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Generic Bifurcation of Steady-State Solutions

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Brunovsky and Chow [1] have recently proved that for a generic C^2 function with the Whitney topology, the "time map" $T(\cdot,f)$ (see [4]) associated with the differential equation $u'' + f(u) = 0$ with homogeneous Dirichlet or Neumann boundary conditions, is a Morse function. In this note we give a simpler proof of this result as well as some new applications. Our method of proof is quite elementary, uses only Sard's theorem and the implicit function theorem for functions in $C^1(\mathbb{R}^2, \mathbb{R})$, and avoids the use of transversality in function spaces.

An annoying difficulty that one has to face is that the domain of T varies with f. We get around this by constructing a continuous function $H(f)$ which, if positive, implies that T is a Morse function. Thus our task is to prove that the set of f with $H(f) > 0$ is generic. Of course, openness is trivial. To prove the denseness, we take any f, perturb it by a monomial cu^n, and consider the map θ: $(u, c) \to T'(u, \tilde{f})$, where $\tilde{f}(u) = f(u) + cu^n$. We show that 0 is a regular value of θ by checking explicitly that the relevant derivative has the form $an + b$, where $a \neq 0$, and b is bounded. Thus for large n, the linear term dominates and this yields the density statement.

One consequence of this result is that if $f(u) < 0$ for $u > M$, then there are a finite number of (positive) stationary solutions of the equation $u_t = u_{xx} + f(u)$, with homogeneous Dirichlet boundary conditions and, generically, we can completely describe all solutions of this partial differential equation.

1. Elementary Facts

We consider the associated first-order system $u' = v$, $v' = -f(u)$, and its flow ϕ_t. Let $F' = f$, $F(0) = 0$; then $F(u) + v^2/2$ is constant on orbits. If for $p > 0$, $\phi_t(0, p) = (0, -p)$, for some $t > 0$, we define the "time map" by $T(p, f) = \inf\{t > 0: \phi_{2t}(0, p) = (0, -p)\}$. We will write $T(p, f) = T(p)$ if there is no chance of confusion. Observe that if p_0 is in the domain of T, and

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$0 < p < p_0$, then the (positive) orbit through $(0, p)$ enters the region bounded by the v axis and the orbit segment $\phi_t(0, p_0)$, and, hence, must either leave this region in positive time, or approach a rest point. This gives

PROPOSITION 1 [1, Lemma 4.1]. *If* $f(u)^2 + f'(u)^2 > 0$, $u \in \mathbb{R}_+$ *the domain of* T *is an open interval* $(0, \beta(f))$ *minus a discrete set, where* $\beta(f)^2 = 2\sup\{F(u): u \in \mathbb{R}_+\}$. *Note that this interval may be finite, infinite, or void.*

If $p \in \text{domain}(T)$, then by symmetry, $\phi_{T(p)}(0, p) = (\alpha(p), 0)$, where $2F(\alpha(p)) = p^2$. This gives (see [4]) the explicit formulas for T and T',

(i) $T(p) = \int_0^{\alpha(p)} (2\Delta F(\xi))^{-1/2}\, du$,

(ii) $T'(p) = p[\alpha(p) f(\alpha(p))]^{-1} \int_0^{\alpha(p)} ((2\Delta F(\xi) - \Delta\xi f(\xi))/(2\Delta F(\xi))^{3/2})\, du$.

In these formulas, we are using the notation $\Delta g(\xi) = g(\alpha(p)) - g(u)$, for any function g.

PROPOSITION 2. [1, Lemma 4.2]. *If* $0 < p_0 < \beta(f)$, $f(u)^2 + f'(u)^2 > 0$, $u \in \mathbb{R}_+$, *and* $p_0 \notin \text{domain}(T)$, *then* $\lim_{p \to p_0} T(p) = \lim_{p \to p_0 -} T'(p) = -\lim_{p \to p_0 +} T'(p) = \infty$. *If* $\lim_{p \to \beta(f)-} \alpha(p) < \infty$, *then* $\lim_{p \to \beta(f)-} T(p) = \lim_{p \to \beta(f)-} T'(p) = \infty$.

Proof. If $\phi_N(0, p_0) = (u, v)$ with $v > 0$, then for p near p_0, $\phi_N(0, p) = (\tilde{u}, \tilde{v})$ with $\tilde{v} > 0$. Hence, $T(p) > N/2$ and $\lim_{p \to p_0} T(p) = \infty$; similarly $\lim_{p \to \beta(f)-} T(p) = \infty$. Next, from (ii)

$$T'(p) = \frac{p\, 2^{-3/2}}{\alpha f(\alpha)} \left[2T(p) + \int_0^{\alpha(p_0) - \varepsilon} \frac{-\Delta \xi f(\xi)}{(\Delta F(\xi))^{3/2}}\, du + \int_{\alpha(p_0) - \varepsilon}^{\alpha(p)} \frac{-\Delta \xi f(\xi)}{(\Delta F(\xi))^{3/2}}\, du \right], \tag{1}$$

where $\alpha(p_0) = \lim_{p \to p_0 -} \alpha(p)$ and $\varepsilon > 0$ is chosen so that $f(x) + xf'(x) < 0$ for $\alpha(p_0) - \varepsilon \leqslant x \leqslant \alpha(p_0)$. (Note that $f(\alpha(p_0)) = 0$, and $f'(\alpha(p_0)) < 0$.) Then $-\Delta\xi f(\xi) = \int_u^\alpha - (f(x) + xf'(x))\, dx > 0$, for $\alpha(p_0) - \varepsilon \leqslant u \leqslant \alpha(p) < \alpha(p_0)$. Thus the second integral in (1) is nonnegative, the first is bounded on $[\alpha(p_0) - \varepsilon/2, \alpha(p_0)]$, and since $T(p) \to \infty$ as $p \to p_{0-}$, we see $T'(p) \to \infty$ as $p \to p_0$. The same argument with $p_0 = \beta(f)$ shows $T'(p) \to \infty$ as $p \to \beta(f)-$.

Finally, for $p > p_0$, let $\bar{\alpha}(p_0) = \lim \alpha(p)$ as $p \to p_0+$, and write

$$T'(p) = \frac{p}{f(\alpha(p))} \left[\int_0^{\alpha(p_0) - \varepsilon} + \int_{\alpha(p_0) + \varepsilon}^{\alpha(p_0) - \varepsilon} + \int_{\bar{\alpha}(p_0) - \varepsilon}^{\alpha(p)} + \int_{\alpha(p_0) - \varepsilon}^{\alpha(p_0) + \varepsilon} \right],$$

and note that the first two integrals are bounded for $p > p_0$, as is the third if $f(\bar{\alpha}(p_0)) \neq 0$. If ε is small, we may estimate f by its linear part near $\alpha(p_0)$ in the last integral $(a(u-\alpha)p_0)) \leqslant f(u) \leqslant b(u-\alpha(p_0)))$ and explicitly evaluate the integral. This gives $\lim_{p \to p_0+} \int_{\alpha(p_0)-\varepsilon}^{\alpha(p_0)+\varepsilon} = -\infty$, and if $f(\bar{\alpha}(p_0)) = 0$, the same argument works for the third integral.

2. The Whitney Topology

Let $C^k(\mathbb{R}_+)$ denote C^k functions $f: \mathbb{R}_+ \to \mathbb{R}$ with the C^k-Whitney topology, that is, U is a neighborhood of f if there is a function $\varepsilon: \mathbb{R}_+ \to \mathbb{R}_+ \backslash \{0\}$ such that

$$\left\{ g \in C^k : \sum_{i=0}^{k} | g^{(i)}(u) - f^i(u)| < \varepsilon(u) \right\} \subseteq U.$$

Let $A = \{f \in C^2(\mathbb{R}_+) : f(x)^2 + f'(x)^2 > 0, \forall x \in \mathbb{R}_+\}$; then it is easy to see that A is open and dense in $C^2(\mathbb{R}_+)$.

If $f_1, f_2 \in C^2(\mathbb{R}_+)$, and $|f_1(u) - f_2(u)| < (1+u^2)^{-1}$, then their respective primitives, F_1, F_2 are simultaneously bounded or unbounded from above. Thus $C^2(\mathbb{R}_+) = \mathscr{B} \cup \mathscr{U}$, where $\mathscr{B}$ and $\mathscr{U}$ are open and consist of those f's which have bounded and unbounded primitives, respectively.

For $f \in \mathscr{U} \cap A$, we define $H(f): \mathbb{R}_+ \backslash \{0\} \to \mathbb{R}$ by

$$\begin{aligned} H(f)(p) &= \min(1, T'(p)^2 + T''(p)^2) && \text{if } p \in \text{domain } T, \\ &= 1 && \text{otherwise,} \end{aligned}$$

where T is the associated time map. Proposition 2 implies that $H(f)$ is continuous. Similarly, if $f \in \mathscr{B} \cap A$, we define the continuous function $H(f): (0, 1) \to \mathbb{R}$, by

$$\begin{aligned} H(f)(t) &= \min[1, T'(t\beta(f))^2 + T''(t\beta(f))^2] && \text{if } \quad t\beta(f) \in \text{domain } T \\ &= 1 && \text{otherwise.} \end{aligned}$$

For any compact C in $\mathbb{R}_+ \backslash \{0\}$, $H(f)|C$ is a continuous function of f in $\mathscr{U}$; that is, given $f \in A$, a compact C in the domain of $H(f)$ and $\varepsilon > 0$, there is an open set 0 such that if $\tilde{f} \in 0$, $|Hf(x) - H\tilde{f}(x)| < \varepsilon$ for all x in C. This holds since T' and T'' as well as $\beta(f)$ depend continuously on $f \in C^2(\mathbb{R}_+)$. Similarly, if C is a compact set in $(0, 1)$, and $f \in \mathscr{B}$, then $H(f)|C$ is continuous in $f \in B$.

Now let

$$\mathscr{S} = \{f \in A : H(f) > 0\},$$

and note that $\mathscr{G}$ consists of Morse functions. We can write

$$\mathscr{G} = (\mathscr{G} \cap \mathscr{U}) \cup (\mathscr{G} \cap \mathscr{B}) \equiv \mathscr{G}^{\mathscr{U}} \cup \mathscr{G}^{\mathscr{B}},$$

$$\mathscr{G}_k^{\mathscr{U}} = \{f \in \mathscr{U} : H(f) | [k^{-1}, k] > 0\},$$

and

$$\mathscr{G}_k^{\mathscr{B}} = \{f \in \mathscr{B} : H(f) | [k^{-1}, 1 - k^{-1}] > 0\}.$$

Then we note that $\mathscr{G}^{\mathscr{U}} = \bigcap_{k \in \mathbb{Z}_+} \mathscr{G}_k^{\mathscr{U}}, \mathscr{G}^{\mathscr{B}} = \bigcap_{k \in \mathbb{Z}_+} \mathscr{G}_k^{\mathscr{B}}$. Also by the above remark, $\mathscr{G}_k^{\mathscr{U}}$ is open since $[k^{-1}, k]$ is compact; similarly, $\mathscr{G}_k^{\mathscr{B}}$ is open. We will show that for all k, $\mathscr{G}_k^{\mathscr{U}}$ is dense in $\mathscr{U}$ and $\mathscr{G}_k^{\mathscr{B}}$ is dense in $\mathscr{B}$. Then by the Baire category theorem, $\mathscr{G}$ is residual in $C^2(\mathbb{R}_+)$.

THEOREM 3. [1, Theorem 3.1]. *$\mathscr{G}$ is residual in $C^2(\mathbb{R}_+)$.*

Proof. We will show that $\mathscr{G}_k^{\mathscr{U}} \cap \mathscr{O} \neq \phi$ for any nonvoid open set $\mathscr{O} \subset \mathscr{U}$; the proof for $\mathscr{B}$ is virtually identical. Choose $f_0 \in \mathscr{O} \cap A$, and let $M = [k^{-1}, k] \backslash \{p : H(f_0)(p) \geqslant \frac{1}{2}\}$. Consider the map $\theta: M \times (-\varepsilon, \varepsilon) \to \mathbb{R}$, defined by $\theta(p, x) = T'(p, \tilde{f})$, where $\tilde{f}(u) = f(u) + xb_n(u)$, and

$$\begin{aligned} b_n(u) &= u^n && \text{if} \quad 0 \leqslant u \leqslant k+1, \\ &= 0 && \text{if} \quad u \geqslant k+2, \end{aligned}$$

is a C^2 function; b_n has compact support, but for computational purposes, it is just u^n. Note that θ is C^1. (If $\mathscr{O} \subset \mathscr{B}$, replace k in the definition of b_n by $\beta(f_0)$.)

We claim that 0 is a regular value of θ if ε is sufficiently small and n is sufficiently large. By differentiating under the integral, we have

$$\frac{\partial T'(p, \tilde{f})}{\partial x} = \frac{p}{\alpha f(\alpha)} \int_0^\alpha \frac{\Delta \xi^{n+1}}{(n+1)(2\Delta F(\xi))^{3/2}} \left\{ \frac{3}{2} \frac{\Delta \xi f(\xi)}{\Delta F(\xi)} - (n+2) \right\} du,$$

when $x = 0$ and $T' = 0$. Note that for $p \in \bar{M}, f(\alpha(p)) > 0$ so $\Delta \xi f(\xi)/\Delta F(\xi)$ is bounded if $0 \leqslant u \leqslant \alpha(p)$, by L'Hospital's rule. Thus for large n, $\partial T'/\partial x < 0$ if $p \in \bar{M}$, $T'(p, \tilde{f}) = 0$, and $x = 0$. Thus $\partial T'/\partial x < 0$ for $p \in \bar{M}$, $T'(p, \tilde{f}) = 0$, and $|x| < \varepsilon$ for some $\varepsilon > 0$. Thus 0 is a regular value of θ, and so $\theta^{-1}(0)$ is a C^1 curve C in $M \times (-\varepsilon, \varepsilon)$. Note that if x_0 is a regular value of the projection map $\pi: C \to (-\varepsilon, \varepsilon)$, then the map $p \to T'(p, \tilde{f})$ has 0 as a regular value for $p \in M$. But by Sard's theorem, the regular values of π are dense in $(-\varepsilon, \varepsilon)$ so we may choose x_0, a regular value of π sufficiently small so that $\tilde{f} \in \mathscr{O}$ and $H(\tilde{f})(p) > 0$ if $H(f)(p) \geqslant \frac{1}{2}$. Then $H(\tilde{f})(p) > 0$ for $p \in M$ and for $H(f_0(p) > \frac{1}{2}$; that is, for any $p \in [k^{-1}, k]$ and, hence, $\tilde{f} \in \mathscr{G}_k^{\mathscr{U}} \cap \mathscr{O}$. This completes the proof.

COROLLARY 4. *Suppose that* $T(p_i) = L$ *for* $i = 1, 2, 3, \ldots$ *Then* $\alpha(p_i) \to \infty$ *as* $i \to \infty$ *if either*: (a) $f \in \mathscr{G}$ *or* (b) L *is a regular value of* T.

Proof. Suppose that $\alpha(p_i) \leqslant N$ for all i; then $p_i^2 \leqslant \sup\{2|F(u)|: u \leqslant N\}$. Suppose hypothesis (a) holds. Then by Rolle's theorem, there exist $\bar{p}_i$, $p_i < \bar{p}_i < p_{i+1}$ with $T'(\bar{p}_i) = 0$, or T is not defined for $\bar{p}_i$. Since f has only a finite number of zeros on $0 \leqslant u \leqslant N$, it follows that there are only a finite number of $\bar{p}_i$ in the latter class. Let U be a neighborhood of $P = \{p_i : i \in \mathbb{Z}_+\}$ with $T' \neq 0$ on $U \backslash P$, and let $c = \lim p_i$. Then $T|(C \backslash U)$ is a Morse function and has only finitely many points where $T' = 0$; thus P is finite. In case (b), we approximate f by $\tilde{f}$ such that L is a regular value for $T(\cdot, \tilde{f})$, and $\tilde{f} \in \mathscr{G}$, and then use part (a). This is possible since P is a compact subset of domain (T). Thus we may say that $\alpha(p_i) \to \infty$ as $i \to \infty$ for generic f and generic L.

3. REMARKS AND APPLICATIONS

(1) If we consider the Neumann problem, we may write (cf. [4]), $T(p) = T_1(p) + T_2(p)$, where T_2 is the time map we have just considered, and $T_1(p) = \inf\{t: \phi_{-2t}(p) = (0, -p)\}$. Defining the map θ as in the proof of the theorem, (and setting $b_n(u) = 0$ for $u \leqslant 0$) we see that T_1 is independent of x so that $\partial T/\partial x = \partial T_2/\partial x$, 0 is again a regular value, and thus our result also holds for Neumann boundary conditions.

(2) When T is a Morse function, the critical points of T can accumulate, a priori, only at 0 or at $\beta(f)$. In fact, if $f(0) > 0$, one can define $T(0) = 0$ and then T is differentiable from the right with $T'(0) \neq 0$. If $f(0) < 0$, one again has $T'(0) \neq 0$. If $f(0) = 0$ and $f'(0) > 0$, then generically, in the C^2 topology, $T'(0) \neq 0$ by the results in [3]. If $f(0) = 0$ and $f'(0) < 0$ then, barring a saddle to saddle connection, $T'(0) \neq 0$ and, hence, generically for the Dirichlet problem, T is a Morse function on $[0, \beta(f))$ minus a discrete set. Note however, if $\beta(f) < \infty$, and $\alpha(\beta(f)) < \infty$ this discrete set is finite since f can have only a finite number of critical points on $[0, \alpha(\beta(f))]$ for $f \in A$. If, however, $\beta(f) < \infty$ and $\alpha(\beta(f)) = \infty$, then the critical points of T

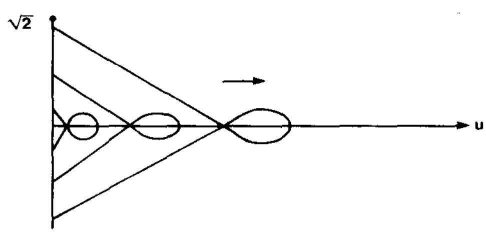

FIGURE 1

can accumulate at β. For example, if $F(u)=(1-e^{-u})\cos u$, we find $\beta=\sqrt{2}$ and $\alpha(\sqrt{2})=\infty$, see Fig. 1. Thus $T'(p)\not\to\infty$ as $p\to\beta-$. This example contradicts Lemma 4.2(ii) of [1].

Note also that the perturbation which we have defined in the proof of the theorem does not affect $f(0)$.

(3) For the Neumann problem, critical points of T may accumulate at 0 if $f(0)=0$, but, using the results in [4], these can be generically eliminated in the C^3 topology (cf. [2]), by using a cubic approximation of f near 0.

(4) We turn now to some new applications.

PROPOSITION 5. *Suppose that* $\overline{\lim}\, F(u)/u^2\leqslant 0$ *or* $\lim |F(u)|/u^2=0$ *as* $u\to\infty$, *where* $F'=f$. *Then the equation* $T(p)=L$ *has at most a finite number of solutions p for generic L or generic f.*

Proof. Let $T(p_i)=L$, $\alpha_i=\alpha(p_i)$; then $\lim\alpha_i=\infty$ by Corollary 4. But

$$\sqrt{2}\,T(p_i)=\int_0^{\alpha_i}\frac{du}{(F(\alpha_i)-F(u))^{1/2}}$$
$$=\int_0^{\alpha_i}\frac{1}{\sqrt{(F(\alpha_i)-F(u))/\alpha_i-u}}\,\frac{du}{\sqrt{\alpha_i-u}}=\frac{2\sqrt{\alpha_i}}{\sqrt{f(\xi_i)}}$$

for $0<\xi_i<\alpha$, by the mean-value theorem. In case (a),

$$\frac{\alpha_i}{f(\xi_i)}=\frac{\alpha_i}{\xi_i}\,\frac{\xi_i}{f(\xi_i)}\to\infty\qquad\text{as}\quad i\to\infty.$$

In case (b), we have, for $0<\xi_i<\alpha_i$,

$$\sqrt{2}\,T(p_i)=\int_0^{\alpha_i}\frac{du}{\alpha_i\sqrt{F(\alpha_i)-F(u)}}=1/\sqrt{F(\alpha_i)-F(\xi_i)}\to\infty$$

as $i\to\infty$.

COROLLARY 6. *If f is any polyonomial then* (a) *for generic L,* $T(p)=L$ *has only a finite number of solutions; and* (b) *if* $f\neq\lambda^2u$, *then f may be approximated arbitrarily close in* C^2 *by an* $\tilde f$ *with* $T(p,\tilde f)=L$ *having only a finite number of solutions for any L.*

Proof. (a) If f is not linear, then $2F(u)-uf(u)$ is monotone for large u and so from (2) $T'\neq 0$ for large α. Also, $f(\alpha)\neq 0$ for large α so we cannot have $T(\alpha_i)=T(\alpha_j)$ for $\alpha_i,\alpha_j\geqslant N$. Thus the result follows from Corollary 4. If f is linear, then $T(\cdot,f)\equiv c$, and so $T(p)=L$ has no solutions if $L\neq c$.

(b) Choose $\tilde f\in\mathscr{G}$ near f. Then for large u we still have $\tilde f(u)\neq 0$, and $2\tilde F(u)-u\tilde f(u)$ is monotone. The conclusion again follows from Corollary 4.

Note that $\lambda^2 u$ can be approximated by, say $\lambda^2 u + \varepsilon(1 + u^2)^{-1}$ which has a monotone time map T; thus $T(p) = L$ has at most one solution. But this approximation is not small in the C^2 Whitney topology. We do not know if $\lambda^2 u$ can be approximated arbitrarily close in $C^2(\mathbb{R}_+)$ with $T(p) = L$ having only a finite number of solutions for all L.

Finally, we remark that these last results enable us to generically give a complete qualitative description of all the solutions of the associated parabolic equation $u_t = u_{xx} + f(u)$, having homogeneous Dirichlet boundary conditions. This follows from Theorem 24.15 of [5].

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