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FRICTIONAL SLIP AND SEPARATION IN THE TRANSONIC RANGE CAUSED BY A PLANE STRESS PULSE

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Localized frictional slip and separation slip between two contacting solids caused by an incident plane elastic wave of arbitrary form was studied in reference [1]. In that paper the angle of incidence of the wave was restricted to produce a disturbance propagating along the interface with supersonic speed. The results were obtained in closed form and convenient graphical representations were given. Here the investigation is continued in the transonic range by using a dislocation formulation that leads to singular integral equations. Examples of localized slip, separation–slip and slip–separation–slip are given for identical materials. Some of the results are in closed form, while other results are obtained by numerical integration.

1. INTRODUCTION

When an incident plane stress pulse strikes the frictional interface between two solids in contact, a disturbance consisting of localized slip and separation zones may propagate along the interface. The speed v with which the disturbance propagates is

$$v = c_0/\sin\theta_0, \tag{1}$$

where c_0 is the phase velocity of the incident wave and θ_0 the angle of incidence. For identical materials v is either supersonic ($v > c_L$) or transonic ($c_T < v < c_L$, $c_0 = c_T$). Localized separation of a *frictionless* interface in the supersonic and transonic range was studied in references [2] and [3]. The effect of friction was considered in reference [1] for the supersonic range. Here the investigation of reference [1] is continued in the transonic range, both slip and separation being considered. The formulation resembles that of reference [3]. Solids with dissimilar material properties have been considered in references [1] and [2].

2. FORMULATION

The geometry of the problem with the incident plane wave and the applied normal and shear tractions p^∞ and q^∞ is shown in Figure 1. The elastic constants and wave speeds are denoted by λ, μ, c_L and c_T in the usual manner. If slip and separation are not considered, the normal and shear tractions transmitted by the interface due to the pulse only are [4]

$$\sigma_{yy}(\eta) = \mathscr{A}_0 f(\eta), \qquad \sigma_{xy}(\eta) = -\mathscr{A}_0 \cot(2\theta_0) f(\eta), \tag{2, 3}$$

where $f(\eta)$ is a normalized function determining the shape of the pulse, $\mathscr{A}_0$ is a real constant with dimensions of stress and η is a dimensionless co-ordinate moving with velocity v. To prevent global slip the restriction $|q^\infty| < fp^\infty$ ($p^\infty > 0$) is imposed, where f

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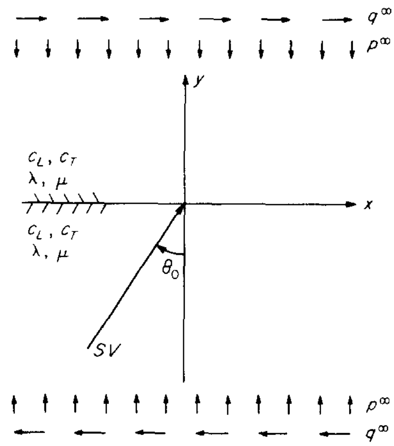

Figure 1. Geometry of the problem.

is the coefficient of friction. No separation will occur if $p^\infty > \mathscr{A}_0$, it being assumed that $\max f(\eta) = 1$.

To allow for the possibility of separation and slip one can introduce arrays of climb and glide dislocations with densities $B_y(\eta)$ and $B_x(\eta)$, moving along the interface with speed v. The dislocation densities are related to the gap $g(\eta)$ and relative tangential shift $h(\eta)$ by

$$B_y(\eta) = -\mathrm{d}g/\mathrm{d}\eta, \qquad B_x(\eta) = -\mathrm{d}h/\mathrm{d}\eta. \tag{4, 5}$$

The tractions due to the dislocation arrays have been given in reference [5] as

$$\sigma_{xy}^D(\eta) = -\frac{\mu(1-\zeta_T^2)^2}{2(1+\zeta_T^2)\zeta_T} B_x(\eta) + \frac{2\mu}{\pi} \frac{\zeta_L^*}{1+\zeta_T^2} \int_{-\infty}^{\infty} \frac{B_x(\xi)}{\eta - \xi} \mathrm{d}\xi, \tag{6}$$

$$\sigma_{yy}^D(\eta) = -\frac{2\mu\zeta_T}{1+\zeta_T^2} B_y(\eta) - \frac{\mu}{2\pi} \frac{(1-\zeta_T^2)^2}{(1+\zeta_T^2)\zeta_L^*} \int_{-\infty}^{\infty} \frac{B_y(\xi)}{\eta-\xi} \mathrm{d}\xi, \tag{7}$$

where

$$\zeta_T = \{(v^2/c_T^2) - 1\}^{1/2}, \qquad \zeta_L^* = \{1 - v^2/c_L^2\}^{1/2}. \tag{8}$$

The total tractions at the interface are then

$$N(\eta) = -p^\infty + \mathscr{A}_0 f(\eta) + \sigma_{yy}^D(\eta), \qquad S(\eta) = q^\infty - \mathscr{A}_0 \cot(2\theta_0) f(\eta) + \sigma_{xy}^D(\eta). \tag{9, 10}$$

3. SLIP ONLY

The case of a slip zone only is shown in Figure 2(a). Then

$$B_y(\eta) = 0. \tag{11}$$

For Coulomb's law of friction, the boundary conditions are

$$N(\eta) \leq 0, \quad -\infty < \eta < \infty, \qquad |S(\eta)| = f|N(\eta)|, \quad \alpha < \eta < \beta, \tag{12, 13}$$

$$|S(\eta)| < |N(n)|, \quad \eta > \beta, \eta < \alpha, \qquad \operatorname{sgn} S(\eta) = \operatorname{sgn} V(\eta), \quad \alpha < \eta < \beta, \tag{14, 15}$$

where $V(\eta)$ is the slip velocity,

$$V(\eta) = \mathrm{d}h/\mathrm{d}t = k_0 c_T B_x(\eta), \tag{16}$$

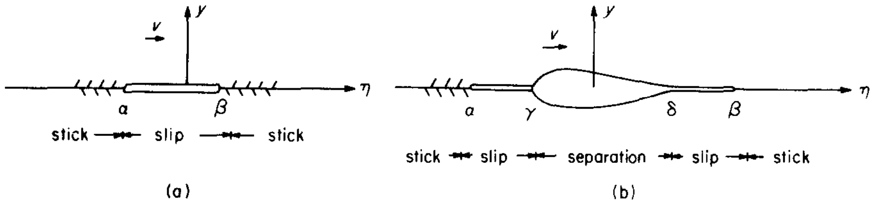

Figure 2. Arrangement of zones.

and k_0 is a normalizing constant with dimension $[L]^{-1}$, characteristic of the pulse. Introducing

$$\rho = f \operatorname{sgn} V(\eta) \tag{17}$$

one can replace expression (14) by

$$S(\eta) = -\rho N(\eta). \tag{18}$$

By using equations (6), (9), (10) and (11), equation (18) can be written as

$$B_x(\eta) + \frac{1}{\pi}\frac{4\zeta_T\zeta_L^*}{(1-\zeta_T^2)^2}\int_\alpha^\beta \frac{B_x(\xi)}{\xi-\eta}\,\mathrm{d}\xi$$
$$= \frac{2\zeta_T(1+\zeta_T^2)}{\mu(1-\zeta_T^2)^2}\{q^\infty - \mathscr{A}_0 \cot(2\theta_0) f(\eta) + \rho[\mathscr{A}_0 f(\eta) - p^\infty]\}, \qquad \alpha < \eta < \beta. \tag{19}$$

The solution of equation (19) bounded at both ends is [6]

$$B_x(\eta) = \frac{2\zeta_T(1+\zeta_T^2)}{\mu(1-\zeta_T^2)^2}\cos^2(\pi A)\mathscr{A}_0$$
$$\times\left\{G(\eta) + \frac{\tan(\pi A)}{\pi}\,w(\eta)\int_\alpha^\beta \frac{G(\xi)\,\mathrm{d}\xi}{(\xi-\eta)w(\xi)}\right\}, \qquad \alpha<\eta<\beta, \tag{20}$$

$$G(\eta) = (1/\mathscr{A}_0)\{q^\infty - \rho p^\infty + \mathscr{A}_0[\rho - \cot(2\theta_0)]f(\eta)\}, \qquad w(\eta) = (\beta-\eta)^A(\eta-\alpha)^{1-A}, \tag{21, 22}$$

$$A = (1/\pi)\tan^{-1}[-4\zeta_T\zeta_L^*/(1-\zeta_T^2)^2], \qquad \tfrac{1}{2} < A < 1. \tag{23}$$

The total shift caused by slip is

$$D = k_0\int_\alpha^\beta B_x(\xi)\,\mathrm{d}\xi \tag{24}$$

and the condition for bounded $B_x(\eta)$ (consistency condition) is

$$\int_\alpha^\beta \frac{G(\xi)}{w(\xi)}\,\mathrm{d}\xi = 0. \tag{25}$$

Substituting equation (20) into equation (24) gives

$$D = \frac{2\zeta_T(1+\zeta_T^2)}{\mu(1-\zeta_T^2)^2}\mathscr{A}_0 k_0 \cos(\pi A)\int_\alpha^\beta \frac{\xi G(\xi)}{w(\xi)}\,\mathrm{d}\xi. \tag{26}$$

Inequalities (14), (15) and equation (25) are sufficient for the determination of the endpoints α and β of the slip zone. The results can be illustrated by an example. For a parabolic pulse

$$f(\eta)=\begin{Bmatrix} 1-\eta^2, & |\eta|\leq 1 \\ 0, & |\eta|>1 \end{Bmatrix}, \tag{27}$$

as in reference [3], and the case in which the slip zone is completely contained within the pulse extent ($|\alpha|, |\beta|<1$), the integrations can be carried out and the results are in closed form. Equations (25), (26) and (20) become

$$(q^\infty-\rho p^\infty)/\mathscr{A}_0[-\cot(2\theta_0)+\rho]=-1+\alpha^2+2\alpha(\beta-\alpha)A+\tfrac{1}{2}(\beta-\alpha)^2A(1+A), \tag{28}$$

$$D=-[2\zeta_T(1+\zeta_T^2)/\mu(1-\zeta_T^2)^2][-\cot(2\theta_0)+\rho]\mathscr{A}_0k_0A(1-A)(\beta-\alpha)$$
$$\times[\alpha+\tfrac{1}{3}(1+A)(\beta-\alpha)], \tag{29}$$

$$B_x(\eta)=[2\zeta_T(1+\zeta_T^2)/\mu(1-\zeta_T^2)^2][-\cot 2\theta_0+\rho]\mathscr{A}_0\cos(\pi A)w(\eta)$$
$$\times[\eta+\alpha+(\beta-\alpha)A], \qquad \alpha<\eta<\beta. \tag{30}$$

The normal tractions are

$$N(\eta)=-p^\infty+\mathscr{A}_0 f(\eta), \qquad -\infty<\eta<\infty, \tag{31}$$

and the shear tractions in the stick region are

$$S(\eta)=q^\infty-\mathscr{A}_0\cot(2\theta_0)f(\eta)+\mathscr{A}_0[-\cot(2\theta_0)+\rho]$$
$$\times\left\{-1+\eta^2-\frac{q^\infty-\rho p^\infty}{\mathscr{A}_0[-\cot(2\theta_0)+\rho]}-(\eta-\alpha)[\eta+\alpha+(\beta-\alpha)A]\left|\frac{\eta-\beta}{\eta-\alpha}\right|^A\right\},$$
$$\eta>\beta,\ \eta<\alpha. \tag{32}$$

Checking inequalities (14) and (15) shows that they are contradictory unless

$$2\alpha+(\beta-\alpha)A=0. \tag{33}$$

Therefore, β is given by

$$\beta=-\alpha(2-A)/A, \qquad \alpha\leq 0. \tag{34}$$

Substituting expression (34) into equations (28) and (30) gives

$$(q^\infty-\rho p^\infty)/\mathscr{A}_0[-\cot(2\theta_0)+\rho]=-1+\alpha^2(2-A)/A \tag{35}$$

$$D=-\tfrac{4}{3}[\zeta_T(1+\zeta_T^2)/\mu(1-\zeta_T^2)^2]\,k_0\mathscr{A}_0[-\cot(2\theta_0)+\rho][\alpha^2(2-A)/A](1-A) \tag{36}$$

$$=-\tfrac{4}{3}[\zeta_T(1+\zeta_T^2)/\mu(1-\zeta_T^2)^2]\,k_0(1-A)[-\mathscr{A}_0\cot(2\theta_0)+q^\infty+\rho(\mathscr{A}_0-p^\infty)], \tag{37}$$

$$B_x(\eta)=-[2\zeta_T(1+\zeta_T^2)/\mu(1-\zeta_T^2)^2]\,\mathscr{A}_0[-\cot(2\theta_0)+\rho]\cos(\pi A)$$
$$\times(\beta-\eta)^A(\eta-\alpha)^{2-A}, \qquad \alpha<\eta<\beta. \tag{38}$$

The parameter α is now determined from equation (35). The shear tractions in the slip zones can also be simplified by using equation (35):

$$S(\eta)=q^\infty-\mathscr{A}_0\cot(2\theta_0)f(\eta)+\mathscr{A}_0[-\cot(2\theta_0)+\rho]$$
$$\times[\eta^2+\alpha\beta-|\eta-\beta|^A|\eta-\alpha|^{2-A}], \qquad \eta>\beta,\ \eta<\alpha. \tag{39}$$

It can be verified from equation (39) that the shear tractions tend to q^∞ as $\eta\to\infty$. It is also clear from equation (39) that the shear tractions have a continuous derivative at the trailing edge of the slip zone and a discontinuous derivative at the leading edge, confirming results obtained in quasistatic situations [7].

The case $|\alpha|>1$ can be treated similarly by numerical evaluation of the integrals involved, as in the next section, but it is of no particular interest and it is not pursued further.

4. SLIP AND SEPARATION

For the case of slip and separation one can consider a tensile pulse with amplitude $\mathscr{A}_0 > p^\infty$ so that separation occurs in the interval $\gamma<\eta<\delta$ (see Figure 2(b)). One must require

$$N(\eta)=0, \qquad \gamma<\eta<\delta, \qquad N(\eta)\leqslant 0, \qquad \eta<\gamma,\ \eta>\delta. \tag{40, 41}$$

In addition, the boundary conditions (14), (15) and (18) of the previous section remain valid. Although the slip zone depends on the separation zone, the latter can be determined independently and the results are identical to those of reference [3], where the frictionless interface was considered. For the case in which the separation zone is contained within the pulse extent ($|\gamma|$, $|\delta|<1$), one obtains closed form expressions. The results for a parabolic pulse (from reference [3], with minor notation modifications) are

$$\delta=[(1-C)/(2+C)]\gamma, \qquad \gamma\leqslant 0, \qquad p^\infty/\mathscr{A}_0=1-[(1-C)/2(2+C)]\gamma^2 \tag{42, 43}$$

$$N(\eta)=-\mathscr{A}_0\hat{f}(\eta)-\mathscr{A}_0\left|\frac{\eta-\delta}{\eta-\gamma}\right|^C\left[\eta^2-\frac{3C}{2+C}\gamma\eta-\frac{(1+C)(1+2C)}{(2+C)^2}\gamma^2\right], \qquad \eta>\delta,\ \eta<\gamma, \tag{44}$$

$$\text{where } C=A-1/2,\ \hat{f}(\eta)=1-\eta^2-f(\eta) \tag{45, 46}$$

and $f(\eta)$ is given by equation (27). This solution is valid for

$$3(1+C)/2(2+C)<p^\infty/\mathscr{A}_0<1. \tag{47}$$

For lower applied pressures the solution is obtained numerically [3]. The solution of equation (18) with $N(\eta)$ given by equation (44) is still given by equation (20) but with $G(\eta)$ replaced by $R(\eta)$:

$$R(\eta)=(1/\mathscr{A}_0)[q^\infty-\mathscr{A}_0\cot(2\theta_0)+\rho N(\eta)]. \tag{48}$$

The consistency condition can be solved for q^∞ yielding

$$q^\infty=\frac{\sin(\pi A)}{\pi}\int_\alpha^\beta\frac{-\rho N(\xi)+\mathscr{A}_0\cot(2\theta_0)f(\xi)}{w(\xi)}\,\mathrm{d}\xi. \tag{49}$$

The shear tractions in the stick region are

$$S(\eta)=q^\infty-\mathscr{A}_0\cot(2\theta_0)f(\eta)$$
$$+\mathscr{A}_0\frac{\sin(\pi A)}{\pi}(\eta-\alpha)\left|\frac{\eta-\beta}{\eta-\alpha}\right|^A\int_\alpha^\beta\frac{R(\xi)}{w(\xi)(\xi-\eta)}\,\mathrm{d}\xi, \qquad \eta>\beta, \qquad \eta<\alpha. \tag{50}$$

For the numerical integration, the interval (α, β) is normalized by the change of variables

$$\eta=\tfrac{1}{2}(\beta-\alpha)\,s+\tfrac{1}{2}(\beta+\alpha), \tag{51}$$

and then the Jacobi quadrature given in reference [8] is applied. The discretized expressions of the relevant quantities are

$$B_x(s_k)=\frac{\zeta_T(1+\zeta_T^2)}{\mu(1-\zeta_T^2)^2}\tan(\pi A)\frac{\mathscr{A}_0}{\pi}w(s_k)\sum_{i=1}^{n}\frac{A_i^{(n)}R(u_i)}{u_i-s_k},\qquad k=1,2,\ldots,n-1, \quad (52)$$

$$q^\infty=\frac{\sin(\pi A)}{A}\sum_{i=1}^{n}[-\rho N(u_i)+\mathscr{A}_0\cot(2\theta_0)f(u_i)]A_i^{(n)}, \quad (53)$$

$$S(\eta)=q^\infty-\mathscr{A}_0\cot(2\theta_0)f(\eta)+\frac{\sin(\pi A)}{\pi}(\eta-\alpha)\left|\frac{\eta-\beta}{\eta-\alpha}\right|^A\sum_{i=1}^{n}\frac{A_i^{(n)}R(u_i)}{(\xi_i-\eta)}, \quad (54)$$

where the u_i are the roots of the Jacobi polynomial $P_n^{(-A,-1+A)}$, the $A_i^{(n)}$ are the coefficients of the corresponding quadrature,

$$P_{n-1}^{(A,1-A)}(s_k)=0,\qquad k=1,2,\ldots n-1,\qquad \xi_i=\tfrac{1}{2}(\beta-\alpha)\,u_i+\tfrac{1}{2}(\beta+\alpha), \quad (55,56)$$

and the same symbols are used for the functions in the normalized variables.

As an example, suppose

$$\theta_0=35°,\qquad c_T/c_L=\tfrac{1}{2},\qquad f=0{\cdot}5. \quad (57)$$

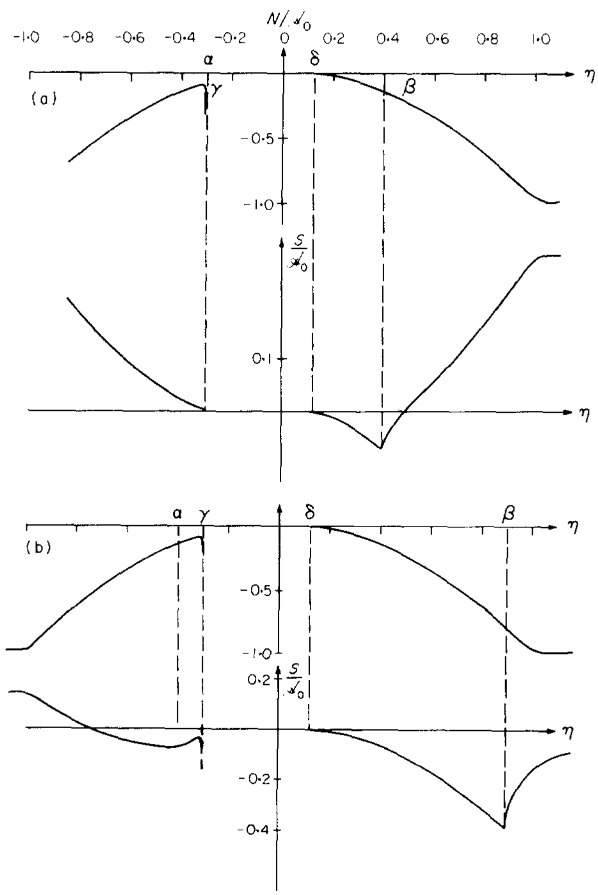

Figure 3. Normal and shear tractions for (a) $\beta=0{\cdot}4$ and (b) $\beta=0{\cdot}9$.

Then $\rho=-f$ and $V(\eta)<0$. This inequality and equation (14) determine α. Equation (53) can be used to determine β; however it is simpler to assign β and compute q^{∞} from equation (53) instead. For a fixed separation zone, it is found that for small β the slip zone extends over to the right of the separation zone, $\alpha=\gamma$, and as β increases another slip zone appears next to trailing edge of the separation zone. In terms of q^{∞} one finds that, for $q^{\infty}>0$, as q^{∞} increases the slip zones decrease and eventually the left slip zone disappears. This is because in the present example slip is dominated by the shear tractions due to the pulse, which are of sign opposite to that of q^{∞}. For $\gamma=-0{\cdot}3$ the left slip zone occurs for $0{\cdot}83<\beta<0{\cdot}84$, and for $\gamma=0{\cdot}1$ for $0{\cdot}27<\beta<0{\cdot}28$. Figures 3(a) and (b) show the normal and shear tractions for $\gamma=-0{\cdot}3$ and $\beta=0{\cdot}4$ and $0{\cdot}9$, respectively.

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