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NON-EXISTENCE OF RENORMALIZABLE SELF-INTERACTION IN $N = 2$ SUPERSYMMETRY FOR SCALAR HYPERMULTIPLETS

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We prove that the assumption of invariance under "isospin rotations", i e automorphisms of $N = 2$ supersymmetric charges, implies that there is no local renormalizable interaction among scalar hypermultiplets

1. Introduction

Amongst various motivations in recent studies of extended supersymmetry, one that entreats us is the possibility of building finite (for a review of finite theories see [1]) realistic four-dimensional models For this purpose, a preliminary step is to ascertain what couplings and soft breakings are allowed, as such theories are very restrictive.

Let us focus our attention on $N = 2$ supersymmetric models in which the dimensions of the operators in the lagrangians are not greater than four We shall adhere to the conventional terminology and call such models renormalizable, although in extended supersymmetry there may be other possibilities

It is known how to introduce gauge interactions [2] On the other hand, there have been remarks of inability to construct other renormalizable interactions between scalar multiplets [3] It is fair to say that the impossibility has not been substantiated with published systematic analysis The purpose of this note is to give a proof that there can be no self-interaction between an arbitrary number of scalar hypermultiplets Here, by a scalar hypermultiplet, we mean the $N = 2$ supersymmetry (SS) algebra in which either the scalars (A_i) or the spinors (ψ_i) are isodoublets with respect to the SU(2) automorphisms of SS charges The superpartners ψ_α, $\bar{\phi}_\alpha$ and A_{ij} are respectively singlets and $1 \oplus 0$ SU(2) representations

While for the multiplet (A_i, ψ, ϕ) it is trivial to see that no self-interaction can be written which is SU(2) invariant (because of the impossibility of forming SU(2) covariant Yukawa coupling), for the multiplet (A_{ij}, ψ_i) this is not evident, as we can in principle construct SU(2) covariant coupling. $A^{ij}\psi_i\psi_j$ The same is true for bosonic trilinear couplings, which can appear after the elimination of auxiliary fields.

We know in several cases that the internal symmetry due to extended supersymmetry is manifest only when the lagrangians are written in terms of physical component fields [4] Thus, when one performs a general analysis about interactions, one should not make assumptions about the isospin assignments in the auxiliary fields* In fact, we shall do away with the auxiliary fields altogether and work only with physical fields.

We shall show that the $N=2$ supersymmetry algebra already implies the non-existence of self-interaction** This is accomplished by writing down the most general transformations possible on the physical fields, consistent with isospin invariance, locality and renormalizability The algebra is then forced to have only free field transformations.

2. Proof

In this section, we prove that there is no renormalizable Yukawa interaction among N_1 Φ_i type and N_2 Φ_{ij} type hypermultiplets One assumption we need to make for our proof is that the set of transformations on the $N=2$ indices (e g i and ij above) constitutes a good symmetry. We call this the SU(2) isospin symmetry. This implies that all equations must be covariant in isospin. We will show that the most general supersymmetric transformations on these fields, consistent with the extended algebra, isospin covariance and renormalizability, correspond to an interaction free theory

For notational convenience, we group all N_1 Φ_i into one vector and all N_2 Φ_{ij} into another vector We choose to work with "real" Φ_{ij}, this should place no limitations on our proof, because we can always decompose each complex Φ_{ij} into two real Φ_{ij}.

We have not assumed any further internal symmetry among these hypermultiplets, but it is clear that our proof goes through when such symmetry exists, because that will just be tantamount to applying a general result to a particular case For example, when Φ_i falls into a representation of sp(n), one can further impose a

* In fact it is enough to make the assumption of SU(2) invariance only for the hard part of the interactions

** We emphasize that no lagrangian formulation is assumed in what follows, as the proof relies on the algebra and its representation only (We wish to thank the referee for reminding us to stress this point) reality condition

$$\left(A_{i}{}^{L}\right)^{*}=\Omega^{LM}\varepsilon^{ij}A_{jM}$$

Correspondingly, we must halve the number of spinor components through another relation

$$\psi_{L}=\Omega_{LM}\phi^{M}$$

Because the symplectic matrix commutes with SU(2), this essentially means that we can pass to a Φ_i with fewer components by destroying the manifest invariance under sp(n). In what follows, we assume that this reduction has been made

We take the phase convention*

$$\left(A_{i}{}^{j}\right)^{*}=A^{*i}{}_{j} \tag{1}$$

Writing

$$A_{i}{}^{j}=A_{0}\delta_{i}^{j}+A\quad\tau_{i}{}^{j}, \tag{2}$$

we impose that

$$A_{0}=-A_{0}{}^{*},\qquad A=A^{*} \tag{3}$$

It follows then

$$A_{i}^{j}=A^{*}{}_{i}^{j},\qquad\left(A_{ij}\right)^{*}=-A^{ij} \tag{4}$$

Based on dimensional and isospin considerations, we have the following general supersymmetric transformations

$$\delta A_{i}=\xi_{i}^{\alpha}\mathcal{D}\psi_{\alpha}+\bar{\xi}_{\alpha i}\bar{\mathcal{D}}\bar{\phi}^{\alpha}, \tag{5}$$

$$\delta A^{*i}=\bar{\xi}_{\alpha}^{i}\mathcal{D}^{*}\bar{\psi}^{\alpha}+\xi^{\alpha i}\bar{\mathcal{D}}^{*}\phi_{\alpha}, \tag{6}$$

$$\delta\psi_{\alpha}=\xi_{\alpha}^{k}\mathcal{M}_{k}+i\left(\sigma_{\mu}\partial^{\mu}\bar{\xi}\right)_{\alpha k}\bar{\mathcal{N}}^{k}. \tag{7}$$

* We have

$$\left(\tau_{i}{}^{j}\right)^{*}=\tau^{i}{}_{j},\qquad\left(g_{ij}\right)^{*}=-g^{ij}$$

$$\left(\xi^{\alpha}{}_{k}\right)^{*}=\bar{\xi}^{\alpha k},\qquad\left(\xi^{\alpha k}\right)^{*}=-\bar{\xi}^{\alpha}{}_{k}$$

and a similar one for $\delta\bar{\phi}_\alpha$, where

$$\mathcal{M}_k = m\left(\mathcal{E}_1 A_k + \mathcal{E}_2 A_k^*\right) + \mathcal{S}_1 A_k A_m^m + \mathcal{S}_2 A_k^* A_m^m + \mathcal{T}_1 A^m A_{mk} + \mathcal{T}_2 A^{*m} A_{mk}, \quad (8)$$

$$\bar{\mathcal{N}}^k = \bar{\mathcal{B}} A^k + \bar{\mathcal{C}} A^{*k} \quad (9)$$

We also have

$$\delta A_{ij} = \xi^{\alpha k} \psi_{\alpha k ij} + \bar{\xi}_{\alpha k} \bar{\psi}^{\alpha k}{}_{ij}, \quad (10)$$

with

$$\psi_{\alpha k ij} = -D_1 g_{ki} \psi_{\alpha j} - D_2 g_{kj} \psi_{\alpha i},$$

$$\bar{\psi}^k_{\alpha ij} = -D_1^* \delta_i^k \bar{\psi}_{\alpha j} - D_2^* \delta_j^k \bar{\psi}_{\alpha i} \quad (11)$$

We shall write down the transformation for $\psi_{\alpha i}$ later In the above, ξ and $\bar{\xi}$ are the infinitesimal anticommuting parameters. $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{T}$, $\mathcal{S}$ and D's are matrices with appropriate dimensions in rows and columns m is a mass parameter We have also used the phase conventions

$$(\psi_i^\alpha)^* = \bar{\psi}^{\alpha i}, \qquad (\psi^{\alpha i})^* = -\bar{\psi}_i^\alpha \quad (12)$$

Now, the extended supersymmetry algebra demands that

$$\left[\delta_\xi, \delta_{\bar{\xi}}\right] \psi_\alpha = 2i \xi^{\beta k} \left(\sigma_\mu \partial^\mu\right)_{\beta\dot{\beta}} \bar{\xi}_k^{\dot{\beta}} \psi_\alpha, \quad (13)$$

whereas, by applying the above transformations in successive orders, one obtains

$$\left[\delta_\xi, \delta_{\bar{\xi}}\right] \psi_\alpha = -i \xi^{\beta k} \left(\sigma_\mu \partial^\mu \bar{\xi}\right)_{\alpha k} \left(\bar{\mathcal{B}} \mathcal{D} \psi_\beta + \bar{\mathcal{C}} \bar{\mathcal{D}}^* \phi_\beta\right) + \xi_\alpha^k \bar{\xi}_{\alpha l} P^{\alpha l}{}_k, \quad (14)$$

where

$$P^{\alpha l}{}_k = m\left(\mathcal{E}_1 \bar{\mathcal{D}} \bar{\phi}^\alpha + \mathcal{E}_2 \mathcal{D}^* \bar{\psi}^\alpha\right) \delta_k^l$$

$$+ \mathcal{S}_1 \bar{\mathcal{D}} \bar{\phi}^\alpha A_m{}^m \delta_k^l + \mathcal{S}_1 A_k \bar{\psi}^{\alpha l}{}_m{}^m + \mathcal{S}_2 \mathcal{D}^* \bar{\psi}^\alpha A_m{}^m \delta_k^l + \mathcal{S}_2 A_k^* \bar{\psi}^{\alpha l}{}_m{}^m$$

$$- \mathcal{T}_1 \bar{\mathcal{D}} \bar{\phi}^\alpha A_k^l + \mathcal{T}_1 A^m \bar{\psi}^{\alpha l}{}_{mk} - \mathcal{T}_2 \mathcal{D}^* \bar{\psi}^\alpha A_k^l + \mathcal{T}_2 A^{*m} \bar{\psi}^{\alpha l}{}_{mk} \quad (15)$$

Comparing eqs (13), (14), we have

$$2i\left(\sigma_\mu \partial^\mu\right)_{\beta\dot{\beta}} \psi_\alpha \delta_k^l = -i\left(\sigma_\mu \partial^\mu\right)_{\alpha\dot{\beta}} \left(\bar{\mathcal{B}} \mathcal{D} \psi_\beta + \bar{\mathcal{C}} \mathcal{D}^* \phi_\beta\right) \delta_k^l + \varepsilon_{\alpha\beta} \varepsilon_{\dot{\alpha}\dot{\beta}} P^{\alpha l}{}_k \quad (16)$$

The part which is symmetric in α and β of the last equation gives

$$2i(\sigma_\mu\partial^\mu)_{\beta\beta}\psi_\alpha + 2i(\sigma_\mu\partial^\mu)_{\alpha\beta}\psi_\beta = -i(\sigma_\mu\partial^\mu)_{\alpha\beta}\left(\overline{\mathcal{B}}\mathcal{D}\psi_\beta + \bar{\mathcal{C}}\mathcal{D}^*\phi_\beta\right)$$
$$-i(\sigma_\mu\partial^\mu)_{\beta\beta}\left(\overline{\mathcal{B}}\mathcal{D}\psi_\alpha + \bar{\mathcal{C}}\mathcal{D}^*\phi_\alpha\right), \qquad (17)$$

which, upon multiplying with sides by $(\bar\sigma_\nu\partial^\nu)^{\beta\beta}$, results in

$$6i\psi_\alpha = 3i\left(\overline{\mathcal{B}}\mathcal{D}\psi_\alpha + \bar{\mathcal{C}}\mathcal{D}^*\phi_\alpha\right), \qquad (18)$$

or

$$\overline{\mathcal{B}}\mathcal{D} = 2, \qquad (19)$$

$$\bar{\mathcal{C}}\mathcal{D}^* = 0 \qquad (20)$$

As eq (19) asserts that $\mathcal{D} \neq 0$, we must conclude from eq (20) that

$$\bar{\mathcal{C}} = 0 \qquad (21)$$

Likewise, we lower the index l in eq (16) and obtain from the symmetric part in k and l

$$P^{\alpha}{}_{lk} + P^{\alpha}{}_{kl} = 0 \qquad (22)$$

By equating coefficients of the various products of A and ψ, we have*

$$\mathcal{T}_1\overline{\mathcal{D}} \otimes I = 0, \qquad (23)$$

$$\mathcal{T}_2\mathcal{D}^* \otimes I = 0, \qquad (24)$$

$$\mathcal{S}_1 I \otimes (-D_1^* + D_2^*) - \mathcal{T}_1 I \otimes D_1^* = 0, \qquad (25)$$

$$\mathcal{S}_2 I \otimes (-D_1^* + D_2^*) - \mathcal{T}_2 I \otimes D_1^* = 0 \qquad (26)$$

Eqs (23) and (24) give

$$\mathcal{T}_1 = \mathcal{T}_2 = 0 \qquad (27)$$

As we can show later on that

$$-D_1^* + D_2^* \neq 0, \qquad (28)$$

* Direct product ($\otimes$) occurs, because we have products of fields

which, because of eqs (25),(26), leads to

$$\mathbb{S}_1 = \mathbb{S}_2 = 0. \tag{29}$$

Altogether, eqs. (21), (27) and (29) ensure that A_i and ψ_α obey free field transformation laws. In the same manner, $\bar{\phi}_\alpha$ can be shown to transform as a free field

Now that we have shown that Φ_i's do not interact, we can take the following as the general transformation

$$\delta\psi_{\alpha i} = \xi^k_\alpha M_{ki} + i\left(\sigma_\mu \partial^\mu \bar{\xi}_k\right)_\alpha \bar{N}^k{}_i, \tag{30}$$

$$\delta\bar{\psi}^i_\alpha = -\bar{\xi}_{\alpha k}\bar{M}^{ki} + i\left(\xi^k \sigma_\mu \partial^\mu\right)_\alpha N_k{}^i, \tag{31}$$

where

$$M_{ki} = m\left(EA_{ki} + Fg_{ki}A_l{}^l\right) + G_1 A_{ki}A_l{}^l + G_2 g_{ki}A_{lm}A^{lm} + G_3 g_{ki}A_l{}^l A_m{}^m, \tag{32}$$

$$\bar{N}^k{}_i = \bar{A}A^k{}_i + \bar{B}\delta_i^k A_l{}^l, \tag{33}$$

$$\bar{M}^{ki} = (M_{ki})^*, \qquad N_k{}^i = \left(\bar{N}^k{}_i\right)^* \tag{34}$$

Here, the coefficients $\bar{A}$, $\bar{B}$ etc. are either $N_2 \times N_2$ or $(N_2)^2 \times (N_2)^2$ matrices

As before, we compare the consequences due to extended supersymmetry algebra on the one hand and those due to the transformation on the other

$$\begin{aligned}\left[\delta_\xi, \delta_{\bar{\xi}}\right]\psi_{\alpha i} &= 2i\xi^{\beta k}\left(\sigma_\mu\partial^\mu\right)_{\beta\beta}\bar{\xi}^\beta_k\psi_{\alpha i} \\ &= -i\left(\sigma_\mu\partial^\mu\bar{\xi}_k\right)_\alpha\left(\bar{A}\xi^{\beta l}\psi_{\beta l}{}^k{}_i + \bar{B}\delta_i^k\xi^{\beta l}\psi_{\beta lm}{}^m\right) + \xi^k_\alpha\bar{\xi}_{\alpha l}P^{\alpha l}{}_{ki},\end{aligned} \tag{35}$$

where

$$\begin{aligned}P^{\alpha l}{}_{ki} &= m\left(E\bar{\psi}^{\alpha l}{}_{ki} + Fg_{ki}\bar{\psi}^{\alpha l}{}_m{}^m\right) + G_1\bar{\psi}^{\alpha l}{}_{ki}\otimes IA_l{}^l \\ &\quad + G_1A_{ki}I\otimes\bar{\psi}^{\alpha l}{}_m{}^m + G_2g_{ki}A^{mn}I\otimes\bar{\psi}^{\alpha l}{}_{mn} + G_2g_{ki}\bar{\psi}^{\alpha l}{}_{mn}\otimes IA^{mn} \\ &\quad + G_3g_{ki}A_m{}^mI\otimes\bar{\psi}^{\alpha l}{}_n{}^n + G_3g_{ki}\bar{\psi}^{\alpha l}{}_n{}^n\otimes IA_m{}^m.\end{aligned} \tag{36}$$

This results in the equation

$$2i\left(\sigma_\mu\partial^\mu\right)_{\beta\beta}\psi_{\alpha i}\delta_k^l = i\left(\sigma_\mu\partial^\mu\right)_{\alpha\beta}\left(\bar{A}\psi_{\beta l}{}^k{}_i + \bar{B}\delta_i^k\psi_{\beta lm}{}^m\right) + P^{\alpha l}{}_{ki}\varepsilon_{\alpha\beta}\varepsilon_{\alpha\beta} \tag{37}$$

The part symmetric in α and β is

$$\bar{A}\psi_{\alpha k}{}^{l}{}_{i} + \bar{B}\delta_{i}^{l}\psi_{\alpha k m}{}^{m} = 2\delta_{k}^{l}\psi_{\alpha i}. \tag{38}$$

Using eq (11), we deduce

$$\bar{A}(D_1 + D_2) = 2, \tag{39}$$

$$(\bar{A} - 2\bar{B})(D_1 - D_2) = 2. \tag{40}$$

Note that eq (40) asserts that $D_1 - D_2 \neq 0$, a property which was used earlier

We now make use of the part of eq (37) which is anti-symmetric in α and β We simplify it with eqs (39) and (40) into

$$-2i\left(\sigma_{\mu}\partial^{\mu}\right)_{\alpha\beta}\psi_{i}^{\alpha}g_{lk} = P_{\beta lki} \tag{41}$$

From this, the piece which is symmetric in k and l must vanish, i.e

$$P_{\beta lki} + P_{\beta kli} = 0 \tag{42}$$

In particular, the portion which is totally symmetric in k, l and i gives

$$G_1\left(I \otimes \left(-D_1^* + D_2^*\right)\right) = 0, \tag{43}$$

or

$$G_1 = 0 \tag{44}$$

The other equations one can obtain from eq (42) are by requiring the coefficients of various products of A and ψ to vanish

$$G_2\left(I \otimes \left(-D_1^* + D_2^*\right) + \left(-D_1^* + D_2^*\right) \otimes I\right) = 0, \tag{45}$$

$$\left(\frac{G_2}{2} + G_3\right)\left(I \otimes \left(-D_1^* + D_2^*\right) + \left(-D_1^* + D_2^*\right) \otimes I\right) = 0, \tag{46}$$

$$-ED_2^* + F\left(-D_1^* + D_2^*\right) = 0 \tag{47}$$

Eqs. (45) and (46) give

$$G_2 = G_3 = 0 \tag{48}$$

However, eqs (44) and (48) are just the statement that A_{ij} and $\psi_{\alpha i}$ transform like free fields, viz eqs (10), (30)–(34) This establishes the proof

3. Conclusion

We have seen that the assumption of the SU(2) invariance of the hard coupling essentially renders the interaction of the hypermultiplets zero Whether our result changes by giving up the SU(2) covariance remains unanswered

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