

Multi-stage production with variable lot sizes and transportation of partial lots

Z. DREZNER

School of Management, University of Michigan - Dearborn, Dearborn, MI 48128, U.S.A.

A.Z. SZENDROVITS and G.O. WESOLOWSKY

Faculty of Business, McMaster University, Hamilton, Ontario, Canada L8S 4M4

Received November 1982

Revised March 1983

This paper describes a model for a multi-stage production/inventory system where lots may be of different sizes. In addition, either completed lots or partial lots, called batches, may be transported to succeeding stages. The model incorporates constraints on lot and batch-sizes and thus provides a rather comprehensive set of possibilities for organizing a production/inventory system. A heuristic solution procedure is developed and is shown to be 'close to optimal' by bounding.

1. Introduction

When the rate of continuous demand is smaller than the manufacturing rate for a product, intermittent manufacturing in economic lot-sizes is usually justified. Economic lot-sizes are also important when the continuous demand of an assembly line is fed by a part which is manufactured intermittently. The single-stage lot-size model is frequently misused in multi-stage manufacturing because it ignores the work-in-process inventory and overstates the optimal lot-size.

Lot-size models impose a constraint on the scheduling of production facilities which are shared by several products because the manufacturing of lots must be scheduled with priority. Facility scheduling is a lesser problem in single-stage production models than in multi-stage models. Even in the multi-stage case, if a relatively small portion of all products have scheduling priority, there is ample room for manipulating the schedule of the rest of the product line. The key problem is to identify those products which constitute a substantial part of the work-in-process and represent a relatively small portion of the total production capacity. If such products are scheduled according to an appropriate lot-size model, the process inventory and the total inventory cost can be reduced considerably.

The state of the art of production/inventory research is well summarized in a recent comprehensive survey of the literature [10]. Since the mid 70's multi-stage production/inventory models have gained increasing attention. The terminology used in the literature varies substantially. In this paper we call a quantity produced with one set-up at a stage a 'lot' and a portion of a lot transported to the next stage a 'batch'. Deterministic lot-size models for multi-stage serial and assembly systems represent a variety of process organizations. Two classes of these models, both based on an infinite time horizon, can be distinguished in the literature. One class, which we call 'variable lot-size models' [1,2,5,9], allows different and non-increasing lot-sizes across stages. Only complete lots are transported to the next stage and the lot-size of a stage is an integer multiple of the lot-size that follows it. One variable lot-size model [4] does not have integrality restrictions; it is analytically tractable because any portion of a lot can be transported

This research was supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

North-Holland

European Journal of Operational Research 17 (1984) 227-237

0377-2217/84/\$3.00 © 1984, Elsevier Science Publishers B.V. (North-Holland)

to the next stage at zero cost. Another class, which we call 'batch shipment models' [6,7,8], has uniform lot-sizes at all stages but allows portions of a lot to be transported to the next stage in equal-sized batches at some cost per batch.

In this paper, we present a lot-size model for a single product that is manufactured in a serial system through a large number of stages. The model and its assumptions are briefly summarized as follows:

(a) Lot-sizes can be variable and subject to constraints, but must be integer multiples of the lot-size at the following stage. Starting a new lot involves a fixed set-up cost.

(b) Either complete lots or equal-sized batches can be shipped to the next stage at a transportation cost for each shipment. The unit cost of transportation is related to the load capacity of the transport equipment.

(c) Lot sizes and batch-sizes are not restricted to integers. Set-up times of lots and transportation times of batches are assumed to be negligible.

(d) Deterministic (constant) production and demand rates are assumed over an infinite horizon. No backlogging (deliberate shortage) of inventory is permitted in the system.

(e) Linear inventory-holding costs are assumed at all stages. The cost of holding one unit of process inventory is related to the stage which has been completed and cannot decrease at the following stage (due to the added value of the product).

Our model combines the features of the variable lot-size and the batch shipment models and uses the conventional assumptions in the literature. The use of variable lot-sizes balances multiple set-up costs at some stages against the decreased size and cost of the process inventory. Transporting batches instead of complete lots may result in higher transportation costs. On the other hand, production at subsequent stages might be scheduled with overlap on the same lot to further reduce the size and the cost of the average process inventory. As an additional element of flexibility beyond that in existing models, our model can accommodate constraints on lot-sizes that may result from limited production or storage capacity as well as constraints on batch-sizes that may be caused by limited load-capacity of the transport equipment.

Usually, our model would yield a lower (certainly never higher) cost than either of the variable lot-size or the batch shipment models. Hence, it is an economical substitute for these models whenever they are applicable to a serial production system. This is certainly the case in the automobile and generally in the machine industry where many parts of a complex product involve a series and usually large number of production operations (stages). Part manufacturing is seldom synchronized with the rhythm of the assembly line; therefore, a single part is produced in lots intermittently. While a single part may not be sold directly, the assembly of the product incorporating this part could create a continuous demand over time. This is the reason for the applicability of these lot-size models.

Although the generalizations of previous models that are presented in this paper are straightforward, they add considerable realism to the representation of the process organization. At the same time, they increase substantially the difficulty of solving a traditionally formidable optimization problem.

2. Constructing the cost function

First, we define our notation. The stages in the production system are $i = 1, 2, \dots, n$; the final stage, the one which meets the demand for the finished product, is stage 1. Other symbols are as follows:

- D = demand (consumption) rate of the final product (at stage 1),
- P_i = production rate at stage i (note that $P_i > D$),
- F_i = fixed (set-up) cost per lot at stage i ,
- T_i = transportation cost of one load from stage i to the next,
- g_i = the maximum size of one load (load capacity) transported from the i th stage to the next,
- c_i = unit inventory-holding cost per unit time, at stage i ,
- Q_i = the lot-size at stage i ,
- L_i = the maximum lot size permitted at stage i ,

- $S_i = Q_{i+1}/Q_i$ (note that S_i is required to be integer),
- $b_i =$ the number of batches in the lot at stage i (note that b_i is integer),
- $x_i = Q_i/b_i$, the size of batches in the lot at stage i (note that the sizes of the batches are equal).

All parameters above are greater than zero. Also, it should be noted that

- (i) a non-integer value, A , 'rounded up' to the nearest integer is denoted by $\lceil A \rceil$,
- (ii) the 'rounded-down' value is denoted by $\lfloor A \rfloor$,
- (iii) $\lceil \lceil A \rceil \rceil$ denotes conventional rounding to the nearest integer,
- (iv) an integer rounded is the integer.

For the convenience of our equations we define

$$Q_0 = Q_1, \quad c_{n+1} = 0, \quad P_{n+1} = 0, \quad P_0 = D, \quad S_n = 1.$$

To derive the cost function, we start by examining Fig. 1 which illustrates the production/inventory model of two adjacent stages i and $i - 1$ for the case when $P_i < P_{i-1}$. The upper dashed line with slope P_i shows the uninterrupted production build-up of lot-size Q_i at stage i . From this stage b_i number of equal-sized batches, each of size $x_i = Q_i/b_i$, are transported to the next stage, $i - 1$. The rectangular vertex of the triangle (the sides of which form a step function) immediately below the dashed line represents the point at which each batch is available for production at stage $i - 1$. The other step function with slope P_{i-1} shows the production build-up of Q_{i-1} lot sizes at stage $i - 1$. The dashed line below this step function, the slope of which is D , indicates the cumulative demand which must be matched by the cumulative production at every stage. It can be seen that the inventory at stage $i - 1$ cycles back to zero every Q_{i-1}/D units of time.

Let R_{i-1} be the earliest possible time for the start of production at stage $i - 1$. To find an analytical expression for R_{i-1} we denote the sequence of batches shipped from stage i with $j = 1, 2, \dots, b_i$.

Clearly, we wish to start producing at stage $i - 1$ as soon as possible to minimize the size of the process inventory, but R_{i-1} is constrained by the point where the step functions touch. As can be seen in Fig. 1 this happens when $j + 1 = 6$ batches are completed at stage i during $(j + 1)x_i/P_i$ units of time. During this time $\lfloor jx_i/Q_{i-1} \rfloor = 2$ lots are completed at stage $i - 1$ requiring $\lfloor jx_i/Q_{i-1} \rfloor Q_{i-1}/P_{i-1}$ units of time. These lots

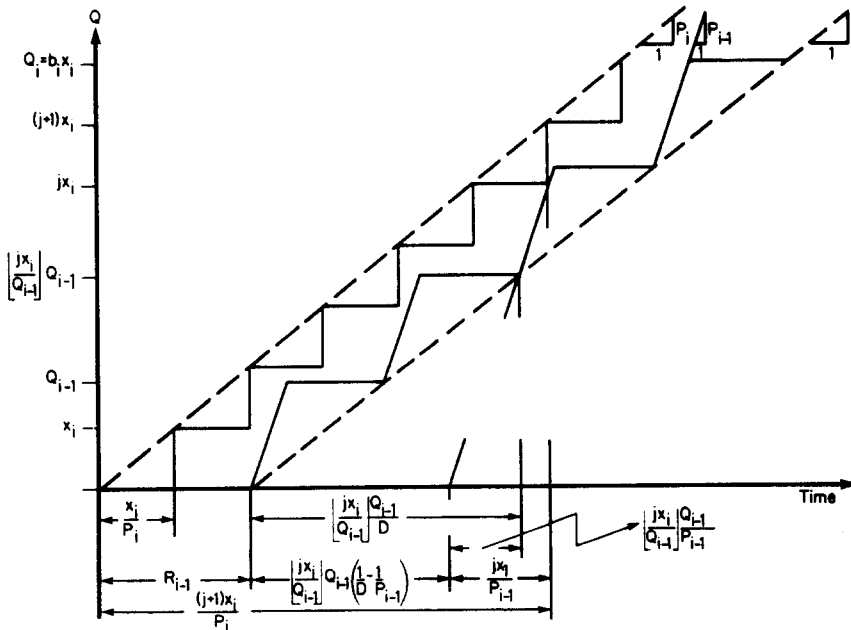


Fig. 1. Build-up of inventory between adjacent stages when $P_i < P_{i-1}$.

satisfy demand during $\lfloor jx_i/Q_{i-1} \rfloor Q_{i-1}/D$ time periods. It is easy to verify that jx_i/P_{i-1} units of time are needed to produce jx_i units at stage $i-1$. From these distances on the time scale we find the following analytical expression:

$$R_{i-1} = (j+1)x_i/P_i - jx_i/P_{i-1} - \lfloor jx_i/Q_{i-1} \rfloor Q_{i-1}(1/D - 1/P_{i-1}).$$

Therefore, as is seen in Fig. 1, to keep stage $i-1$ production supplied it must be true that

$$R_{i-1} + jx_i/P_{i-1} + \lfloor jx_i/Q_{i-1} \rfloor Q_{i-1}(1/D - 1/P_{i-1}) \geq (j+1)x_i/P_i \quad \text{for } 0 \leq j \leq b_i - 1.$$

Of course, this will be true if

$$R_{i-1} = \max_{0 \leq j \leq b_i - 1} \{ (j+1)x_i/P_i - jx_i/P_{i-1} - \lfloor jx_i/Q_{i-1} \rfloor Q_{i-1}(1/D - 1/P_{i-1}) \}.$$

Rearranging the expression for R_{i-1} we obtain

$$R_{i-1} = x_i/P_i + \max_{0 \leq j \leq b_i - 1} \{ \theta(j) \}, \quad (1)$$

where

$$\theta(j) = jx_i(1/P_i - 1/P_{i-1}) - \lfloor jx_i/Q_{i-1} \rfloor Q_{i-1}(1/D - 1/P_{i-1}),$$

$j = 0, 1, \dots, b_i - 1$ and $j = \text{integer}$.

One could illustrate graphically that if $P_i \geq P_{i-1}$, stage $i-1$ can be started as soon as the first batch is completed at stage i , which is after x_i/P_i units of time. It is also easy to verify that expression (1) is also valid for this case. When $P_i \geq P_{i-1}$ the quantity $(1/P_i - 1/P_{i-1})$ is non-positive and $j=0$ provides the maximum in (1); thus

$$R_{i-1} = x_i/P_i \quad \text{for } P_i \geq P_{i-1}.$$

It will be useful to examine the lower bound on R_{i-1} . Naturally, in the case above the lower bound on R_{i-1} is

$$\hat{R}_{i-1} = x_i/P_i \quad \text{for } P_i \geq P_{i-1}.$$

It is also obvious from Fig. 1 that if $P_i < P_{i-1}$ the shortest time for R_{i-1} will occur when the step functions touch during the production of the first lot at stage $i-1$. That is when $j = (\lfloor Q_{i-1}/x_i \rfloor - 1)$. Substituting this for j in (1) we obtain the lower bound on R_{i-1} which is

$$\hat{R}_{i-1} = x_i/P_i + \theta(\lfloor Q_{i-1}/x_i \rfloor - 1) \quad \text{for } P_i < P_{i-1}.$$

Since part of $\theta(j)$ in (1) is $\lfloor jx_i/Q_{i-1} \rfloor$ which becomes $\{(\lfloor Q_{i-1}/x_i \rfloor - 1)x_i/Q_{i-1}\} = 0$, the analytical expression for the lower bound on R_{i-1} is

$$\hat{R}_{i-1} = x_i/P_i + (\lfloor Q_{i-1}/x_i \rfloor - 1)x_i(1/P_i - 1/P_{i-1}) \quad \text{for } P_i < P_{i-1}.$$

The two lower bounds can be combined in one general expression for \hat{R}_{i-1} :

$$\hat{R}_{i-1} = x_i[1/P_i + (l_i - 1)(1/P_i - 1/P_{i-1})] \quad (2)$$

where

$$l_i = \begin{cases} 1 & \text{if } P_i \geq P_{i-1}, \\ \lfloor Q_{i-1}/x_i \rfloor & \text{if } P_i < P_{i-1}. \end{cases}$$

Actually, this rather than the case in Fig. 1 is usual. R_{i-1} is not equal to \hat{R}_{i-1} only if D is 'nearly equal' to P_i .

Having found the analytical expression for R_{i-1} we can determine the inventory at stage i as shown in Fig. 2.

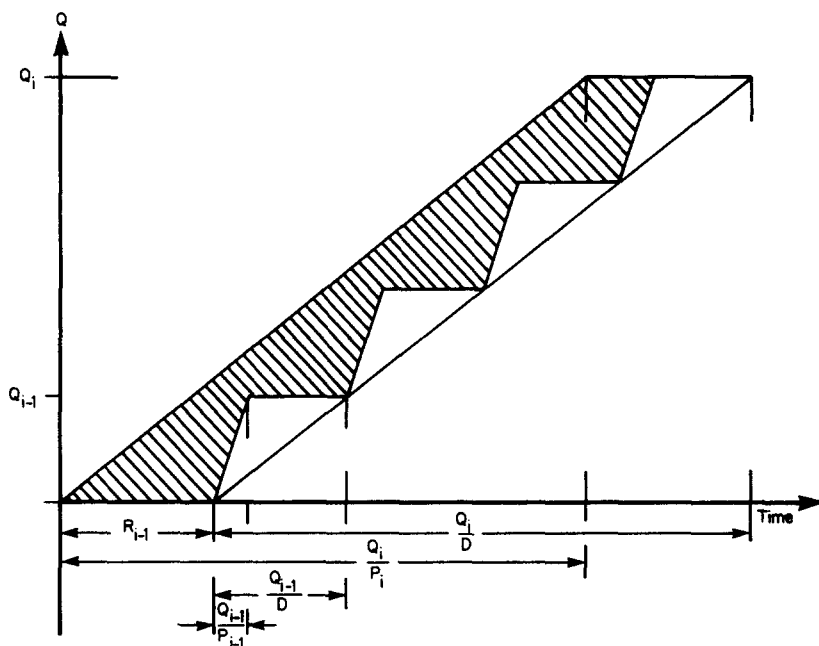


Fig. 2. Time-weighted inventory of stage i .

To find the average inventory-holding cost at stage i , we start by finding the time-weighted inventory (shaded area in Fig. 2). This is done by subtracting triangles from a trapezium. Then, we divide the area by Q_i/D to obtain the average inventory and multiply by c_i to obtain the inventory-holding cost per unit time. Thus C_i , the average inventory-holding cost of stage i is

$$C_i = \frac{1}{2}c_i D [2R_{i-1} + Q_i(1/D - 1/P_i) - Q_{i-1}(1/D - 1/P_{i-1})]. \quad (3)$$

Since $P_0 = D$ the expression for C_i holds for all stages.

It is interesting to consider the possibility of using batch-sizes that are larger than the load capacity of the transport equipment, $x_i > g_i$. Each batch would thus require $\lceil x_i/g_i \rceil$ loads and hence the transportation cost per Q_i lot-size would be $T_i \lceil Q_i/x_i \rceil \lceil x_i/g_i \rceil$. If batch-sizes are limited to $x_i \leq g_i$, the transport cost per Q_i lot-size would be $T_i \lceil Q_i/g_i \rceil$ which is obviously smaller. It also can be verified, by examining Fig. 1, that increasing the number of batches and hence decreasing the batch-size involves a smaller R_{i-1} (earlier start of production at stage $i-1$) and thus saves inventory-holding cost. The constraint $x_i \leq g_i$ allows us to simplify the statement of the cost function.

Recall that F_i is the set-up cost per lot at stage i . The sum of the set-up and transportation costs, $(F_i + b_i T_i)$, divided by Q_i/D gives the average fixed cost per unit time.

The total cost of the system can be obtained by summing the fixed and inventory-holding costs:

$$TC = D \sum_{i=1}^n \left\{ (F_i + b_i T_i)/Q_i + \frac{1}{2}c_i [2R_{i-1} + Q_i(1/D - 1/P_i) - Q_{i-1}(1/D - 1/P_{i-1})] \right\}. \quad (4)$$

Since $c_{n+1} = 0$, we can rearrange (4) and express the total cost in terms of Q_i . Thus, the optimization problem is as follows:

$$\begin{aligned} \text{minimize} \quad & TC = D \sum_{i=1}^n \{ F_i/Q_i + Q_i(1/D - 1/P_i)(c_i - c_{i+1})/2 + c_i R_{i-1} + T_i/x_i \}, \\ \text{subject to} \quad & x_i \leq g_i && \text{for } i = 1, \dots, n, \\ & Q_i \leq L_i && \text{for } i = 1, \dots, n, \\ & Q_{i+1}/Q_i = S_i = \text{positive integer} && \text{for } i = 1, \dots, n-1, \\ & Q_i/x_i = b_i = \text{positive integer} && \text{for } i = 1, \dots, n. \end{aligned} \quad (5)$$

3. Bounding

In order to obtain a lower bound on the cost of the optimal solution to problem (5) we relax the integrality constraints on S_i , l_i and b_i , replace R_{i-1} with \hat{R}_{i-1} and solve the problem using $x_i = Q_i/b_i$. Thus we have

$$l_i^c = \begin{cases} 1 & \text{if } P_i \geq P_{i-1}, \\ Q_{i-1}/x_i & \text{if } P_i < P_{i-1}. \end{cases} \quad (6)$$

Hence

$$\hat{R}_{i-1}^c = x_i [1/P_i + (l_i^c - 1)(1/P_i - 1/P_{i-1})]. \quad (7)$$

Now, we introduce a new indicator variable

$$\delta_i = \begin{cases} 0 & \text{if } P_i \geq P_{i-1} \text{ or } i = n + 1, \\ 1 & \text{otherwise} \end{cases} \quad (8)$$

which we can use in (7); therefore

$$\hat{R}_{i-1}^c = (1 - \delta_i)x_i/P_i + \delta_i [x_i/P_{i-1} + Q_{i-1}(1/P_i - 1/P_{i-1})]. \quad (9)$$

Substituting (9) into (5) and letting $\delta_{i+1} = 0$, we obtain the following total cost expressed in terms of Q_i and x_i :

$$\begin{aligned} \text{TC}^c = D \sum_{i=1}^n \{ & F_i/Q_i + Q_i [(1/D - 1/P_i)(c_i - c_{i+1})/2 + \delta_{i+1}c_i(1/P_{i+1} - 1/P_i)] \\ & + x_i c_i [(1 - \delta_i)/P_i + \delta_i/P_{i-1}] + T_i/x_i \}. \end{aligned} \quad (10)$$

This can be written as

$$\text{TC}^c = D \sum_{i=1}^n \{ A_i/Q_i + B_i Q_i + H_i x_i + G_i/x_i \} \quad (11)$$

where the A_i , B_i , H_i and G_i coefficients are evident from (10). Since $\hat{R}_{i-1}^c \leq \hat{R}_{i-1}$, $\text{TC}^c \leq \text{TC}$ for all values of Q_i and x_i , now we have the problem

$$\begin{aligned} & \text{minimize} && \text{TC}^c, \\ & \text{subject to} && Q_i \leq L_i, \\ & && x_i \leq g_i, \\ & && Q_{i-1} \leq Q_i, \\ & && x_i \leq Q_i, \\ & && Q_i, x_i \geq 0. \end{aligned} \quad (12)$$

The solution to this problem provides the lower bound on the solution to (5). Since TC^c is convex, as can be seen from (11), we could use nonlinear programming. An efficient method is given in [3].

4. A heuristic solution procedure

First, we find integers S_i by the following cumulative rounding procedure. Let Q_1^c, \dots, Q_n^c be the optimal Q_i values found in problem (12); then the S_i 's are determined in sequence by

$$S_k = \left\lceil \left\lfloor \frac{Q_{k+1}^c}{Q_i^c(S_1 S_2 \cdots S_{k-1})} \right\rfloor \right\rceil \quad \text{for } k = 1, \dots, n-1 \quad (13)$$

where $S_0 = 1$.

Since $L_i/(S_1 S_2 \cdots S_{i-1})$ is the maximum allowable Q_1 , given the lot-size constraint L_i , at stage i ,

$$Q_{1u} = \min_{1 \leq i \leq n} \{L_i/(S_1 S_2 \cdots S_{i-1})\} \tag{14}$$

is the upper bound for Q_1 . It can be shown that $Q_1 \leq Q_{1u}$ if and only if $Q_i \leq L_i$ for $i = 1, \dots, n$. Therefore we choose

$$Q_i = \min\{Q_{1u}, Q_i^c\}, \quad Q_i = Q_1 S_1 \cdots S_{i-1}. \tag{15}$$

Next, we determine the best integer b_i 's for the Q_i 's found in (15). First, we consider the case where $R_{i-1} = \hat{R}_{i-1}$. In (5) we convert each x_i to Q_i/b_i ; hence each term of the sum now containing b_i can be optimized separately with respect to b_i . Terms not containing b_i (constants) are ignored and we have n problems of the type

$$\text{minimize } f(b_i) = (T_i/Q_i)b_i + c_i \hat{R}_{i-1}.$$

Substituting (2) this can be written as follows:

$$\begin{aligned} \text{minimize } & f(b_i) = (T_i/Q_i)b_i + (Q_i/b_i)c_i [1/P_i + (l_i - 1)(1/P_i - 1/P_{i-1})], \\ \text{subject to } & b_i \geq Q_i/g_i, \end{aligned} \tag{16}$$

where

$$l_i = \begin{cases} 1 & \text{if } P_i \geq P_{i-1}, \\ \lceil b_i/S_i \rceil & \text{if } P_i < P_{i-1}. \end{cases}$$

If $P_i \geq P_{i-1}$, then $l_i = 1$ and $f(b_i)$ is convex. We find the lowest cost integer on either side of the non-integer solution $b_i = Q_i/(P_i T_i)^{1/2}$. Then, we compare this integer with $\lceil Q_i/g_i \rceil$ and take the larger of the two.

If $P_i < P_{i-1}$, the problem is more difficult. In this case,

$$f(b_i) = (T_i/Q_i)b_i + (Q_i/b_i)c_i [1/P_i + (r_i + b_i/S_i - 1)(1/P_i - 1/P_{i-1})] \tag{17}$$

where $r_i = \lceil b_i/S_i \rceil - b_i/S_i$ and $0 \leq r_i \leq 1$.

Let $y_i = b_i/S_i$, and consider the function

$$h(y_i, r_i) = (u_i + w_i r_i)/y_i + v_i y_i + w_i \tag{18}$$

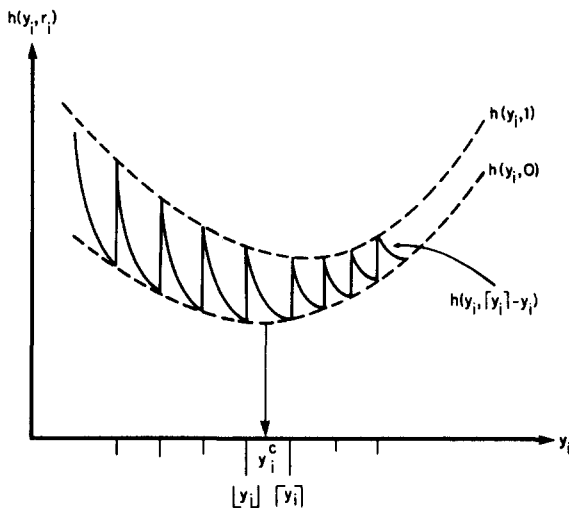


Fig. 3. Illustration of a 'typical' function $h(y_i, r_i)$ in Eq. (18).

where

$$u_i = (Q_i/S_i)(1/P_{i-1})c_i, \quad v_i = T_i S_i/Q_i, \quad w_i = (Q_i/S_i)(1/P_i - 1/P_{i-1})c_i.$$

Note that $h(y_i, 0) \leq f(b_i) \leq h(y_i, 1)$ and $f(b_i) = h(y_i, \lceil y_i \rceil - y_i)$. A 'typical' function $h(y_i, r_i)$ is plotted in Fig. 3. It is shown there as a continuous function although it exists only when $b_i = y_i S_i$ is an integer. Since S_i is an integer, an integer y_i sets b_i also to an integer.

In Fig. 3, y_i^c indicates the minimum of the envelope $h(y_i, 0)$. It can be shown that, when $r_i = 0$, the minimum of function (18) occurs at

$$y_i^c = (u_i/v_i)^{1/2} \quad (19)$$

and that the optimum integer y_i must be in the region $[\lceil y_i^c \rceil, \lfloor y_i^c \rfloor]$.

In order to find the optimum value of b_i when $\lceil y_i^c \rceil S_i \geq Q_i/g_i$, we need only search for the lowest cost integers in the interval $[\lceil y_i^c \rceil S_i, \lfloor y_i^c \rfloor S_i]$. If $\lceil y_i^c \rceil S_i \leq Q_i/g_i$ we would search integer b_i 's from $\lceil Q_i/g_i \rceil$ to $\lfloor \lceil Q_i/g_i \rceil / S_i \rfloor S_i$.

Refinements to this procedure are possible. For instance, let y_i^* be the value of y that minimizes $h(y_i, \lceil y_i \rceil - y_i)$, considered as a continuous function. Since $r_i = \lceil y_i^c \rceil - y_i$, it can be shown from (18) that if $\lceil y_i^c \rceil \leq y_i^* \leq \lfloor y_i^c \rfloor$ then y_i^* is given by

$$y_i^* = ((u_i + w_i \lceil y_i^c \rceil) / v_i)^{1/2}. \quad (20)$$

Only integer b_i 's on either side of $y_i^* S_i$ would be checked if the constraint $b_i \geq Q_i/g_i$ did not interfere. However, unless S_i values are very large such refinements are not necessary.

When $R_{i-1} \neq \bar{R}_{i-1}$ (note that this happens only when $j \neq (\lceil Q_{i-1}/x_i \rceil - 1)$, expression (1) is used for R_{i-1} and problem (16) becomes:

$$\text{minimize } \bar{f}(b_i) = (T_i/Q_i)b_i + c_i \max_{0 \leq j \leq b_i - 1} \left\{ jx_i(1/P_i - 1/P_{i-1}) - \left\lfloor \frac{jx_i}{Q_{i-1}} \right\rfloor (1/D - 1/P_{i-1}) \right\}. \quad (21)$$

Let b_i^* be the optimum number of batches in (16) and b_i^{**} the optimum number of batches in (21). Note that $\bar{f}(b_i^*)$ is an upper bound for $\bar{f}(b_i^{**})$.

As can be seen from (1), $R_{i-1} \geq x_i/P_i$; therefore

$$\bar{f}(b_i) \geq (T_i/Q_i)b_i + (c_i Q_i/P_i)/b_i.$$

We now search for b_i^{**} in the region of $\bar{f}(b_i)$ where

$$(T_i/Q_i)b_i + (c_i Q_i/P_i)/b_i \leq \bar{f}(b_i^*). \quad (22)$$

The resulting range $[b_{i(\min)}^{**}, b_{i(\max)}^{**}]$ is obtained by setting the preceding inequality (22) to an equality and solving the resulting quadratic equation.

Now that the S_i 's and b_i 's are known, Q_1 may be updated to its 'current-best' value as follows. Given $S_1 \cdots S_n$ and $b_1 \cdots b_n$ eq. (10) can be put in the form

$$\text{TC} = U/Q_1 + VQ_1 \quad (23)$$

where

$$\begin{aligned} U &= D \sum_{i=1}^n (F_i + b_i T_i) / (S_1 S_2 \cdots S_{i-1}), \\ V &= D \sum_{i=1}^n S_1 S_2 \cdots S_{i-1} \{ (1/D - 1/P_i)(c_i - c_{i+1})/2 \\ &\quad + [\delta_{i+1}(1/P_{i+1} - 1/P_i) + (1 - \delta_i)/(P_i b_i) + \delta_i/(P_{i-1} b_i)] c_i \}, \\ \delta_{n+1} &= 0 \text{ and the } \delta_i \text{ indicator variable is given in (8).} \end{aligned}$$

The lot size that minimizes the total cost is expressed by $(U/V)^{1/2}$. Therefore, we can set the new $Q_1 = \min(Q_{1u}, (U/V)^{1/2})$ and the procedure is ready to loop again. It was found empirically that when g_i 's are relatively small and binding on the value of x_i , the best value of Q_1 is often an integer multiple of some g_i . Therefore, in addition of the value of Q_1 found above, the closest such integer multiplies are checked. Since the cost is non-increasing and the number of possible b_i 's is finite the procedure must converge.

5. Computational example and conclusions

Table 1 presents the problem parameters used in the example. The solution to the problem is shown in Table 2.

The first two sections of Table 2 show the solutions obtained by the heuristic procedure presented in this paper (without and with constraints). The total costs in each case are denoted by TC and the lower bounds on the costs are denoted by TC_s. The third section of the table contains results for a typical variable lot-size model [5] when batch shipments are not allowed. The 'optimal' total cost here is denoted by TC*. For each of the cases 'Δ cost-ratio' indicates the percent cost in excess over the lower bound cost.

Note that the total costs in Table 2 for both the unconstrained and constrained cases are very close to the lower bound. Considering the fact that the lower bound on cost is a hypothetical result (i.e. very seldom attainable due to integrality requirements), the accuracy of the heuristic procedure is favourably reflected by the example. As a further test, 100 cases of each kind were computed with uniformly randomized input, the results of which are summarized in Table 3.

The results in Table 3 support confidence in the accuracy of the heuristic procedure. Except for very few cases, the Δ cost-ratio is very moderate. For the unconstrained problem 95 percent of the cases are 2.23 or less percent above the lower bound; for the constrained problem we found this percent to be 3.88 or less. This accuracy is especially remarkable if one considers that even for an optimal solution (for $Q_i = x_i$) 95 percent of the cases are 1.5 percent or less above the lower bound costs.

The computation of the heuristic procedure is rather efficient. Its time was between 0.16 and 0.22 CPU seconds for a large sample of 12 stage cases on a CYBER computer. Experiments with very large numbers of stages, $10 \leq n \leq 300$, have shown that the computational time increases nearly proportionally with the number of stages up to $n = 100$. For example, the following CPU seconds were needed (average of five problems) for some large numbers of stages: 0.41 for $n = 50$, 0.98 for $n = 100$, 1.90 for $n = 150$, 4.15 for $n = 200$ and 6.02 for $n = 300$.

Last but certainly not least, a noteworthy comparison can be made from Table 2 between the optimal total cost of a variable lot-size model (\$15 245.52) and the heuristic total cost (\$12 265.51) for the more flexible model presented in this paper. Note that both results include the same transportation cost per shipment whether it is a lot or a batch. The model which accommodates simultaneously both variable

Table 1
Problem parameters

i	c_i	P_i	F_i	T_i	g_i	L_i
1	2.5	250 000	1.0	0.6	250	1500
2	2.4	500 000	6.0	0.6	250	1500
3	2.2	375 000	17.0	0.6	250	1500
4	2.1	200 000	8.0	2.2	250	1500
5	1.7	150 000	26.0	1.3	500	5000
6	1.6	225 000	9.0	3.2	500	5000
7	1.5	275 000	17.0	1.6	500	5000
8	1.4	125 000	24.0	1.0	500	5000
9	0.6	175 000	46.0	0.8	500	5000
10	0.4	525 000	18.0	1.2	5000	5000
11	0.3	800 000	16.0	8.0	6000	∞
12	0.1	600 000	60.0	5.8	6000	∞

Number of stages: $n = 12$; Demand rate: $D = 60\,000$.

Table 2
Solution results

$n = 12$ i	Unconstrained Q_i and x_i				Constrained $Q_i \leq L_i, x_i \leq g_i$				Unconstrained $Q_i = x_i$		
	Q_i	S_i	x_i	b_i	Q_i	S_i	x_i	b_i	Q_i	S_i	b_i
1	1309.526	1	261.905	5	1250.0	1	250.0	5	428.37	3	1
2	1309.526	1	327.381	4	1250.0	1	250.0	5	1285.11	1	1
3	1309.526	1	327.381	4	1250.0	1	250.0	5	1285.11	1	1
4	1309.526	2	654.763	2	1250.0	2	250.0	5	1285.11	1	1
5	2619.052	1	436.502	6	2500.0	1	416.6	6	1285.11	1	1
6	2619.052	1	654.763	4	2500.0	1	500.0	5	1285.11	1	1
7	2619.052	1	523.810	5	2500.0	1	500.0	5	1285.11	1	1
8	2619.052	2	436.509	6	2500.0	2	416.6	6	1285.11	3	1
9	5238.104	1	476.191	12	5000.0	1	500.0	10	3855.33	1	1
10	5238.104	1	1309.526	4	5000.0	1	1250.0	4	3855.33	1	1
11	5238.104	2	4238.104	1	5000.0	2	5000.0	1	3855.33	2	1
12	104476.207	1	5238.104	2	10000	1	5000.0	2	7710.66	1	1
TC = \$12265.51				TC = \$12515.90				TC* = \$15245.52			
TC _c = \$12212.85				TC _c = \$12458.13				TC _c = \$15135.91			
Δ cost-ratio = 0.43%				Δ cost-ratio = 0.46%				Δ cost-ratio = 0.72%			

Table 3
 Δ cost-ratio percentiles

No. of cases	Type of cases randomized	Percent of cases				Max.	Min.	Mean
		≤ 25	≤ 50	≤ 75	≤ 95			
Δ Cost-ratios (in percentages)								
100	Unconstrained Q_i and x_i	0.31	0.51	1.01	2.23	5.16	0.13	0.77
100	Constrained $Q_i \leq L_i, x_i \leq g_i$	0.92	1.47	1.94	3.88	11.90	0.28	1.80
100	Unconstrained $Q_i = x_i$	0.30	0.70	1.00	1.50	1.60	0.01	0.72

Data used (square brackets denote ranges): $n = 12, D = 60000, F_i = [1.0 - 50.0], T_i = [0.1 - 10.0], c_i = [0.1 - 7.5], P_i = [65000 - 950000], L_i = 1500, g_i = [100 - 1000$ (in 100's)], for $i = 1, 2, \dots, n$.

lot-size and batch shipment creates its savings by decreasing the set-up and/or the inventory-holding costs. The 24 percent cost savings in the example speaks for itself.

References

- [1] W.B. Crowston, M.H. Wagner and A. Henshaw, Comparison of exact and heuristic routines for lot-size determination in multi-stage assembly systems, *AIIE Trans.* 4 (1972) 313-317.
- [2] W.B. Crowston, M.H. Wagner and J.F. Williams, Economic lot size determination in multi-stage assembly systems, *Management Sci.* 19 (5) (1973) 517-527.
- [3] Z. Drezner, A.Z. Szendrovits and G.O. Wesolowsky, Multi-stage production with variable lot sizes and transportation of partial lots, Faculty of Business, McMaster University, Research and Working Paper Series No. 184 (1982) 1-24.
- [4] P.A. Jensen and H.A. Kahn, Scheduling in a multi-stage production system with set-up and inventory costs, *AIIE Trans.* 4 (2) (1972) 126-133.
- [5] L.B. Schwarz and L. Schrage, Optimal and system myopic policies for multi-echelon production/inventory assembly systems, *Management Sci.* 21 (11) (1975) 1285-1294.
- [6] A.Z. Szendrovits, Manufacturing cycle time determination for a multi-stage economic production quantity model, *Management Sci.* 22 (3) (1975) 298-308.

- [7] A.Z. Szendrovits, On the optimality of sub-batch sizes for a multi-stage EPQ model – a rejoinder, *Management Sci.* 23 (3) (1976) 334–338.
- [8] A.Z. Szendrovits and Z. Drezner, Optimizing multi-stage production with constant lot size and varying numbers of batches, *Omega* 8 (6) (1980) 623–629.
- [9] H.A. Taha and R.W. Skeith, The economic lot sizes in multi-stage production systems, *AIIE Trans.* 2 (2) (1970) 157–162.
- [10] H.M. Wagner, Research portfolio for inventory management and production planning systems, *Operations Res.* 28 (3) Part I (1980) 445–475.