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TWO-LOOP LARGE HIGGS MASS CORRECTION TO THE ρ-PARAMETER

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The two-loop correction to the ρ-parameter in the limit of the large Higgs mass is calculated. Numerically it is found that the two-loop contribution is equal in magnitude, but of opposite sign to the one-loop correction if the Higgs mass is about 137 times the vector boson mass. The calculation suggest a breakdown of perturbation theory if the Higgs mass is larger than 3 TeV. There is no direct correspondence between our results and the poles at $n = 3$ of the simple non-linear σ-model at the two-loop level.

1. Introduction

In considering present day gauge theory one cannot escape the idea that the subject can be divided broadly into two areas: the successful and the not so successful domain. Keywords such as cosmological constant, axions, monopoles, strong CP-violation identify the latter area: it relates to properties of the vacuum and the Higgs system. While there is no hard fact confronting theory and experiment here, it is also true that one has to twist and bend the theory in order to avoid such a conflict. Absurdities like the "invisible axion" have made their appearance in the physics literature, and perhaps some day invisible proton decay will save grand unification!

Now no one knows what is really going on above 100 GeV. At the very least we might say that the above mentioned lack of success makes the Higgs sector suspect. On the other hand, the Higgs is there as a technical means to guarantee renormalizability, and the principle of renormalizability has most certainly proven to be a very successful guideline. It is a very difficult dilemma, and it is improbable that we will get any further without experimental input. It has been argued that some manifestation of the Higgs system, or whatever goes for it, must show up at around 1 TeV or below [1]. Thus a successful quest requires accelerators in the multi-TeV domain.

For some time to come most of the experimental information will come from the experiments at relatively low energies, that is below 100 GeV. This may or may not give clues to our present day problems.

As it happens there is at least one measurement that may give us some clue, and that is a high-precision measurement of the ratio of charged and neutral vector boson masses, nowadays called the ρ-parameter. Several years ago [2] it was pointed out that this parameter is sensitive to the mass spectrum above 100 GeV, and also it is the only measurable quantity at low energy that is really finite (up to one loop) due to the Higgs system. Unfortunately even that one-loop dependence is only logarithmic, and only a very brutal deviation from the currently envisaged Higgs systems would influence this quantity in a measurable way.

However, the situation changes when including two-loop contributions. They are expected to grow like m^2 (where m is the Higgs mass), and may therefore be of more relevance. For one thing, if they become larger than the one-loop contribution, then perturbation theory is no more valid, and any experimental observation along these lines would be extremely important. Here we are talking about minute effects: of the order of 0.1% on the ρ-parameter.

The substance for the argument that the two-loop corrections grow like m^2 for large m comes from previous studies on massive Yang-Mills theories [3] and equivalently, the gauged non-linear σ-model [4]. The non-linear σ-model is what the Higgs sector of the standard model becomes if one lets the Higgs mass m go to infinity at the lagrangian (= tree) level. This non-linear σ-model is non-renormalizable, and gives rise to infinities in the ρ-parameter: logarithmic (simple poles in $n-4$ using dimensional regularization) at the one-loop level and quadratic (poles at $n-3$) at the two-loop level. At the one-loop level the coefficients of the simple pole terms are precisely equal to the coefficients of the terms proportional to $\ln m$ in the standard model [1,4], and it is on this basis that one guesses a similar relation on the two-loop level.

An explicit calculation of the large Higgs mass effects at the two-loop level may therefore be of both direct experimental and more indirect theoretical interest. Are there really other than logarithmic effects if the Higgs mass is made large? If there are only logarithmic effects then this may well change our view on the necessity of the Higgs system. Of course, there is still the question of uncontrollable growth of the W-W scattering amplitude at the tree-level (the so-called unitarity limit), but this problem is not really fully analyzed and understood. The only thing one can say is that higher-order effects become important at this energy if the Higgs mass is large, but this means that the perturbation expansion breaks down, and not that the Higgs mass cannot be large.

In this paper we present the results of a calculation of large Higgs mass effects to the ρ-parameter at the two-loop level. The work itself is quite substantial, and requires also a precise understanding of the complexities of gauge theory renormalization. Most things turn out the way one would expect from the beginning, with one exception: while the two-loop behaviour is indeed like m^2, it is clearly not related in any obvious way to the non-linear σ-model. This aspect, very interesting and intriguing, will not be discussed in any detail in this paper, although we will point out where the origin is of this anomalous behaviour.

In sect. 2 the model and our notations are defined. In sect. 3 the essential ingredients of the necessary one-loop calculations are presented. Sect. 4 contains a discussion of the renormalization procedure that we adopted, and in sect. 5 the required calculations are outlined. In sect. 6 details of the evaluation of two-loop self-energy diagrams are given, and in sect. 7 reducible and counterterm contributions are discussed, including a more detailed discussion of quadratic counterterms. In sect. 8 results are summarized, in particular the correction to the ρ-parameter. Appendix A gives some details on the evaluation of the occurring integrals, and the necessary equations are listed.

While large parts of the calculation were done by hand also, we have relied heavily on the use of the computer program Schoonschip. Appendix B gives some details.

2. The model

This section can be brief. We use the lagrangian and notation as given in ref. [5]. There is only one point on which we want to elaborate, and that is the gauge fixing terms and the gauge transformations leading to the ghost lagrangian as given in that paper. The gauge fixing term is*

$$\mathcal{L}_{\mathrm{g.f.}} = -C^{+}C^{-} - \tfrac{1}{2}(C^{3})^{2} - \tfrac{1}{2}(C^{A})^{2}. \tag{2.1}$$

With (C^{-} follows by hermitian conjugation):

$$C^{+} = -\partial_{\mu}W_{\mu}^{+} + M\phi^{+},$$

$$C^{3} = -\partial_{\mu}W_{\mu}^{0} + \frac{M}{c}\phi^{0},$$

$$C^{A} = -\partial_{\mu}A_{\mu}. \tag{2.2}$$

In this paper we will drop the subscript θ from c_{θ} and s_{θ}, referring to cos and sin of the weak mixing angle. The ghost lagrangian follows by subjecting the C to an infinitesimal gauge transformation. Thus:

$$\mathcal{L}_{\mathrm{ghost}} = +\partial_{\mu}\overline{X^{+}}\left(\delta W_{\mu}^{+}\right) + M\overline{X^{+}}\left(\delta\Phi^{+}\right) + \partial_{\mu}\overline{X^{-}}\left(\delta W_{\mu}^{-}\right) + M\overline{X^{-}}\left(\delta\phi^{-}\right)$$
$$+\partial_{\mu}\overline{Y^{0}}\left(\delta W_{\mu}^{0}\right) + \frac{M}{c}\overline{Y^{0}}\left(\delta\phi^{0}\right) + \partial_{\mu}\overline{Y^{A}}\left(\delta A_{\mu}\right). \tag{2.3}$$

* For an unknown reason these terms were misrepresented in ref. [5].

The quantities δW etc. follow by considering an infinitesimal gauge transformation with gauge parameters Λ:

$$\begin{aligned}
\delta W_\mu^+ &= ig\Lambda^+\left(cW_\mu^0 + sA_\mu\right) - ig(c\Lambda^W + s\Lambda^A)W_\mu^+ - \partial_\mu\Lambda^+ ,\\
\delta W_\mu^0 &= -igc\Lambda^+ W_\mu^- + igc\Lambda^- W_\mu^+ - \partial_\mu\Lambda^W,\\
\delta A_\mu &= -igs\Lambda^+ W_\mu^- + igs\Lambda^- W_\mu^+ - \partial_\mu\Lambda^A,\\
\delta\phi^+ &= \tfrac{1}{2}ig\Lambda^+\phi^0 - \tfrac{1}{2}ig\left(\frac{c^2-s^2}{c}\Lambda^W + 2s\Lambda^A\right)\phi^+ - \tfrac{1}{2}gZ\Lambda^+ - M\Lambda^+ ,\\
\delta\phi^0 &= -\tfrac{1}{2}ig(\Lambda^+\phi^- - \Lambda^-\phi^+) - \frac{g}{2c}\Lambda^W Z - \frac{M}{c}\Lambda^W,\\
\delta Z &= \tfrac{1}{2}g(\Lambda^+\phi^- + \Lambda^-\phi^+) + \frac{g}{2c}\phi^0\Lambda^W.
\end{aligned} \tag{2.4}$$

Note that Z represents the Higgs field, and W^0 the neutral vector boson (often called the Z in the literature). In the above equation one must replace Λ^+ by X^+, Λ^- by X^-, Λ^W by Y^0 and Λ^A by Y^A to obtain the correct ghost lagrangian. The gauge parameters Λ^+, Λ^-, Λ^W and Λ^A are related to the pure SU(2) and U(1) gauge parameters Λ^i and Λ^0 as follows:

$$\begin{aligned}
\Lambda^{\mp} &= \sqrt{\tfrac{1}{2}}\,(\Lambda^1 \mp i\Lambda^2),\\
\Lambda^W &= c\Lambda^3 - s\Lambda^0,\\
\Lambda^A &= s\Lambda^3 + c\Lambda^0.
\end{aligned} \tag{2.5}$$

This is similar in structure to the equation for the vector bosons, see e.g. (2.1) of ref. [5].

The fermion sector is relevant insofar that we will define the ρ-parameter on the basis of low-energy processes such as μ-decay, as described in an earlier paper [2]. In practice no one or two-loop graphs involving fermions need to be calculated, as will be shown later. This is a consequence of the particular procedure followed in this paper. We will work in the limit of small fermion mass, thus neglecting diagrams with the Higgs coupled to the fermions. If ever fermions with a mass comparable to the vector boson mass are discovered then such diagrams must be included.

3. One-loop calculations

As a first step we have calculated:

(i) all infinities (poles in $n-4$) of all one-loop graphs but without fermion internal lines;

(ii) all terms behaving like m^2 of all self-energy graphs (with some exceptions for the Higgs self-energy graphs), including terms proportional to $n-4$;

(iii) the complete set of tadpole diagrams but without fermion internal lines.

The terms proportional to $n-4$ are important, because at the two-loop level they may combine with poles to give a finite contribution. Since we will ignore Higgs-fermion couplings we do not need such terms for the Higgs self-energy diagrams. That we may ignore diagrams with internal fermions will become clear in the following.

Infinities and terms proportional to m^2 are unobservable at the one-loop level* and can thus be absorbed in the parameters of the lagrangian. The most transparent way to represent the results is as follows. Replace in the invariant lagrangian the various fields and parameters as follows:

$$\begin{aligned}
&W_\mu^{\pm} \to W_\mu^{\pm}(1+\delta_c),\\
&W_\mu^0 \to W_\mu^0(1+\delta_0) + \delta_{0A} A_\mu,\\
&A_\mu \to A_\mu(1+\delta_A) + \delta_{A0} W_\mu^0,\\
&\phi \to \phi(1+\delta_Z), \qquad Z \to Z(1+\delta_Z) + \frac{M}{g}\delta_t,\\
&M \to M(1+\delta_M), \qquad m \to m(1+\delta_m),\\
&c \to c(1+\delta_{co}), \qquad g \to g(1+\delta_g).
\end{aligned} \tag{3.1}$$

Instead of δ_M and δ_m we will use quantities δ_1 and δ_2, related as follows:

$$\delta_1 \equiv \delta_M + \tfrac{1}{2}\delta_t, \qquad \delta_2 \equiv \delta_M - \delta_m. \tag{3.2}$$

The immediate advantage of this notation is the simplification in those terms that depend on m^2/M^2.

The terms mentioned in (i) to (iii) are *reproduced* by using the values:

$$\delta_1 = \left(\frac{1}{2c^2} - \tfrac{31}{12}\right)\Delta + \delta_{1f},$$

$$\delta_2 = \left(-\tfrac{25}{12} + \frac{3}{4c^2} - \frac{3M^2}{2m^2} - \frac{3M^2}{4c^4m^2} - \frac{3m^2}{4M^2}\right)\Delta + \delta_{2f},$$

$$\delta_c = \tfrac{19}{12}\Delta, \qquad \delta_0 = \left(\tfrac{1}{6} + \tfrac{3}{2}c^2 - \frac{1}{12c^2}\right)\Delta, \qquad \delta_{0A} = -2cs\Delta,$$

$$\delta_A = \tfrac{3}{2}s^2\Delta, \qquad \delta_{A0} = \left(5sc + \frac{s}{6c}\right)\Delta,$$

* Except for diagrams with external Higgs lines.

$$\delta_Z = \left(\tfrac{1}{2} + \frac{1}{4c^2}\right)\Delta + \delta_{Z\mathrm{f}},$$

$$\delta_t = \left(\frac{3M^2}{2c^4m^2} + \frac{1}{4c^2} + \tfrac{1}{2} + 3\frac{M^2}{m^2} + \frac{3m^2}{4M^2}\right)\Delta + \delta_{t\mathrm{f}},$$

$$\delta_g = -\tfrac{43}{12}\Delta, \qquad \delta_{co} = -\frac{s^2(43 - 42s^2)}{12c^2}\Delta, \qquad \Delta = \frac{g^2}{8\pi^2(n-4)}. \tag{3.3}$$

The finite quantities $\delta_{1\mathrm{f}}$, $\delta_{2\mathrm{f}}$, $\delta_{Z\mathrm{f}}$, and $\delta_{t\mathrm{f}}$, contain no pole terms, and are related to the calculations (ii) and (iii) mentioned above. In order to correctly reproduce the one-loop results one must take:

$$\delta_{1\mathrm{f}} = \delta_{Z\mathrm{f}} = \frac{m^2g^2}{8\pi^2M^2}\left\{\tfrac{1}{32} + \varepsilon\left(\tfrac{1}{64}\log m^2 - \tfrac{3}{128}\right)\right\},$$

$$\delta_{2\mathrm{f}} = \frac{m^2g^2}{8\pi^2M}\left\{-\frac{9\pi}{32\sqrt{3}} + \tfrac{25}{32} - \tfrac{3}{8}\log m^2\right\},$$

$$\begin{aligned}
\delta_{t\mathrm{f}} = \frac{g^2}{8\pi^2}\Bigg\{ & -\frac{M^2}{4m^2c^4} - \frac{1}{8c^2} - \tfrac{1}{4} - \frac{M^2}{2m^2} - \frac{3m^2}{8M^2} + \left(\tfrac{1}{4} + \frac{3M^2}{2m^2}\right)\log M^2 \\
& + \left(\frac{1}{8c^2} + \frac{3M^2}{4c^4m^2}\right)\log\frac{M^2}{c^2} + \frac{3m^2}{8M^2}\log m^2\Bigg\} \\
- \frac{1}{8\pi^2}g^2\varepsilon\Bigg[& \pi^2\left(-\frac{M^2}{32c^4m^2} - \frac{1}{192c^2} - \tfrac{1}{96} - \frac{M^2}{16m^2} - \frac{m^2}{64M^2}\right) \\
& - \frac{M^2}{8c^4m^2} - \frac{1}{16c^2} - \tfrac{1}{8} - \frac{M^2}{4m^2} - \frac{3m^2}{16M^2} \\
& + \left(\tfrac{1}{8} + \frac{M^2}{4m^2}\right)\log M^2 - \left(\tfrac{1}{16} + \frac{3M^2}{8m^2}\right)\log^2 M^2 \\
& + \left(\frac{M^2}{8c^4m^2} + \frac{1}{16c^2}\right)\log\frac{M^2}{c^2} - \left(\frac{1}{32c^2} + \frac{3M^2}{16m^2c^4}\right)\log^2\frac{M^2}{c^2} \\
& + \frac{3m^2}{16M^2}\log m^2 - \frac{3m^2}{32M^2}\log^2 m^2\Bigg].
\end{aligned} \tag{3.4}$$

In the above $\varepsilon = n - 4$. As said before, taking these expressions as given, the one-loop results are reproduced. Giving these terms a minus sign leads to the cancellation of all such terms at the one-loop level.

Concerning δ_{2f}, essentially referring to the renormalization of the Higgs mass m, the following must be noted. In this quantity we have kept only the terms proportional to m^2, and these terms are chosen such that if δ_{2f} is introduced in the lagrangian with a minus sign then a calculation of the physical Higgs mass (the location of the pole of the Higgs propagator) gives the result $m_{\mathrm{p.h.}} = m$ up to leading order in the parameter m.

The infinities in the ghost sector are reproduced as follows. Use the same gauge fixing terms as before, and do *not* replace the W etc. by $W(1+\delta)$ etc. in these functions. However for the gauge transformations one must use "renormalized" gauge transformations, and some subtleties are to be noted here. Start from the "bare" gauge transformations, for example:

$$W_\mu^0 \to W_\mu^0 + \delta W_\mu^0, \quad A_\mu \to A_\mu + \delta A_\mu .$$

Now make the various replacements, $W = W(1+\delta)$ etc. In the example chosen we get:

$$W_\mu^0(1+\delta_0) + \delta_{0A} A_\mu \to W_\mu^0(1+\delta_0) + \delta_{0A} A_\mu + \delta \overline{W}_\mu^0 ,$$

$$A_\mu(1+\delta_A) + \delta_{A0} W_\mu \to A_\mu(1+\delta_A) + \delta_{A0} W_\mu^0 + \delta \overline{A}_\mu .$$

In here $\delta\overline{W}$ and $\delta\overline{A}$ are obtained from δW and δA by making the various replacements. Working out the two equations above one finds as the new gauge transformations:

$$W_\mu^0 \to W_\mu^0 + (1-\delta_0)\delta\overline{W}_\mu^0 - \delta_{0A}\delta\overline{A}_\mu ,$$

$$A_\mu \to A_\mu + (1-\delta_A)\delta\overline{A}_\mu - \delta_{A0}\delta\overline{W}_\mu^0 . \tag{3.5}$$

Here only terms up to order g^3 need be retained.

Furthermore also replacements must be made for the Λ:

$$\Lambda^W \to (1+\delta_0')\Lambda^W + \delta_{0A}'\Lambda^A ,$$

$$\Lambda^A \to (1+\delta_A')\Lambda^A + \delta_{A0}'\Lambda^W ,$$

$$\Lambda^{\pm} \to (1+\delta_c')\Lambda^{\pm} . \tag{3.6}$$

The quantities δ' must be taken as follows:

$$\delta_0' = \left(\tfrac{1}{6} + \tfrac{5}{2}c^2 - \frac{1}{12c^2}\right)\Delta , \qquad \delta_{0A}' = -cs\Delta ,$$

$$\delta_A' = \tfrac{5}{2}s^2\Delta , \qquad \delta_{A0}' = \left(6cs + \frac{s}{6c}\right)\Delta ,$$

$$\delta_c' = \tfrac{31}{12}\Delta . \tag{3.7}$$

The above demonstrates a known fact [6]: to correctly reproduce (or counter) the one-loop infinites etc. one must replace fields by their "renormalized" values, e.g. $W \to W(1+\delta)$, but not in the gauge conditions. This means that in going from the original lagrangian to the above described one also a change of gauge is involved. To be clear, the starting point is:

$$\mathcal{L} = \mathcal{L}_{\text{inv}}(W, M, \ldots) + \mathcal{L}_{\text{g.f.}}(W, M) + \mathcal{L}_{\text{gh.}}(W, M).$$

Step one, replace W etc. by $W(1+\delta)$. Step two, make a change of gauge that eliminates these changes in the gauge fixing lagrangian. Let us consider in some detail what this means to the mass terms in the ghost lagrangian. The original lagrangian contains the gauge fixing term

$$-\partial_\mu W^+_\mu + M\phi^+,$$

and, concentrating on the second term, a ghost mass term induced by a gauge transformation of ϕ^+:

$$-M^2\bar{\chi}^+\chi^+.$$

Now perform step 1, the replacement $M \to M(1+\delta_M)$, $Z \to Z(1+\delta_Z) + (M/g)\delta_t$, and $\phi \to \phi(1+\delta_Z)$ and also a rescaling of the ghost fields whose precise form is not relevant for this discussion. The gauge condition and the ghost mass term become respectively:

$$M\phi^+(1+\delta_M+\delta_Z), \quad -M^2\bar{\chi}^+\chi^-(1+2\delta_M+\tfrac{1}{2}\delta_t).$$

The δ_t term comes from the term $-\frac{1}{2}gM\bar{\chi}^+\chi^+Z$ in the ghost lagrangian. Now change the gauge condition back to its original value by adding on a term $-(\delta_M+\delta_Z)M\phi^+$ to that gauge condition. As a consequence the ghost mass term changes, we get an addition $-(\delta_M+\delta_Z)M^2\bar{\chi}^+\chi^+$. Including a term δ_c' coming from the ghost rescaling we have for the ghost mass term:

$$-M^2\bar{\chi}^+\chi^+(1+\delta_c'+\delta_M+\tfrac{1}{2}\delta_t-\delta_Z) = -M^2\bar{\chi}^+\chi^+(1+\delta_c'+\delta_1-\delta_Z). \quad (3.8)$$

The choice of δ_1 as a parameter instead of δ_M becomes clear here. Similar things happen all around. Furthermore, it is seen that the ghost mass term changes differently from the W mass term; that latter becomes after the various replacements:

$$-M^2W^-_\mu W^+_\mu \to -M^2W^-_\mu W^+_\mu(1+2\delta_c+2\delta_1).$$

Also here there is a tadpole contribution originating from the W^-W^+Z term.

This is perhaps the right moment to emphasize our strategy. Here we do not choose counterterms in any particular way. We have simply listed what contributions of the types mentioned before occur at the one-loop level. The fact that no m^2

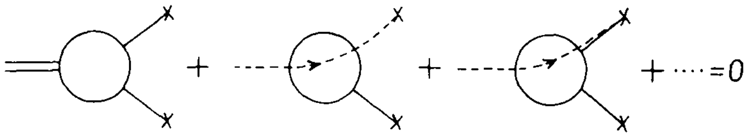

Fig. 1.

terms occur in ghost self-energy diagrams manifests itself here in the fact that the m^2 dependence of δ_1 is equal to that of δ_Z. If the ghost self-energy diagrams would have had such dependences then we would not have been able to reproduce them through a replacement type technique, as done above. The fact that this replacement-gauge-changing technique correctly reproduces all m^2 dependencies shows that at the one-loop level there are no physically observable effects going with m^2.

The latter is actually not strictly true. One-loop diagrams with external Z(= Higgs) lines have m^2 dependencies other than given above in the form of a replacement prescription. The only terms that we have included are those that occur in the Z self-energy diagrams insofar as needed to determine the physical Z-mass (the location of the pole of the propagator).

As noted above, we have explicitly computed the m dependence in self-energy diagrams only. We thus have not explicitly checked that three-point and four-point diagrams have the dependence as suggested by the replacement recipe. That this is true however follows from the Ward identities. As a first step, consider diagrams with three external W-lines. Quite evidently such diagrams have no terms behaving like m^2. Now consider a Ward identity involving two W-sources and one C-line (fig. 1).

Now C involves a W and a ϕ line, thus the left-hand side contains diagrams with three external W-lines as well as with two W-lines and one ϕ line. In addition, there are self-energy type diagrams, such as shown in fig. 2.

If we perform in the lagrangian the various replacements, but with a minus sign, then there is no m^2 dependence in any self-energy diagram, and since there is none to begin with in diagrams with three external W-lines it follows that the total of the diagrams with one ϕ and two W-lines, including counterterms of that type, has no m^2 dependence. This argument can be repeated to cover diagrams with any number of ϕ- and W-lines, but cannot be extended to diagrams with external Z-lines, because for instance diagrams with one Z and two W external lines are not obviously free from m^2 dependence.

4. Renormalization

Questions of convenience and habit determine the renormalization procedure to be followed. The foregoing section is strongly suggestive of the procedure that we followed, which is this.

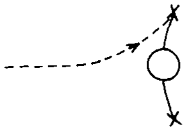

Fig. 2.

The starting point of our two-loop calculations is the lagrangian obtained from the original (bare) lagrangian by performing the various replacements, but with a minus sign, and the described change of gauge. This of course makes no difference as far as the physics is concerned, it only changes the relation between experimental quantities and the parameters in the lagrangian. But this relation becomes in fact simpler, there are no m^2 terms in the relation between the physical W-mass and the parameter M; similarly for the Higgs mass m (at least as far as the relevant leading order in m terms are concerned).

There are some subtleties concerning the replacement recipe. Such replacements generate also quadratic terms, e.g. $W_\mu W_\mu$ becomes $W_\mu W_\mu(1+2\delta_c+\delta_c^2)$. In the WW, WA, and AA terms these quadratic terms, of order g^4, cannot be ignored and must be taken into account.

The lagrangian so obtained has the following virtues:

no m^2 dependence on the one-loop level;

no m^2 dependence in the ghost sector;

no m^2 dependence in two-loop vertex diagrams with outgoing fermion lines.

The latter point needs some elaboration. In principle m^2 dependence arises through diagrams containing W self-energy insertions. One must show that together with the counterterms this dependence vanishes (see fig. 3a, b).

Let us consider such a diagram (fig. 3c). We encounter an integral of the form:

$$\int \mathrm{d}_n p \int \mathrm{d}_n q \frac{\gamma p \gamma p(2p-q)(2p-q)}{(p^2+m_f^2)^2(p^2+M^2)^2(q^2+M^2)((q+p)^2+m^2)}.$$

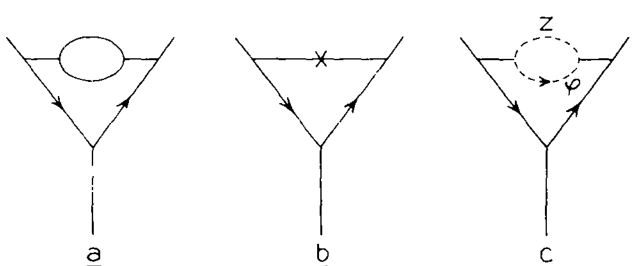

Fig. 3.

We have not specified various indices. The q integral is quadratically divergent, and has an m^2 dependence. At the one-loop level this dependence is extracted by developing the denominator[1]):

$$\int \frac{q^2}{(q^2+M^2)\left((q+p)^2+m^2\right)} = \int \frac{q^2}{(q^2+M^2)(q^2+m^2)}\left\{1 - \frac{2qp+p^2}{q^2+m^2} + \dots\right\}. \tag{4.1}$$

Only the first term in the expansion gives rise to an m^2 dependence, and can be evaluated easily. Unfortunately at the two-loop level this procedure fails, because the p integral becomes more divergent. In the case at hand the p integral is very convergent, and we can permit ourselves one step, thus writing:

$$\frac{1}{(q+p)^2+m^2} = \frac{1}{q^2+m^2} - \frac{p^2+2qp}{(q^2+m^2)\left((q+p)^2+m^2\right)}. \tag{4.2}$$

The first term is easily evaluated; since there is no more overlap in the p and q integrations one reproduces the one-loop result times a p integral. This will cancel precisely against the counterterm. The second term gives rise to an integral of the form:

$$\int \mathrm{d}_4 p\, \mathrm{d}_4 q \frac{\gamma p \gamma p (2p-q)(2p-q)(p^2+2qp)}{(p^2+m_\mathrm{f}^2)^2(p^2+M^2)^2(q^2+M^2)(q^2+m^2)\left((q+p)^2+m^2\right)}.$$

Counting degrees of divergence we see a logarithmic divergence in the q integral, a logarithmic divergent p integral, and the overall integral is also logarithmically divergent. This implies no dependence proportional to m^2 for large m. The treatment holds equally well for fermion self-energy diagrams such as shown in fig. 4. The final integrals are linearly divergent, but that makes no difference to the argument.

This argument shows that the inclusion of the finite terms proportional to m^2 in the lagrangian liberates us from the necessity to evaluate two-loop diagrams with fermion external lines. Important in this respect is the absence of diagrams with the Z coupled directly to the fermions. The small mass fermions known at low energy, to

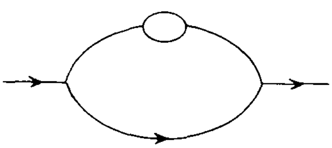

Fig. 4.

be used in our definition of the ρ-parameter, couple sufficiently weakly to allow restricting ourselves such diagrams. Also, one may learn from the above that inclusion of counterterms does not necessarily lead to vanishing m^2 dependence of self-energy diagrams plus counterterms. If those self-energy diagrams plus counterterms occur as part of a sufficiently divergent integral then new m^2 dependencies may arise. This, in fact, is what happens in 2-loop self-energy diagrams.

5. Radiative corrections

To fit the lagrangian parameters to experimental quantities and to calculate the ρ-parameter up to order $g^4 m^2$ one needs to calculate two-loop corrections to Coulomb scattering, e and e^- scattering and to β-decay. Two-loop irreducible corrections to these processes fall in different classes, shown in fig. 5. The unspecified blobs contain irreducible two-loop diagrams. All outgoing lines are fermion lines. With our choice of the lagrangian we need to consider only W and A self-energy diagrams as occurring in fig. 5a. The other diagrams do not give rise to behaviour proportional to m^2 or higher. Most of this is obvious, or has been discussed in sect. 4. Furthermore there are products of one-loop and/or counterterm corrections. From these we must carefully disentangle the contribution of the quadratic counterterms because they do not cancel completely against diagrams containing the product of two single loops.

The essential part of the calculation is the evaluation of the two-loop vector boson diagrams, to be treated in the next section, and the evaluation of one-loop graphs with counterterm insertions, to be treated in sect. 7.

6. Vector boson self-energy diagrams

We need the vector-boson self-energy diagrams in an expansion around $k^2 = 0$, where k is the momentum of the external vector boson. For $W_\mu^+ W_\nu^-$ and $W_\mu^0 W_\nu^0$ we only need the constant term, but for $A_\mu W_\nu^0$ and $A_\mu A_\nu$ we also need the $k^2\delta_{\mu\nu}$ and $k_\mu k_\nu$ terms. No diagrams with internal fermion lines need be considered. The only

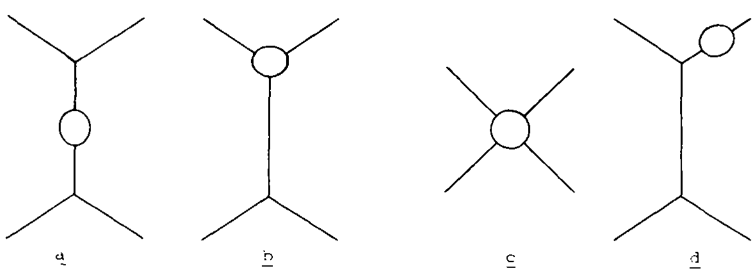

Fig. 5.

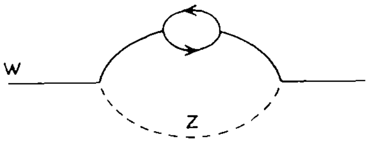

Fig. 6.

diagram for which this is not obvious is of the form shown in fig. 6. While the fermion self-energy diagram is quadratically divergent, the other integrals are not and, at most, logarithmic behavior with respect to m is possible. Also diagrams with internal ghost lines behave at most as $\log m^2$.

There are approximately 300 irreducible diagrams contributing to order $g^4 m^2$ to these self-energies. The relevant topologies are shown in fig. 7. The internal lines may be W, A, ϕ or Z lines. Furthermore there are reducible graphs and graphs containing counterterms. The evaluation of the diagrams proceeds as follows. First the diagrams are reduced to scalar integrals. Only a limited set of integrals occurs. The resulting expressions are added together and finally the expressions for the scalar integrals given in appendix A are substituted.

To show how the procedure works we present explicitly two examples.

(ii) A diagram contributing to the W^+W^- mass, see fig. 8. The corresponding expression is

$$\frac{1}{(2\pi)^4 i}\int \mathrm{d}^n p\,\mathrm{d}^n q\, p_\mu p_\nu \frac{1}{4}\frac{m^4}{M^2}\frac{1}{(p^2+m^2)^2(p^2+M^2)\left((p+q)^2+M^2\right)(q^2+M^2)}.$$

The answer is of the form $A\delta_{\mu\nu}$ and we obtain A by multiplying with $\delta_{\mu\nu}/n$. Substituting next $p^2 \to (p^2+M^2)-M^2$ gives

$$\frac{1}{(2\pi)^4 i}\frac{m^4}{4M^2}\frac{\delta_{\mu\nu}}{n}\left[(2m|M|M)-M^2(2mM|M|M)\right].$$

The notation and further evaluation is given in appendix A.

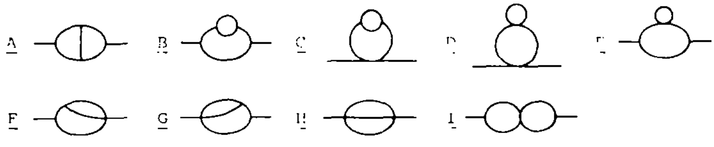

Fig. 7.

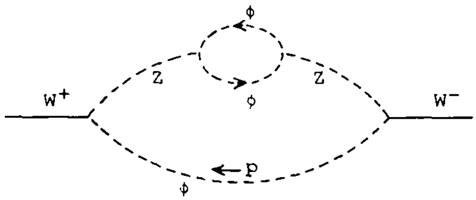

Fig. 8.

(ii) A photon-photon self-energy diagram, see fig. 9. The corresponding expression is

$$\frac{s^2m^4}{4M^2}\cdot\frac{1}{(2\pi)^4 i}\cdot\int \mathrm{d}^n p\,\mathrm{d}^n q$$

$$\frac{(2p+k)_\mu(2q+k)_\nu}{\left((p+k)^2+M^2\right)\left(p^2+M^2\right)\left((q+k)^2+M^2\right)\left(q^2+M^2\right)\left((p-q)^2+m^2\right)}.$$

In this case we use a trick: we introduce Feynman parameters x, y for the propagators with $p+k$, p respectively $q+k$, q. Then we shift the integration parameters

$$p\to p-kx, \qquad q\to q-kx.$$

The integrand becomes

$$\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y$$

$$\frac{[2p+k(1-2x)]_\mu[2q+k(1-2x)]_\nu}{\left[p^2+M^2+k^2x(1-x)\right]^2\left[(q+k(y-x))^2+k^2y(1-y)+M^2\right]\left[(p-q)^2+m^2\right]}.$$

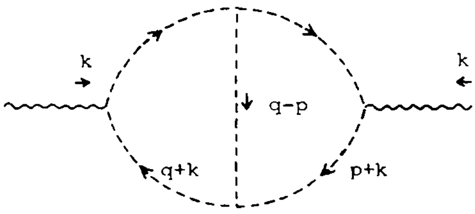

Fig. 9.

Next we expand in k, keep the quadratic terms and do the x, y integration. This gives

$$\frac{1}{(2\pi)^4 i}\frac{s^2 m^4}{4M^2}\left[4p_\mu q_\nu\left\{\frac{-\frac{1}{3}k^2}{(p^2+M^2)^3(q^2+M^2)^2}+\frac{2(k\cdot q)^2}{(p^2+M^2)^2(q^2+M^2)^4}\right.\right.$$

$$\left.-\frac{\frac{2}{3}k^2}{(p^2+M^2)^2(q^2+M^2)^3}\right\}$$

$$\left.+\frac{\frac{1}{3}k_\mu k_\nu}{(p^2+M^2)^2(q^2+M^2)^2}\right]\left((p-q)^2+m^2\right)^{-1}.$$

Next we use

$$\int \mathrm{d}^n p\,\mathrm{d}^n q\,F(p,q)\,p_\mu q_\nu q_\alpha q_\beta=\int \mathrm{d}^n p\,\mathrm{d}^n q\,F(p,q)(p\cdot q)q^2\frac{(\delta_{\mu\nu}\delta_{\alpha\beta}+\delta_{\mu\alpha}\delta_{\nu\beta}+\delta_{\mu\beta}\delta_{\nu\alpha})}{n^2+2n}.$$

Subsequently one expresses everything in terms of p^2, q^2 and $r^2=(p-q)^2$. One obtains terms in the numerator of the form r^2q^2, p^2q^2, q^4 etc. These are worked out through substitutions

$$r^2\to(r^2+m^2)-m^2,\qquad p^2\to(p^2+M^2)-M^2,\qquad q^2\to(q^2+M^2)-M^2.$$

Finally keeping only terms behaving like m^2 one obtains the expression:

$$\frac{1}{(2\pi)^4 i}\frac{s^2m^4}{M^2}k^2\delta_{\mu\nu}\left[\frac{1}{2n+4}\left\{(3M|2M)-m^2(3M|2M|m)-(2M|2M|m)\right.\right.$$

$$\left.-(3M|M|m)+2M^2(3M|2M|m)\right\}$$

$$+\frac{M^2}{n^2+2n}\left\{(4M|2M)-m^2(4M|2M|m)-(3M|2M|m)\right.$$

$$\left.\left.-(4M|M|m)+2M^2(4M|2M|m)\right\}\right]$$

$$
\begin{aligned}
&+\frac{1}{(2\pi)^4 i}\frac{s^2m^4}{M^2}k_\mu k_\nu\left[\frac{-1}{n^2+2n}\{(3M|2M)-m^2(3M|2M|m)-(2M|2M|m)\right.\\
&\qquad\qquad -(3M|M|m)+2M^2(3M|2M|m)\}\\
&\qquad +\frac{M^2}{n^2+2n}\{(4M|2M)-m^2(4M|2M|m)-(3M|2M|m)\\
&\qquad\qquad -(4M|M|m)+2M^2(4M|2M|m)\}\\
&\qquad \left.+\tfrac{1}{3}(2M|2M|m)\right].
\end{aligned}
$$

A few remarks are in order. We did not include diagrams containing tadpoles. The inclusion of the complete tadpole counterterm cancels all one-loop tadpole contributions and the two-loop tadpole diagrams couple to W^+W^- and W^0W^0 with relative strength c^2, and therefore do not contribute to the ρ-parameter. Furthermore one easily sees that the topology (I) does not contribute to the W masses, while topologies C, D, H and I do not contribute to the AA and AW wave-function expressions (terms proportional to k^2 and $k_\mu k_\nu$).

7. Reducible and counterterm contributions

These contain products of one-loop contributions. Since the only place where m^2 corrections appear in one loop, relevant for four-fermion scattering, is in the W self-energy, one must consider one-loop insertions. However we have chosen the counterterms in such a way that such terms are cancelled. As an example consider fig. 10.

This also demonstrates the need to include terms proportional to $\varepsilon = n - 4$ in the counterterms. If this were not the case the pole term in the vertex multiplied with the

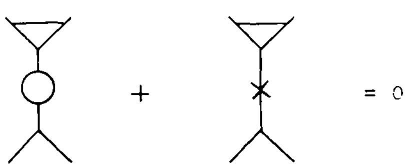

Fig. 10.

Fig. 11.

εm^2 term of the self-energy diagram would give an m^2 contribution in the amplitude. A peculiar fact is the need to include quadratic counterterms. The relevant m^2 dependent quadratic counterterms in the lagrangian are:

$$\frac{4M^2 s}{c}\Delta\delta_{1\mathrm{f}}A_\mu W_\mu^0 + M^2 W_\mu^+ W_\mu \left\{\left(-\frac{1}{c^2}\delta_{1\mathrm{f}} - \tfrac{7}{6}\delta_{1\mathrm{f}} + \tfrac{43}{12}\delta_t\right)\Delta - \delta_{1\mathrm{f}}^2\right\}$$

$$+\frac{M^2}{2c^2}W_\mu^0 W_\mu^0\left\{\left(-\frac{1}{c^2}\delta_{1\mathrm{f}} - \tfrac{55}{6}\delta_{1\mathrm{f}} + 8c^2\delta_{1\mathrm{f}} + \tfrac{43}{12}\delta_t\right)\Delta - \delta_{1\mathrm{f}}^2\right\}. \qquad (7.1)$$

They do not cancel against products of one-loop diagrams and must be explicitly included. All other one-loop diagrams and counterterms cancel, like for instance the set shown in fig. 11. To show the situation with respect to the quadratic counterterms consider the W-A amplitude. Including quadratic counterterms we have the set of diagrams shown in fig. 12. Diagrams a and c, b and e cancel; also d and g and f and g, and thus i survives. We may therefore ignore diagrams a–h, but must include i explicitly in our calculations. We emphasize that the above obtains by a replacement operation in the lagrangian; if we did not include diagram i we would not have the same physics as with the original (bare) lagrangian.

Concerning diagrams (see for example fig. 13) with counterterm insertions we can be brief. The only noteworthy fact is that there are counterterms not corresponding to vertices in the bare lagrangian. The replacements $W \rightarrow (1+\delta)W + \delta A$ and similarly for A^W generate such vertices. Specifically one has $A\phi^0 Z$, AW^0Z, $A\phi^0\phi^0W^0$, AW^0ZZ and $\bar{Y}^0Y^AZ$ terms, apart from various two-point counter vertices.

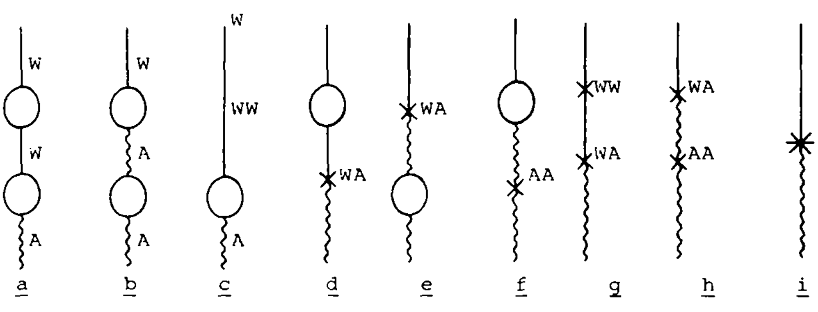

Fig. 12.

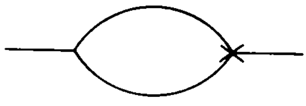

Fig. 13.

8. Results

Adding all contributions (except two-loop tadpoles, as noted above) we find the following amplitudes

$$A_\mu A_\nu = \frac{-1}{64} i \frac{s^2 m^2}{2} \left(k_\mu k_\nu - k^2 \delta_{\mu\nu} \right) \equiv f_{AA} \left(k_\mu k_\nu - k^2 \delta_{\mu\nu} \right), \tag{8.1}$$

$$A_\mu W_\nu^0 = \frac{-1}{256} i \frac{m^2}{M^2} \left(\frac{s}{c} - \frac{2s^3}{c} \right) \left(k_\mu k_\nu - k^2 \delta_{\mu\nu} \right) \equiv f_{AW_0} \left(k_\mu k_\nu - k \delta_{\mu\nu}^2 \right), \tag{8.2}$$

$$\begin{aligned} W_\mu^+ W_\nu = {} & \tfrac{1}{4} i \delta_{\mu\nu} \frac{m^4}{M^2} \left[\frac{3}{128\varepsilon} + \tfrac{3}{128} \log m^2 - \tfrac{1}{768} \pi^2 + \tfrac{3}{256} \pi \sqrt{3} - \tfrac{9}{64} C - \tfrac{3}{1024} \right] \\ & + \tfrac{1}{4} i \delta_{\mu\nu} m^2 \left[\frac{43}{16\varepsilon^2} + \frac{43}{32} \frac{\log m^2}{\varepsilon} - \frac{301}{192\varepsilon} + \tfrac{43}{128} \log^2 m^2 - \left(\tfrac{277}{384} - \frac{1}{32c^2} \right) \log m^2 \right. \\ & \left. + \tfrac{61}{768} \pi^2 + \tfrac{9}{128} \pi \sqrt{3} - \tfrac{27}{32} C - \frac{3}{32c^2} + \tfrac{73}{128} \right] \equiv f_c \delta_{\mu\nu}. \end{aligned} \tag{8.3}$$

$$\begin{aligned} W_\mu^0 W_\nu^0 = {} & \frac{i\delta_{\mu\nu}}{4c^2} \frac{m^4}{M^2} \left[\frac{3}{128\varepsilon} + \tfrac{3}{128} \log m^2 - \tfrac{1}{768} \pi^2 + \tfrac{3}{256} \pi \sqrt{3} - \tfrac{9}{64} C - \tfrac{3}{1024} \right] \\ & + \frac{i\delta_{\mu\nu}}{4c^2} m^2 \left[\frac{43}{16\varepsilon^2} + \frac{43}{32} \frac{\log m^2}{\varepsilon} - \frac{301}{192\varepsilon} + \tfrac{43}{128} \log^2 m^2 \right. \\ & - \left(\tfrac{277}{384} - \frac{1}{32c^2} \right) \log m^2 + \tfrac{43}{768} \pi^2 + \frac{3\pi^2}{128c^2} \\ & \left. + \frac{9\pi\sqrt{3}}{128c^2} - \frac{27}{32} \frac{C}{c^2} + \tfrac{167}{256} - \frac{45}{256c^2} \right] \equiv f_0 \delta_{\mu\nu}. \end{aligned} \tag{8.4}$$

with $C = \sqrt{\tfrac{1}{3}}\, \mathrm{Cl}(\tfrac{1}{3}\pi)$ where $\mathrm{Cl}(x)$ is the so-called Clausen function, related to the

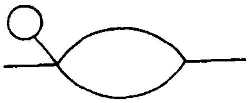

Fig. 14.

Spence function:

$$\mathrm{Cl}(\tfrac{1}{3}\pi) = \mathrm{Re}\left[\frac{1-x}{2i}\left\{-\mathrm{Sp}\left(1-\frac{1}{x}\right)+\mathrm{Sp}\left(-\frac{x}{1-x}\right)\right\}\right], \qquad x = \tfrac{1}{2}(1+i\sqrt{3}\,). \tag{8.5}$$

Notice that both the $A_\mu A_\nu$ and $A_\mu W_\nu$ terms are transversal. For the $A_\mu A_\nu$ term this follows from the Ward identity for the photon propagator. For the $A_\mu W_\nu^0$ term this follows from the $\bar{\nu}\nu A$ Ward identity, which effectively tells us that the neutrino charge is zero. Combined with the fact that in our renormalization scheme there is no g^2m^2 contribution in the vertex correction, the result follows. As an extra check we also calculated the topologies A to I and the diagrams containing tadpoles as shown in fig. 14 for the AA propagator. The sum has to be transversal because of the photon-photon Ward identity. This indeed turned out to be the case. We now turn to the evaluation of the correction to the ρ-parameter.

There are four free parameters g, s, M and m. They can be fixed by comparison with four experimental quantities for which we take the electric charge e, the Fermi-coupling constant G from μ-decay, the neutral cross-section ratio $\sigma(\bar{\nu}_\mu \mathrm{e})/\sigma(\nu_\mu \mathrm{e})$ and the physical Higgs mass. Our choice of counterterms is such that $m^2_{\mathrm{exp}} = m^2$. The electric charge is defined through Coulomb scattering. The following diagrams contribute (see fig. 15). This leads to:

$$e^2_{\mathrm{exp}} \equiv g^2_{\mathrm{exp}} s^2_{\mathrm{exp}} = g^2 s^2\left(1+\frac{f_{\mathrm{AA}}}{(2\pi)^4 i}\right), \qquad \frac{e^2_{\mathrm{exp}}}{4\pi} = \alpha \simeq \tfrac{1}{137}$$

with f_{AA} defined above. For μ-decay one has the diagram shown in fig. 16. This leads to the relation

$$G_{\mathrm{F}} = \frac{g^2}{8M^2}\left(1+\frac{f_{\mathrm{c}}}{(2\pi)^4 iM^2}\right).$$

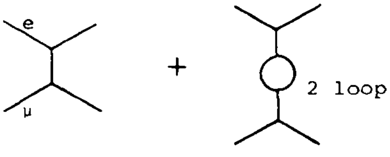

Fig. 15.

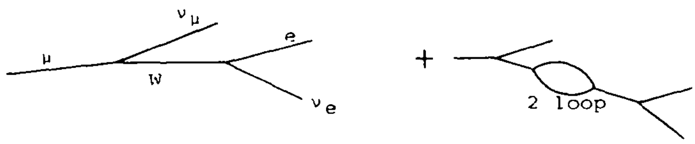

Fig. 16.

For ν_μe scattering we have the diagrams of fig. 17. This leads to the amplitude

$$\frac{g^2}{16M^2}\left(1+\frac{c^2 f_c}{M^2(2\pi)^4 i}\right)\left(\bar{\nu}_\mu\gamma^\alpha(1+\gamma_5)\nu_\mu\right)\left(\bar{e}\gamma^\alpha(\xi+\gamma_5)e\right),$$

with

$$\xi = 1-4s^2+\frac{4scf_{\mathrm{AW_0}}}{(2\pi)^4 i}.$$

The ratio of the total cross sections for ν_μe and $\bar{\nu}_\mu$e can be calculated from this expression. It is

$$\frac{\sigma^{\bar{\nu}e}}{\sigma^{\nu e}} = \frac{\xi^2-\xi+1}{\xi^2+\xi+1}.$$

Thus:

$$s^2_{\exp} \equiv \tfrac{1}{4}(1-\xi)|_{\exp} = s^2 - \frac{scf_{\mathrm{AW0}}}{(2\pi)^4 i}.$$

Comparing the expressions for μ-decay and νe scattering we find that the ρ-parameter is given by

$$1+\frac{c^2 f_0}{(2\pi)^4 iM^2}-\frac{f_c}{(2\pi)^4 iM^2} = 1+\frac{g^4m^2}{\pi^4M^2}\mathrm{tg}^2\theta\left\{\frac{-21}{16384}-\tfrac{27}{2048}C+\tfrac{9}{8192}\pi\sqrt{3}+\tfrac{3}{8192}\pi^2\right\}$$

$$= 1+\frac{g^4m^2}{\pi^4M^2}\mathrm{tg}^2\theta(5.854\cdot 10^{-4}).$$

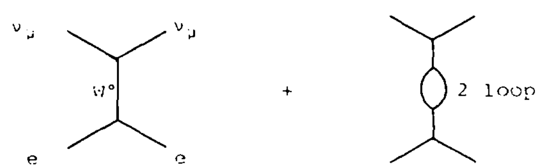

Fig. 17.

where we used for the constant C, given before, the numerical value 0.586. For s we used the value $s^2 = 0.23$. Together with the one-loop results we have the following for large m^2.

$$\rho = 1 - \frac{3\alpha}{16\pi c^2}\ln\frac{m^2}{M^2} + 9.49 \cdot 10^{\;4}\frac{\alpha^2}{s^2c^2}\frac{m^2}{M^2} = 1 - 5.66 \cdot 10^{-4}\log\frac{m^2}{M^2}$$

$$+ 2.85 \cdot 10^{\;7}\frac{m^2}{M^2}\,.$$

In this expression one can take the various parameters to be the experimental ones. This is due to the renormalization scheme we adopted. If we had not explicitly subtracted the Higgs self-energy terms one would have $m^2_{\exp} = m^2 + O(g^2m^4/M^2)$. Substitution of this expression in the one-loop result would have given an extra g^4m^2 type contribution. Our subtraction scheme evades this problem.

To get some idea of the order of magnitude we have listed in table 1 some numerical values. There is in fact quite a strong cancellation among the various terms in the second-order expression for $\delta\rho$ (individual terms are about 10 times larger than the total). The Higgs mass must be very large before the second-order correction is as large as the first order, in fact this happens for $m \sim 137\ M \simeq 10$ TeV. At that point there is a complete cancellation, and no correction results, but this is quite meaningless since by then the next order is expected to dominate. If we extrapolate our result for the magnitude of the two-loop corrections to a guess for the three-loop corrections then we expect the three-loop correction g^6m^4/M^4 to be approximately the same as the two-loop results if $m/M \gtrsim 45$, i.e. $m \sim 3.3$ TeV. Then perturbation theory breaks down. It is very unclear what happens if the Higgs mass is in this area.

We finally comment on the result as compared to the two-loop quadratic divergencies of the non-linear σ-model. These we have computed also, and they are unrelated to the above results. Partly the correspondence is lost because one cannot exchange loop integrations and the large-m limit. But more specifically, in the

TABLE 1
Corrections to ρ as a function of m/M

m/M	δ_{ρ_1}	δ_{ρ_2}	$\delta_{\rho_1} + \delta_{\rho_2}$
5	$-0.18 \cdot 10^{\;2}$	$0.71 \cdot 10^{\;5}$	$0.18 \cdot 10^{\;2}$
10	$0.26 \cdot 10^{\;2}$	$0.29 \cdot 10^{\;4}$	$\cdot\ 0.26 \cdot 10^{\;2}$
20	$-0.34 \cdot 10^{\;2}$	$0.11 \cdot 10^{\;3}$	$-0.33 \cdot 10^{\;2}$
50	$-0.44 \cdot 10^{\;2}$	$0.71 \cdot 10^{\;3}$	$0.37 \cdot 10^{\;2}$
100	$-0.52 \cdot 10^{\;2}$	$0.29 \cdot 10^{\;2}$	$0.24 \cdot 10^{\;2}$
140	$0.56 \cdot 10^{\;2}$	$0.56 \cdot 10^{\;2}$	0

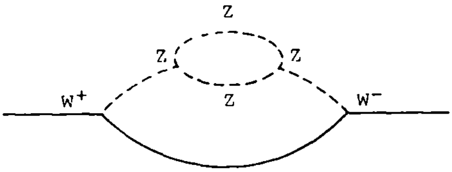

Fig. 18.

non-linear σ-model all knowledge of the Higgs self-interactions is lost. For example, the diagram of fig. 18 has no correspondence to anything in the non-linear σ-model. Such diagrams give, among other things, rise to the constant C, containing the Clausen function. It is trivial to see that no such Clausen functions arise in the non-linear σ-model.

One might well ask if our results could be further influenced by adding other Higgs self-interactions. One could imagine particles that may or may not be heavy, and couple to the Higgs but not to the other particles. It is not easy to see what this could do. Essentially only one guess can be made. If such particles could be very heavy one could apply the Appelquist-Carrazone-Symanzik decoupling theorem to argue that their influence can be neglected. This is certainly true for finite m. However, in the limit of large m one might again run into complications concerning the exchange of limits. The subject clearly needs further investigation.

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Appendix A

FORMULAE FOR SCALAR INTEGRALS

We define

$$(M_{11}, M_{12} \dots M_{1n_1} | M_{21}, \dots M_{2n_2} | M_{31}, \dots M_{3n_3})$$

$$= \int \mathrm{d}^n p \int \mathrm{d}^n q \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} \prod_{k=1}^{n_3} \frac{1}{(p^2 + M_{1i}^2)} \frac{1}{(q^2 + M_{2j}^2)} \frac{1}{((p+q)^2 + M_{3k}^2)} . \tag{A.1}$$

The basic formula one needs is $(M, M | M_1 | M_2)$. All other expressions can be found from this one through the use of partial fractions, differentiation and partial p. For instance by partial p (see ref. [10]):

$$(M_0|M_1|M_2) = \frac{1}{3-n}\{M_0^2(M_0M_0|M_1|M_2)$$

$$+M_1^2(M_1M_1|M_0|M_2)+M_2^2(M_2M_2|M_0|M_1)\}. \tag{A.2}$$

Another useful relation is

$$-(n-3)(M_1M_1|M_2|M_3)-2M_2^2(M_1M_1|M_2M_2|M_3)-(M_1|M_2|M_3M_3)$$

$$+(M_1M_1)(M_3M_3)+(M_1^2-M_2^2-M_3^2)(M_1M_1|M_2|M_3M_3)=0, \tag{A.3}$$

where

$$(M_1M_1)=\int \mathrm{d}_n p \frac{1}{(p^2+M_1^2)^2}. \tag{A.4}$$

By partial fractions

$$(M_0, M_1|M_2|M_3)=\frac{1}{M_0^2-M_1^2}((M_1|M_2|M_3)-(M_0|M_2|M_3)),$$

$$(M, M|M_1, M_1|M_2)=-\frac{\partial}{\partial M_1^2}(M, M|M_1|M_2).$$

As a shorthand we sometimes write $(M, M|M|m)=(2M|M|m)$ where this use is unambiguous. Using Feynman parameters one finds

$$(M, M|M_1|M_2)=\frac{-\pi^4(\pi M^2)^{n-4}\Gamma(2-\frac{1}{2}n)}{\Gamma(3-\frac{1}{2}n)}$$

$$\times\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\,(x(1-x))^{n/2-2}y(1-y)^{2-n/2}$$

$$\times\left[\Gamma(5-n)\frac{\mu^2}{(y+\mu^2(1-y))^{5-n}}+\tfrac{1}{2}n\Gamma(4-n)\frac{1}{y+\mu^2(1-y)}\right].$$

with

$$\mu^2=\frac{ax+b(1-x)}{x(1-x)}, \qquad a=\frac{M_1^2}{M^2}, \qquad b=\frac{M_2^2}{M^2}.$$

Define $n = 4 + \varepsilon$. Performing the y integration gives

$$(M, M|M_1|M_2) = \pi^4\left[\frac{-2}{\varepsilon^2} + \frac{1}{\varepsilon}\left(1 - 2\gamma_E - 2\log(\pi M^2)\right)\right]$$
$$+\pi^4\left[-\tfrac{1}{2} - \tfrac{1}{12}\pi^2 + \gamma_E - \gamma_E^2 + (1 - 2\gamma_E)\log(\pi M^2)\right.$$
$$\left.-\log^2(\pi M^2) - f(a, b)\right] + O(\varepsilon)$$

where we used

$$\Gamma(z) = \tfrac{1}{z} - \gamma_E + \left(\tfrac{1}{12}\pi^2 + \tfrac{1}{2}\gamma_E^2\right)z + O(z^2), \qquad (z \to 0).$$

γ_E is the Euler constant $\gamma_E = 0.577215665$ and

$$f(a, b) = \int_0^1 \mathrm{d}x\left(\mathrm{Sp}(1 - \mu^2) - \frac{\mu^2 \log \mu^2}{1 - \mu^2}\right).$$

$\mathrm{Sp}(x)$ is the Spence function or di-logarithm defined by

$$\mathrm{Sp}(x) = \int_0^x -\frac{\ln(1 - y)}{y}\,\mathrm{d}y.$$

Performing the x integration we find

$$f(a, b) = \tfrac{1}{2}\log^2 a - \mathrm{Sp}\left(\frac{a - b}{a}\right)$$
$$+\left(\frac{a + b - 1}{2\sqrt{\ }} - \tfrac{1}{2}\right)\left[\mathrm{Sp}\left(\frac{b - a}{x_1}\right) - \mathrm{Sp}\left(\frac{a - b}{1 - x_1}\right)\right.$$
$$\left.-\mathrm{Sp}\left(1 - \frac{1}{x_1}\right) + \mathrm{Sp}\left(\frac{-x_1}{1 - x_1}\right)\right]$$
$$-\left(\frac{a + b - 1}{2\sqrt{\ }} + \tfrac{1}{2}\right)\left[\mathrm{Sp}\left(\frac{b - a}{x_2}\right) - \mathrm{Sp}\left(\frac{a - b}{1 - x_2}\right)\right.$$
$$\left.-\mathrm{Sp}\left(1 - \frac{1}{x_2}\right) + \mathrm{Sp}\left(\frac{-x_2}{1 - x_2}\right)\right],$$

$$\sqrt{\ } = \left(1 - 2(a + b) + (a - b)^2\right)^{1/2},$$

$$x_1 = \tfrac{1}{2}\left(1 + b - a + \sqrt{\ }\right), \qquad x_2 = \tfrac{1}{2}\left(1 + b - a - \sqrt{\ }\right).$$

Through explicit symmetrization in a and b and the use of the relations:

$$\mathrm{Sp}(1-x) = -\mathrm{Sp}(x) - \log x \log(1-x) + \tfrac{1}{6}\pi^2,$$

$$\mathrm{Sp}\left(\frac{1}{x}\right) = -\mathrm{Sp}(x) - \tfrac{1}{2}\log^2(-x) - \tfrac{1}{6}\pi^2,$$

we can simplify this formula to:

$$f(a,b) = -\tfrac{1}{2}\log a \log b$$
$$-\left(\frac{a+b-1}{\sqrt{\ }}\right)\left[\mathrm{Sp}\left(\frac{-x_2}{y_1}\right) + \mathrm{Sp}\left(\frac{-y_2}{x_1}\right) + \tfrac{1}{4}\log^2\frac{x_2}{y_1} + \tfrac{1}{4}\log^2\frac{y_2}{x_1}\right.$$
$$\left. + \tfrac{1}{4}\log^2\frac{x_1}{y_1} - \tfrac{1}{4}\log^2\frac{x_2}{y_2} + \zeta(2)\right],$$

$$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \tfrac{1}{6}\pi^2,$$

with

$$y_1 = \tfrac{1}{2}[1 + a - b + \sqrt{\ }\,], \qquad y_2 = \tfrac{1}{2}[1 + a - b - \sqrt{\ }\,].$$

In the actual calculation we need the expansion of these equations for the large m = Higgs mass. As it happens these integrals occur with coefficients containing m^2, and in order to have an answer correct to order m^2 we need the following expansions:

$$(m, m|M_1|M_2), \qquad \text{to order } 1/m^6,$$

$$(M_1, M_2|M_1|m), \qquad \text{to order } 1/m^4,$$

$$(m, m|m|M_1), \qquad \text{to order } 1/m^2,$$

$$(M, M|m|m), \qquad \text{to order } 1.$$

In addition we need $(m, m|m|m)$. In the following we will give these expansions going two orders further than needed for the purposes of this paper. Also we ignore the constant γ and the factor π^2 appearing in the logarithms, which effectively means that one must read $\gamma + \log(\pi M^2)$ for $\log M^2$. In the final result we will have only logarithms of ratios of masses, and this replacement becomes irrelevant.

(i) Using the parameters $a = M_1^2/m^2$ and $b = M_2^2/m^2$ we have

$$
\begin{aligned}
(mm|M_1|M_2) = \pi^4 \Big[& -\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon} - \frac{2}{\varepsilon}\log m^2 + \log m^2 - \log^2 m^2 - \tfrac{1}{4}\pi^2 \\
& - \tfrac{1}{2} + a + b + \tfrac{1}{4}a^2 + \tfrac{1}{4}b^2 + ab + \tfrac{1}{9}a^3 + 3a^2b + 3ab^2 + \tfrac{1}{9}b^3 \\
& + \tfrac{1}{16}a^4 + \tfrac{11}{2}a^3b + \tfrac{49}{4}a^2b^2 + \tfrac{11}{2}ab^3 + \tfrac{1}{16}b^4 \\
& + \tfrac{1}{25}a^5 + \tfrac{74}{9}a^4b + 35a^3b^2 + 35a^2b^3 + \tfrac{74}{9}ab^4 + \tfrac{1}{25}b^5 \\
& - \left(\log a \log b + \tfrac{1}{3}\pi^2\right)\big(ab + 2ab^2 + 2a^2b + 3ab^3 + 9a^2b^2 + 3a^3b \\
& \qquad + 4ab^4 + 24a^2b^3 + 24a^3b^2 + 4a^4b\big) \\
& - \log(a)\big(a + \tfrac{1}{2}a^2 + ab + \tfrac{1}{3}a^3 + 4a^2b + ab^2 + \tfrac{1}{4}a^4 + 8a^3b \\
& + \tfrac{21}{2}a^2b^2 + ab^3 + \tfrac{1}{5}a^5 + \tfrac{38}{3}a^4b + 40a^3b^2 + 20a^2b^3 + ab^4\big) \\
& - \log(b)\big(b + \tfrac{1}{2}b^2 + ab + \tfrac{1}{3}b^3 + 4ab^2 + a^2b + \tfrac{1}{4}b^4 + 8ab^3 \\
& + \tfrac{21}{2}a^2b^2 + a^3b + \tfrac{1}{5}b^5 + \tfrac{38}{3}ab^4 + 40a^2b^3 + 20a^3b^2 + a^4b\big)\Big].
\end{aligned}
$$

(ii) Using the parameters $a = M_1^2/m^2$ and $b = M_2^2/m^2$:

$$
\begin{aligned}
(M_1M_1|M_2|m) = \pi^4 \Big[& -\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon} - \frac{2}{\varepsilon}\log M_1^2 + \log M_1^2 - \log^2 M_1^2 - \tfrac{1}{2}\log^2 a \\
& - \tfrac{1}{2} - a - b - \tfrac{1}{4}a^2 - 3ab - \tfrac{9}{4}b^2 - \tfrac{1}{9}a^3 - \tfrac{11}{2}a^2b - 9ab^2 - \tfrac{47}{18}b^3 \\
& - \tfrac{1}{16}a^4 - \tfrac{74}{9}a^3b - \tfrac{107}{4}a^2b^2 - 20ab^3 - \tfrac{401}{144}b^4 \\
& + \tfrac{1}{3}\pi^2\big(\tfrac{1}{4} + b + 2ab + b^2 + 6ab^2 + 3a^2b \\
& \qquad + b^3 + 12ab^3 + 18a^2b^2 + 4a^3b + b^4\big)
\end{aligned}
$$

$$+\log(a)\big(a+3ab+\tfrac{1}{2}a^2+\tfrac{1}{3}a^3+7a^2b+5ab^2$$

$$+\tfrac{1}{4}a^4+\tfrac{35}{3}a^3b+\tfrac{51}{2}a^2b^2+7ab^3\big)$$

$$+\log(b)\big(b+ab+\tfrac{5}{2}b^2+a^2b+8ab^2$$

$$+\tfrac{10}{3}b^3+a^3b+\tfrac{33}{2}a^2b^2+23ab^3+\tfrac{47}{12}b^4\big)$$

$$+\log(a)\log(b)\big(b+2ab+b^2+3a^2b+6ab^2+b^3$$

$$\left.+4a^3b+18a^2b^2+12ab^3+b^4\big)\right].$$

(iii) With $b=M^2/m^2$:

$$(mm|m|M)=\pi^4\left[-\frac{2}{\varepsilon^2}+\frac{1}{\varepsilon}-\frac{2}{\varepsilon}\log m^2+\log m^2-\log^2 m^2-\tfrac{1}{12}\pi^2\right.$$

$$\left.-\tfrac{1}{2}+b+\tfrac{5}{36}b^2+\tfrac{47}{1800}b^3+\log b\big(-\tfrac{1}{2}b-\tfrac{1}{12}b^2-\tfrac{1}{60}b^3\big)\right].$$

(iv) The expression for $(MM|m|m)$ may be obtained as follows. Use eq. (A.3) for the case $M_1=M$, $M_2=M_3=m$. One needs $(MM|mm|m)=(MM|m|mm)$, given above. The expressions for MM and mm are obtained by differentiating the equation for $F(m)$ given below.

(v) Finally $(m|m|mm)$ is given by:

$$(m|m|mm)=\pi^4\left[-\frac{2}{\varepsilon^2}+\frac{1}{\varepsilon}-\frac{2}{\varepsilon}\log m^2+\log m^2-\log^2 m^2-\tfrac{1}{2}-\tfrac{1}{12}\pi^2+\tfrac{2}{\sqrt{3}}\,\mathrm{Cl}\big(\tfrac{1}{3}\pi\big)\right].$$

with

$$\mathrm{Cl}(x)=\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}.$$

One computes $\tfrac{2}{\sqrt{3}}\,\mathrm{Cl}(\tfrac{1}{3}\pi)=1.171953619344$, with quite unnecessary precision.

Also appearing are expressions involving the product of two one-loop integrals.

$$(M_{11}M_{12}\dots M_{1n_1})(M_{21}\dots 2n_2)=\int \mathrm{d}^n p\prod_{i=1}^{n_1}\frac{1}{(p^2+M_{1i}^2)}\int \mathrm{d}^n q\prod_{i=1}^{n_2}\frac{1}{(q^2+M_{2i}^2)}.$$

To evaluate these one needs

$$F(m)=\int\frac{\mathrm{d}^4p}{p^2+m^2}=\frac{2i\pi^2m^2}{\varepsilon}+i\pi^2m^2\left[\gamma-1+\log(\pi m^2)\right]$$
$$+\varepsilon i\pi^2m^2\left[\frac{\pi^2}{24}+\tfrac{1}{4}\gamma^2-\tfrac{1}{2}\gamma+\tfrac{1}{2}\right.$$
$$\left.+\tfrac{1}{2}(\gamma-1)\log(\pi m^2)+\tfrac{1}{4}\log^2(\pi m^2)\right]+\mathrm{O}(\varepsilon^2),$$

where as usual $\varepsilon=n-4$.

Appendix B

In this appendix we will discuss some of the technical details related to the use of the computer program Schoonschip.

The major problem here is the proliferation of particles and diagrams. Not counting the fermions there are 16 different particles, and 4 different masses. Of the order of 300 essentially different two-loop self-energy diagrams contribute up to order m^2 and must be calculated (which is just about possible to do by hand.) That is, up to writing things in terms of the expressions $(m|M|M_0)$ etc. defined in appendix A. The further work, substituting the expansions in terms of m^2, makes this problem really too big to do by hand, if we want any certainty that no errors come in.

Also the evaluation of diagrams with counterterm insertions is quite cumbersome. The evaluation of one-loop three- and four-point diagrams is very cumbersome, involving the evaluation of thousands of diagrams, but we need only the infinite parts of these diagrams, and that can also be worked out from the self-energy diagrams and a few vertex diagrams using gauge invariance. We have nevertheless done that work, mainly as a check on our computer programs, including the complete list of vertices of the standard model. This also tests the method used with respect to combinatorial factors used if identical particles occur. The latter seemed to present a formidable complication, till we discovered a general rule that we found out later to be contained also in an old paper by Wu [8]. The way it works is very well suited to computer evaluation.

The first problem was to invent a procedure whereby all possible diagrams would be generated, and where the program would all by itself pick out the correct vertices and work out the diagrams. We will illustrate the method on a simple case.

As a first step the various relevant topologies are written down. Thus for one-loop self-energy diagrams there are two topologies, see fig. 19. The particles are numbered

Fig. 19.

Fig. 20.

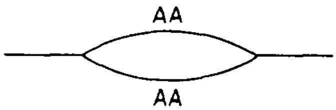

Fig. 21.

from 1 to 16, and with each particle we associate a character (for example A for the photon, U for the W, V for W^- and so on). There are then 16 different propagators (AA, UV, VU etc.). We now write down the general form of some topology with unspecified names for the particles and simply let the computer run through all possible choices for the propagators. That would give rise to $16^2 = 256$ diagrams. Many of these will be zero, because the corresponding vertices do not exist. The computer program looks up the vertex table in all these cases, and if the vertices exist the diagram is kept, the expressions for the vertices are substituted and the whole is worked out further.

Now, if we do this we will do double counting, because a diagram with for example an AA and an UV propagator will occur twice (see fig. 20). We therefore must divide the total result by two. This leads to the fact that diagrams that contain two identical particles and occur only once (see for example fig. 21) get a factor $\frac{1}{2}$, which is the correct rule for the combinatorial factor in this case. This procedure is perfectly general, and we simply must for any topology divide by the amount of double counting and then the combinatorial factors are automatically correct.

In order to allow the computer program to go through this in an efficient way it was necessary to build in some special small capabilities into Schoonschip. For instance, a list of charges of the various particles can be given, and the program would first check conservation of charge in the various vertices before searching the list of vertices. The same can be done with respect to ghost lines. In the end a situation resulted whereby the whole two-loop calculation was done in a few minutes. The worst case was the collection of one-loop diagrams with four external lines; the number of diagrams worked out was probably* of the order of 10000. There the evaluation itself was very trivial, because only the infinite part was needed. Again, we could have done without this, but it provided for a very good check of the system.

* We never asked, and the program never said. It took about 3 minutes. The diagrams are generated at a rate of about 10,000/minute.

References

[1] M. Veltman, Acta Phys. Pol. B8 (1977) 475;
B. Lee, C. Quigg and R. Thacker, Phys. Rev. Lett. 38 (1977) 883
[2] D. Ross and M. Veltman, Nucl. Phys. B95 (1975) 135;
M. Veltman, Nucl. Phys. B123 (1977) 89;
M. Chanowitz, M. Furman and I. Hinchliffe, Phys. Lett. 78B (1978) 285
[3] M. Veltman, Nucl. Phys. B21 (1970) 288
[4] T. Appelquist, C. Bernard, Phys. Rev. D22 (1980) 200;
T. Appelquist, C. Bernard, Phys. Rev. D23 (1981) 425;
A. Longhitano, Phys. Rev. D22 (1980) 1166;
A. Longhitano, Nucl. Phys. B188 (1981) 118;
R. Akhoury, Y.P. Yao, Phys. Rev. D25 (1982) 3361
[5] G. Passarino, M. Veltman, Nucl. Phys. B160 (1979) 151
[6] G. 't Hooft and M. Veltman, 1972 Proc. Marseille Conf., Ed. Ch. Korthals-Altes; Nucl. Phys. B50 (1972) 318
[7] G. 't Hooft, M. Veltman, Nucl. Phys. B44 (1972) 189
[8] T.T. Wu, Phys. Rev. 125 (1962) 1436