frac{\partial e}{\partial p}\\&=1+e^2+\frac{2e(\sin f_2-\sin f_1)}{\sin \Delta}.\end{aligned} \quad (81)$$

Combining the last two equations with eqn (71), we now have

$$\begin{aligned}\frac{\partial}{\partial p}(a^{3/2}M)=&\frac{p^{3/2}}{q^2}\left[\frac{\partial f}{\partial p}+\frac{(q+1)}{2\,ep}\sin f\right]\\&+\frac{\sqrt{p}\,A}{2e(1-e^2)}\left[\frac{3eM}{(1-e^2)^{3/2}}\right.\\&\left.-\frac{(q+1)}{q^2}\sin f\right].\end{aligned} \quad (82)$$

Using eqn (19) for $\partial f/\partial p$ and the second form in eqn (81) for A, this equation can be put in the form

$$\begin{aligned}\frac{\partial}{\partial p}(a^{3/2}M)=&\frac{\sqrt{p}}{eq^2}\left[\frac{1}{\sin f}-\cot f\,\frac{(\sin f_2-\sin f_1)}{\sin \Delta}\right.\\&\left.+\frac{(\cos f_2-\cos f_1)}{\sin \Delta}\right]-\frac{\sqrt{p}}{2e^2}\cot f\\&+\frac{\sqrt{p}\,A}{2e(1-e^2)}\left[\frac{3eM}{(1-e^2)^{3/2}}\right.\\&\left.+\frac{p\cos f-2er}{pe\sin f}\right].\end{aligned} \quad (83)$$

Using $f=f_1$ or $f=f_2$ in this equation, the first term vanishes identically. Inserting the remaining terms with the appropriate subscript for f and r in eqn (68) for $\xi=p$, we have the final form

$$\frac{\partial\tau}{\partial p}=\frac{1}{2e^2}\sqrt{\frac{p}{\kappa}}\{\cot f_1-\cot f_2+Y[(1+e^2)\sin\Delta+2e(\sin f_2-\sin f_1)]\}. \tag{84}$$

For a hyperbolic transfer orbit the time of flight is given by eqn (39) with $N=0$, while the mean anomaly M is defined by eqn (41) with the hyperbolic anomaly H related to the true anomaly f by eqns (42). The same type of derivation leads to identical formulas for the derivatives of τ, with the difference that the function Y is now defined as

$$Y=\frac{1}{(e^2-1)\sin\Delta}\left[-3e^2\tau\sqrt{\frac{\kappa}{p^3}}+2e\left(\frac{1}{q_2\sin f_2}-\frac{1}{q_1\sin f_1}\right)+\cot f_1-\cot f_2\right]. \tag{85}$$

In fact, this is also identical to the previous formula (75).

5. PROBLEM SYNTHESIS

In summary, we have derived the necessary conditions for solving the problem of fixed-time optimal two-impulse transfer between non-coplanar elliptical orbits. The solution depends on three variables namely the semi-latus rectum p of the transfer orbit and the true anomalies α_1 and α_2 defining the locations of the impulses on the initial and the final orbit, respectively. It has been shown that all the elements of the transfer orbit, the transfer time, the characteristic velocities and the optimal directions of the impulses can be expressed explicitly in terms of these variables. The solution is obtained by solving the three optimal conditions (47) and, depending on if we have a minimum-time or a minimum-fuel problem, either eqn (43) or (44) for the four unknowns k, p, α_1 and α_2. By eliminating the Lagrange's multiplier k from eqns (47), we have the set of two equations

$$\frac{\partial V}{\partial p}\frac{\partial\tau}{\partial\alpha_1}-\frac{\partial V}{\partial\alpha_1}\frac{\partial\tau}{\partial p}=0$$
$$\frac{\partial V}{\partial p}\frac{\partial\tau}{\partial\alpha_2}-\frac{\partial V}{\partial\alpha_2}\frac{\partial\tau}{\partial p}=0. \tag{86}$$

These two equations with either eqn (43) or (44) constitute a set of three equations for the three unknowns p, α_1 and α_2. If the partial derivatives, as have been evaluated in Section 4, are substituted in the last two equations, after simplification we have explicitly the final optimal conditions

$$(X_1+YZe\sin f_2)(S_1q_1-T_1e\sin f_1)+S_1T_1+W_1\left[\frac{(W_1-W_2)}{\sin\Delta}q_2-W_1\tan\frac{\Delta}{2}\right]-\frac{W_1Ze\,r_1e_1\sin\alpha_1}{q_1p_1\sin f_1\sin\gamma_1}=0 \tag{87}$$

and

$$(X_2+YZe\sin f_1)(S_2q_2-T_2e\sin f_2)+S_2T_2-W_2\left[\frac{(W_2-W_1)}{\sin\Delta}q_1-W_2\tan\frac{\Delta}{2}\right]+\frac{W_2Ze\,r_2e_2\sin\alpha_2}{q_2p_2\sin f_2\sin\gamma_2}=0 \tag{88}$$

where by definition

$$Z=\frac{q_2X_2-q_1X_1+(S_1+S_2)\tan\frac{\Delta}{2}}{\cot f_1-\cot f_2+Y[(1+e^2)\sin\Delta+2e(\sin f_2-\sin f_1)]} \tag{89}$$

with Y given by eqn (75).

For elliptic transfer, the time eqn (39) has provision for N complete revolutions along the transfer orbit in the case of long duration for the transfer. For a direct transfer, such as the case of minimum-time, we have of course $N=0$. In the actual computation, the main difficulty is the evaluation of the arguments of the trigonometric functions involved. We can always choose the unit time and the unit length such that $\kappa=1$ and $p_1=1$. Then the unit velocity is the circular speed of distance p_1, that is $V_c=\sqrt{\kappa/p_1}$.

The transfer time has been defined as the time between the impulses. The problem formulation is general and the only restriction is that the transfer is accomplished by using two impulses. This is a realistic assumption since in the problem where the prescribed time is short a two-impulse transfer is necessary to accommodate this constraint.

To show the general character of this study, we consider in this section some special cases of interest.

Coplanar transfer

In this case, we have identically $W_1=W_2=0$. But, before using this limit in eqns (87) and (88), we must use eqns (63) to replace in the last term of each equation $W_i/\sin\gamma_i$ by $\sqrt{\kappa p_i}/r_iV_i$. Then, we have the resulting equations

$$(X_1+YZe\sin f_2)(S_1q_1-T_1e\sin f_1)+S_1T_1-\sqrt{\frac{\kappa}{p_1}}\frac{Ze\,e_1\sin\alpha_1}{q_1V_1\sin f_1}=0 \tag{90}$$

and

$$(X_2 + YZe\sin f_1)(S_2q_2 - T_2e\sin f_2) + S_2T_2 + \sqrt{\frac{\kappa}{p_2}}\frac{Ze\,e_2\sin\alpha_2}{q_2V_2\sin f_2} = 0. \quad (91)$$

In the planar case, the terminal orbits are defined by the orbital elements p_1, e_1, p_2, e_2 and the angle ω from the initial pericenter to the final pericenter. With $\gamma_1 = \gamma_2 = \phi = 0$, the eqns (10)–(14) become trivial, while eqn (9) is replaced by

$$\Delta = \omega + \alpha_2 - \alpha_1 \quad (92)$$

It is easy to verify that the two necessary conditions (90) and (91) and either eqn (43) or (44) can be ultimately expressed in terms of the chosen independent variables p, α_1 and α_2.

The special case of minimum-fuel, fixed-time transfer between coplanar circular orbits has been discussed in Ref. [2] based on the remarkable Lambert's theorem for the time equation. In the present formulation, the necessary conditions are obtained by putting $e_1 = e_2 = 0$ in eqns (90) and (91). We have then

$$(X_1 + YZe\sin f_2)(S_1q_1 - T_1e\sin f_1) + S_1T_1 = 0$$
$$(X_2 + YZe\sin f_1)(S_2q_2 - T_2e\sin f_2) + S_2T_2 = 0. \quad (93)$$

By eliminating YZe between these two equations we have an identity so that only one of the two equations applies. This equation, together with eqn (44) constitute a system of two equations for two unknowns for solving the problem of minimum-fuel fixed-time transfer between coplanar circular orbits. The reason for only two independent variables is that we now have rotational symmetry in the plane. The first impulse can be initiated anywhere in the initial orbit. In the present formulation, if we choose f_1 and f_2 as independent variables, then since the radii r_1 and r_2 are given, p and e are functions of these variables as seen from eqns (5) and (6).

In the Lambert's formulation, it is known that the time of flight between impulses can can be expressed in terms of two invariants g and ϵ defined as[2]:

$$2g = E_2 - E_1 \text{ or } 2g = H_2 - H_1 \quad (94)$$

and

$$\epsilon = \frac{2\sqrt{r_1r_2}}{(r_1 + r_2)}\cos\frac{\Delta}{2} = \frac{2\sqrt{n}}{(n+1)}\cos\frac{\Delta}{2} \quad (95)$$

where

$$n = \frac{r_2}{r_1} \quad (96)$$

is the ratio of the radii. Then, by defining the normalized time of flight

$$\bar{\tau} = \frac{\tau}{\pi}\sqrt{\frac{2\kappa}{(r_1 + r_2)^3}} \quad (97)$$

we have for elliptic transfer

$$\bar{\tau} = \frac{G^{3/2}}{2\pi}\left[\frac{2g - \sin 2g + 2\pi N}{\sin^3 g} + \frac{2\epsilon}{G}\right] \quad (98)$$

where

$$G = 1 - \epsilon\cos g. \quad (99)$$

The corresponding equations for hyperbolic transfer are

$$\bar{\tau} = \frac{G^{3/2}}{2\pi}\left[\frac{\sinh 2g - 2g}{\sinh^3 g} + \frac{2\epsilon}{G}\right] \quad (100)$$

and

$$G = 1 - \epsilon\cosh g. \quad (101)$$

The semi-major axis, which is an invariant in the sense of Lambert is given by

$$a = \left(\frac{r_1 + r_2}{2}\right)\frac{G}{\sin^2 g}$$
$$a = \left(\frac{r_1 + r_2}{2}\right)\frac{G}{\sinh^2 g}. \quad (102)$$

Since the constraining relation (44), written as

$$\bar{\tau} - \bar{\tau}_0 = 0 \quad (103)$$

is now a function of ϵ and g, it is expected that the necessary condition, namely one of the two eqns (93) can be expressed in terms of these invariants. To prove this, we first observe that for $e_i = 0$

$$S_iq_i - T_ie\sin f_i = \sqrt{q_i}\,S_i. \quad (104)$$

Using these relations in system (93) for simplification, we have

$$YZe\sin f_2 = -X_1 - \frac{T_1}{\sqrt{q_1}}$$
$$YZe\sin f_1 = -X_2 - \frac{T_2}{\sqrt{q_2}}. \quad (105)$$

Instead of using one of these equations, we shall combine them to retain the symmetry of the resulting formula. By multiplying the first equation by $e\cos f_1$ and the second equation by $-e\cos f_2$ and adding, we obtain

$$e^2YZ\sin\Delta = S_2e\sin f_2 - S_1e\sin f_1 + \left(1 + \frac{1}{\sqrt{q_2}}\right)T_2e\cos f_2 - \left(1 + \frac{1}{\sqrt{q_1}}\right)T_1e\cos f_1. \quad (106)$$

Using the definitions (49) of the direction cosines S_i and T_i, the polar eqns (54) and the identities (55) for $e \cos f_i$ and $e \sin f_i$, the right-hand-side of this equation is much simplified and we obtain the resulting equation

$$YZ \sin \Delta = -\sqrt{\frac{\kappa}{p}}\left(\frac{1}{V_1}+\frac{1}{V_2}\right). \tag{107}$$

For the function Z, as defined in eqn (89), we have for the numerator

$$\begin{aligned} q_2X_2 - q_1X_1 + (S_1+S_2)\tan\frac{\Delta}{2} &= q_2T_2 - q_1T_1 + S_2\, e \sin f_2 - S_1\, e \sin f_1 \\ &+ 2(S_1+S_2)\tan\frac{\Delta}{2} = \sqrt{\frac{\kappa}{p}}\left\{\frac{q_1^{3/2}}{V_1}+\frac{q_2^{3/2}}{V_2}\right. \\ &+ \left[(1-e^2) - \frac{2(1-\cos\Delta)}{\sin^2\Delta}(q_1+q_2)\right. \\ &\left.\left.+ 2\tan^2\frac{\Delta}{2}\right]\left(\frac{1}{V_1}+\frac{1}{V_2}\right)\right\}. \end{aligned} \tag{108}$$

For the denominator of the function Z, the coefficient of Y can be put in the form

$$\begin{aligned} (1+e^2)\sin\Delta + 2e(\sin f_2 - \sin f_1) &= 2\sin\Delta - (1-e^2)\sin\Delta - 4\tan\frac{\Delta}{2} \\ &+ \frac{2(1-\cos\Delta)}{\sin\Delta}(q_1+q_2). \end{aligned} \tag{109}$$

Then, upon substituting into eqn (107) and simplifying, we have

$$\begin{aligned} \frac{V_1+V_2}{q_2^{3/2}V_1 + q_1^{3/2}V_2} &= \frac{Y\sin\Delta}{\cot f_2 - \cot f_1} \\ &= -Y\sin f_1 \sin f_2. \end{aligned} \tag{110}$$

This is the final optimal equation for use with eqn (44) in solving the problem of minimum-fuel, fixed-time transfer between coplanar circular orbits if the variables selected are the true anomalies f_1 and f_2 of the impulses on the transfer orbit. In this case, the magnitudes of the impulses are

$$V_i = \sqrt{\frac{\kappa}{p}}\,[3q_i - 2q_i\sqrt{q_i} - (1-e^2)]^{1/2} \tag{111}$$

which are functions of f_1 and f_2 through the elements p and e. If Lambert's invariants ϵ and g are considered, we first write this equation for elliptic orbit

$$V_i^2 = \frac{\kappa}{r_i}\left[3 - \frac{r_i}{a} - 2\sqrt{q_i}\right]. \tag{112}$$

Since $q_i = p/r_i$ and, as shown in [3]:

$$p = a(1-e^2) = \frac{r_1 r_2 \sin^2\frac{\Delta}{2}}{a\sin^2 g} = \frac{2r_1r_2}{(r_1+r_2)G}\sin^2\frac{\Delta}{2} \tag{113}$$

with r_i/a given by eqn (102) and $\sin^2(\Delta/2)$ evaluated from eqn (95), we obtain the expressions for the impulses

$$V_i^2 = \frac{\kappa}{r_i}\left[3 - \frac{2(r_i/r_1)\sin^2 g}{(n+1)G} - 2\sqrt{\frac{4n-(n+1)^2\epsilon^2}{2(r_i/r_1)(n+1)G}}\right]. \tag{114}$$

For hyperbolic orbit, we change $\sin^2 g$ into $-\sinh^2 g$ while using definition (101) for G.

Finally, we now show that the optimal condition (110) can also be expressed in terms of ϵ and g. For this purpose, we write the r.h.s. of the equation

$$\begin{aligned} -Y\sin f_1 \sin f_2 = &\\ -\frac{1}{(1-e^2)\sin\Delta}&\left[3\tau\sqrt{\frac{\kappa}{p^3}}\,e^2\sin f_1\sin f_2\right. \\ &\left.-\frac{2e\sin f_1}{q_2} + \frac{2e\sin f_2}{q_1} - \sin\Delta\right]. \end{aligned} \tag{115}$$

Then, from eqn (113)

$$q_1 = \frac{2n}{(n+1)G}\sin^2\frac{\Delta}{2} = nq_2 \tag{116}$$

and with the aid of eqn (102)

$$(1-e^2) = \frac{4n\sin^2 g}{(n+1)^2G^2}\sin^2\frac{\Delta}{2}. \tag{117}$$

Since $e \sin f_1$ and $e \sin f_2$ can be expressed in terms of q_1 and q_2 by the identities (55), it is clear that expressions (115) can be expressed in terms of the invariants ϵ and g. After using the appropriate relations in eqn (110), we have the optimal relations expressed in terms of ϵ and g

$$\sqrt{\frac{8n^3}{n+1}}\,\frac{V_1+V_2}{V_1+\sqrt{n^3}\,V_2} = \frac{G^{1/2}F_1 + 3\pi\bar{\tau}\,F_2}{\epsilon\sin^2 g\sqrt{4n-(n+1)^2\epsilon^2}} \tag{118}$$

where, by definition

$$\begin{aligned} F_1 &= (n+1)^2\epsilon(4-\epsilon^2) - 4n(\epsilon + 2\cos g) \\ F_2 &= (n+1)^2\epsilon(\epsilon - 2\cos g) + 4n\cos^2 g. \end{aligned} \tag{119}$$

For hyperbolic transfer, we simply change $\cos g$ into $\cosh g$ and $\sin^2 g$ into $-\sinh^2 g$. The optimal relations which have been deduced as special cases of the present theory are in perfect agreement with the equations given in [2].

Free time transfer

It is of interest to consider this important special case. The three necessary optimal conditions are obtained by taking $k = 0$ in system (47). We have

$$q_1 X_1 - q_2 X_2 - (S_1 + S_2)\tan\frac{\Delta}{2} = 0 \qquad (120)$$

$$X_1\left[\frac{(S_1\sin\Delta - T_1\cos\Delta)}{\sin\Delta}q_1 + \frac{T_1}{\sin\Delta}q_2 - T_1\tan\frac{\Delta}{2}\right] + S_1T_1 + W_1\left[\frac{(W_1 - W_2)}{\sin\Delta}q_2 - W_1\tan\frac{\Delta}{2}\right] = 0 \qquad (121)$$

$$X_2\left[\frac{(S_2\sin\Delta + T_2\cos\Delta)}{\sin\Delta}q_2 - \frac{T_2}{\sin\Delta}q_1 + T_2\tan\frac{\Delta}{2}\right] + S_2T_2 - W_2\left[\frac{(W_2 - W_1)}{\sin\Delta}q_1 - W_2\tan\frac{\Delta}{2}\right] = 0. \qquad (122)$$

These three equations can be solved for the three unknowns p, α_1 and α_2. We notice that the equations are linear in the variables q_1 and q_2. Upon solving the first and the last equations we obtain

$$q_1 = \frac{(1-\cos\Delta)[1 - 2S_2^2 - S_1S_2 + \theta T_2(S_1 + S_2)]}{1 + (S_1T_2 - S_2T_1)\sin\Delta - (S_1S_2 + T_1T_2)\cos\Delta - W_1W_2}. \qquad (123)$$

Similarly, by combining the first two equations, we have

$$q_2 = \frac{(1-\cos\Delta)[1 - 2S_1^2 - S_1S_2 - \theta T_1(S_1 + S_2)]}{1 + (S_1T_2 - S_2T_1)\sin\Delta - (S_1S_2 + T_1T_2)\cos\Delta - W_1W_2} \qquad (124)$$

where, by definition,

$$\theta = \tan\frac{\Delta}{2}. \qquad (125)$$

Upon substituting into eqn (120), we have

$$\begin{aligned}&\theta^3(T_2 - T_1)(S_1 + S_2)^2\\ &+ \theta^2(S_1 + S_2)[3 - 2S_1^2 - 2S_2^2 - S_1S_2 - 3T_1T_2 - W_1W_2]\\ &+ \theta[2T_2 - 2T_1 - T_1S_1^2 + T_2S_2^2 + 3T_1S_2^2 - 3T_2S_1^2]\\ &+ (S_1 + S_2)[1 - 2S_1^2 - 2S_2^2 + 3S_1S_2 - T_1T_2 - W_1W_2] = 0.\end{aligned} \qquad (126)$$

Equations (123), (124) and (126) are precisely the remarkable switching relations first discovered by Marchal [4]. The direction cosines of the impulses in these relations satisfy the identities

$$S_i^2 + T_i^2 + W_i^2 = 1. \qquad (127)$$

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