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COUNTING GROVES–LEDYARD EQUILIBRIA VIA DEGREE THEORY

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We study the Groves–Ledyard mechanism for determining optimal amounts of public goods in economies whose agents have the most general class of preferences for which a Pareto amount of public goods can be computed independently of income distribution. We use degree theory on affine spaces to show that the number of equilibria in such economies grows exponentially as the number of agents in the economy increases. The large number of equilibria in such simple economic models raises doubts as to whether the Groves–Ledyard mechanism is a workable solution to the Free Rider Problem since individuals may have incentives to falsify their preferences in order to drive the adjustment process to a preferred Nash equilibrium.

1. Introduction

Groves and Ledyard (1977) introduced a decentralized method for determining optimal levels of public goods. They formulated a government allocation–taxation scheme which has a Nash equilibrium such that (1) the public good is produced at an optimal level, (2) there is neither a budget surplus nor deficit, and (3) consumers find it in their self-interest to reveal their true preferences for public goods. If one is willing to accept Nash equilibrium as the appropriate equilibrium concept, then the Groves–Ledyard mechanism can be regarded as a solution to the classical Free Rider Problem for public goods.

In a later paper, Groves and Ledyard (1980) present general abstract conditions under which equilibrium for their mechanism exists. However, they have no results concerning the multiplicity of equilibria. Multiple Nash equilibria are an especially vexing problem in this case because a practical implementation of the Groves–Ledyard mechanism must incorporate an adjustment process for attaining Nash equilibria. If there are multiple equilibria with differing distributions of utility, then individuals may have an incentive to falsify their preferences in order to drive the adjustment process to a preferred Nash equilibrium. If on the other hand Groves–Ledyard equilibrium is unique, then it is easy to devise adjustment mechanisms which are cheatproof and converge to the Groves–Ledyard equilibrium.

Applied microeconomists studying simulated or actual environments with public goods usually work with specific families of utility functions which are analytically malleable and which behave nicely under aggregation. They are interested in the existence, uniqueness, and characterization of the equilibria which arise in these more concrete situations. In this paper, we study the Groves–Ledyard mechanism for the two most convenient families of preferences for this purpose. These are: (1) quasi-linear utility with constant marginal utility of private goods, and (2) the more general utility functions which are dual to the Gorman polar form for private goods economies. The latter is the most general class of preferences for which a Pareto amount of public goods can be computed independently of income distribution. Both of these environments always have Groves–Ledyard equilibria. However, for the second class of preferences there are multiple equilibria. In fact, the number of equilibria grows exponentially as the number of agents in the economy increases. This suggests that the Groves–Ledyard mechanism may not be a workable solution to the free-rider problem.

In finding and counting the number of Groves–Ledyard equilibria in the more general models, we use some mathematical techniques which we believe to be at least as interesting as the results they lead to and which are powerful yet simple tools for dealing with equilibria in many situations where neither the domain nor the range of the equilibrium map is compact. In our situation, both spaces are linear subspaces of $\boldsymbol{R}^n$. We replace the usual compactness criteria in equilibrium computations with the notation of a 'proper mapping' and then use degree theory to illustrate how knowledge about the behaviour of a mapping at one point in its image can yield lower bounds for the preimages of other points in the target space. In our model, the target space parametrizes the public goods economies and the pre-images of a point in the target space are the Groves–Ledyard equilibria.

2. The general model

Consider a community with a number $I \geqq 3$ of citizens. Each citizen i has a utility function of the form $u_i(X_i, Y)$ where X_i is his consumption of private goods and Y is the amount of public goods supplied to the community. For the present, let us suppose that there is just one private good and one public good so that X_i and Y are simply non-negative real numbers. Let us also suppose that public goods can be obtained in exchange for private goods at a constant unit cost. If we do so, there is no loss of generality in choosing units of measurement so that one unit of private good can be exchanged for exactly one unit of public good. There is a 'government' which collects 'taxes' in the form of private goods from individuals and exchanges its tax revenue for public goods which it provides to the community. The amount of taxes collected from each individual and the amount of public goods provided will be determined by the government as a function of a list of 'messages' that it receives from the citizens. Each citizen sends a message which is a number m_i, positive or negative, that expresses his desired increment in the output of the public good. Let the vector $m=(m_1,\dots,m_I)$ denote the list of messages received by the government. The government's rules of action can then be described by functions $C_i(m)$ for each i and $Y(m)$ where $C_i(m)$ is person i's tax bill and $Y(m)$ is the amount of public goods supplied if the list of messages is m.

If a consumer has wealth W_i before taxes, and the list of messages is m, then his private consumption will equal his after-tax wealth, $W_i - C_i(m)$. Therefore if the list of messages is m, his utility level will be

$$U_i(m) \equiv u_i(X_i(m), Y(m)), \quad \text{where} \tag{1}$$

$$X_i(m) \equiv W_i - C_i(m). \tag{2}$$

A consumer choosing his message is confronted with a game in which each of the I players chooses a strategy m_i and where the payoff function is (1). A Nash equilibrium for this game is a vector $m^* = (m_1^*,\dots,m_I^*)$ such that

$$\begin{aligned} &U_i(m_1^*,\dots,m_{i-1}^*,m_i^*,m_{i+1}^*,\dots,m_I^*) \\ &\geqq U_i(m_1^*,\dots,m_{i-1}^*,m_i,m_{i+1}^*,\dots,m_I^*) \end{aligned} \tag{3}$$

for all real numbers m_i and for each i. Groves and Ledyard study Nash equilibria for a game of this form where the functions $Y(m)$ and $C_i(m)$ are judiciously chosen.

The functions $Y(m)$ and $C_i(m)$ proposed by Groves and Ledyard are

$$Y(m) = \sum_i m_i, \tag{4}$$

and, for all i,

$$C_i(m) = \alpha_i \sum_h m_h + \frac{\gamma}{2}\left[\frac{I-1}{I}(m_i - \mu^i)^2 - \sigma^{i^2}\right], \tag{5}$$

where γ and the α_i's are arbitrarily chosen parameters such that $\gamma > 0$ and $\sum_i \alpha_i = 1$, and where

$$\mu^i = \frac{1}{I-1}\sum_{h \neq i} m_h, \tag{6}$$

$$\sigma^{i^2} = \frac{1}{I-2}\sum_{h \neq i} (m_h - \mu^i)^2. \tag{7}$$

A Nash equilibrium for the game described by eqs. (1) through (5) will be called a Groves–Ledyard equilibrium.

The workings of the Groves–Ledyard government can be described informally. Each citizen is asked to name a single quantity which he would like to add to or subtract from the amount of public good ordered by others. A positive number m_i denotes an addition and a negative m_i a subtraction. The government will supply the sum of the quantities named by the citizens. Citizen i's tax will consist of a predetermined share α_i of the total value of public good supplied plus an amount that is proportional to the squared deviation of his demand from the average of other citizens' m_i's less an amount that is proportional to the variance of the m_i's stated by others. This last term, σ^{i^2}, is entirely independent of i's choice of m_i.

With some algebraic manipulation of expression (4) and (5) it can be shown that

$$Y(m)=\sum_i C_i(m), \tag{8}$$

for all m. Therefore the Groves–Ledyard government always balances its budget. Groves and Ledyard show that if preferences are convex then the Groves–Ledyard equilibrium produces a Pareto optimal allocation.

In this paper we assume that $u_i(X_i, Y)$ is strictly quasi-concave and twice continuously differentiable. We will be primarily interested in 'interior' Groves–Ledyard equilibria. These are equilibria in which $Y(m^*)>0$ and $X_i(m^*)>0$ for all i. In fact, for the class of economies that we study, reasonable economic assumptions can be found which guarantee that all Groves–Ledyard equilibria are interior.

A necessary condition for m^* to be a Groves–Ledyard equilibrium is that $\partial U_i(m^*)/\partial m_i=0$ for all i. Differentiating (1) with respect to m_i we see that this first-order condition is equivalent to

$$\frac{\partial C_i(m^*)}{\partial m_i}=\frac{\partial u_i(X_i(m^*),\ Y(m^*))}{\partial Y}\div\frac{\partial u_i(X_i(m^*),\ Y(m^*))}{\partial X_i}, \tag{9}$$

for every i. In fact, given quasi-concavity of $u_i(X_i, Y)$, eq. (9) is sufficient as well as necessary for m_i^* to satisfy (3).[1]

Differentiating (5) reveals that

$$\frac{\partial C_i(m)}{\partial m_i}=\alpha_i+\gamma\left(\frac{I-1}{I}\right)(m_i-\mu_i)=\alpha_i+\gamma(m_i-\bar{m}), \tag{10}$$

[1]If $u_i(X_i, Y)$ is quasi-concave in X_i and Y, then $U_i^*(m)$ is a quasi-concave function of m_i. This follows from straightforward application of the definition of quasi-concavity, the fact that $C_i(m)$ is a convex function of m_i and that utility is an increasing function of Y. Since $U_i^*(m)$ is quasi-concave, the first-order condition for maximization is sufficient.

where $\bar{m}=(1/I)\sum_i m_i$. Therefore eqs. (9) can be written as

$$\alpha_i+\gamma(m_i^*-\bar{m}^*)=\frac{\partial u_i(X_i(m^*),Y(m^*))}{\partial Y}\div\frac{\partial u_i(X_i(m^*),Y(m^*))}{\partial X_i}. \tag{11}$$

A necessary and sufficient condition for m^* to be a Groves–Ledyard equilibrium is that m^* solve the equations system (2), (4) and (11). We will exploit this fact in solving for and enumerating Groves–Ledyard equilibria.

Pareto efficiency of the Groves–Ledyard equilibrium can be demonstrated by showing that eqs. (2), (4) and (11) imply the well-known Samuelson first-order necessary and sufficient conditions for Pareto optimal allocation when preferences are convex. To see this we sum eqs. (11) over all i to obtain

$$1=\sum_i\left(\frac{\partial u_i(X_i(m^*),Y(m^*))}{\partial Y}\div\frac{\partial u_i(X_i(m^*),Y(m^*))}{\partial X_i}\right). \tag{12}$$

[Here we use the obvious fact that the sum over i of the right-hand in side (10) equals one.] Eq. (12) requires the summed marginal rates of substitution for the public good to equal the marginal rate of transformation between private and public goods. If we add the budget eqs. (2) and substitute from (8), we find that

$$\sum_i X_i(m^*)+Y(m^*)=\sum_i W_i. \tag{13}$$

Since preferences are assumed to be convex, eqs. (12) and (13) imply that $(X_1(m^*),\ldots,X_I(m^*),Y(m^*))$ is a Pareto optimal allocation.

3. Quasi-linear utility

In the simplest models of economies with public goods, all citizens have quasi-linear utility (constant marginal utility of private goods). See, for example, Feldman (1980, ch. 6). In such models, computing Groves–Ledyard equilibrium is particularly simple and it turns out that equilibrium is unique. Quasi-linear utility has the special form

$$u_i(X_i,Y)=X_i+f_i(Y), \tag{14}$$

for some strictly concave function f_i. In this case, the first-order condition (11) specializes to

$$\alpha_i+\gamma(m_i^*-\bar{m}^*)=f_i'(Y(m^*)). \tag{15}$$

Summing eq. (15) over i yields

$$1=\sum_i f_i'(Y(m^*)). \tag{16}$$

Since, by assumption, $f_i''<0$ for all i, there can be at most one value of Y that satisfies (16). Let this value be $Y^*=Y(m^*)$. According to eq. (4), $Y^*=\sum_i m_i^*=I\bar{m}^*$. Therefore eq. (15) can be rearranged as

$$m_i^*=\frac{1}{\gamma}(f_i'(Y^*)-\alpha_i)+\frac{Y^*}{I}, \tag{17}$$

which solves uniquely for m_i^*.

So far we have shown that there can be no more than one interior Groves–Ledyard equilibrium when preferences are quasi-linear. We must also find conditions which insure that there is at least one interior equilibrium. Let us assume that the problem is non-trivial in the sense that Pareto efficiency requires positive aggregate outputs of both public and private goods. Then there will exist some Y^* such that $\sum_i f_i'(Y^*)=1$ and $Y^*<\sum_i W_i$. We show that if this assumption holds, there will always exist some parameters $\alpha_1,\dots,\alpha_n$ and γ for which an interior Groves–Ledyard equilibrium exists.

An interior Groves–Ledyard equilibrium will exist if for m^* defined by (17), we have

$$X_i(m^*)=W_i-C_i(m^*)>0. \tag{18}$$

Using eqs. (4), (5), (7) and (17) we can show that

$$C_i(m^*)\leqq\alpha_i Y^*+\frac{1}{2\gamma}\left(\frac{I}{I-1}\right)(f_i'(Y^*)-\alpha_i)^2. \tag{19}$$

Therefore if $\alpha_i Y^*<W_i$ for all i, then (18) would hold for all sufficiently large γ. But since by assumption, $Y^*<\sum_i W_i$, we could guarantee that $\alpha_i Y^*<W_i$ by setting $\alpha_i=(W_i/\sum_i W_i)$. Therefore there are always some α_i's and a γ for which an interior Groves–Ledyard equilibrium exists.

4. A more general class of preferences

Finding a unique Groves–Ledyard equilibrium in the case of quasi-linear utility was easy because there were no 'income effects' on individual marginal rates of substitution. On the other hand, available empirical evidence convincingly refutes the hypothesis that individual marginal willingness to pay for public goods is independent of the level of one's private consumption. Therefore we study the Groves–Ledyard equilibrium in more general environments. We consider a family of utility functions that lies intermediate in generality between quasi-linear utility and general quasi-concave functions. This class was introduced by Bergstrom and Cornes (1981, 1983) and consists of preferences representable by a utility function of the form

$$U_i(X_i, Y) = A(Y)X_i + B_i(Y), \tag{20}$$

for each i. Bergstrom and Cornes show that this class of preferences, which is dual to the Gorman polar form for private goods is exactly the class for which a Pareto amount of public goods can be computed independently of income distribution. This class is considerably broader than the quasi-linear class and allows individual marginal rates of substitution to depend on consumption of private goods as well as public goods.

If utility functions are of this form, then individual marginal rates of substitution between public good and private goods can be written as

$$\frac{\partial U_i(X_i, Y)}{\partial Y} \div \frac{\partial U_i(X_i, Y)}{\partial X_i} = \left(\frac{A'(Y)}{A(Y)}\right) X_i + \frac{B_i'(Y)}{A(Y)}. \tag{21}$$

Let

$$a(Y) \equiv \frac{A'(Y)}{A(Y)} \quad \text{and} \quad b_i(Y) \equiv \frac{B_i'(Y)}{A(Y)}.$$

Then the equations system (2), (4), (11) that constitutes the first-order conditions for a Groves–Ledyard equilibrium is

$$a(Y(m^*))X_i(m^*) + b_i(Y(m^*)) = \alpha_i + \gamma(m_i^* - \bar{m}^*), \tag{22}$$

$$\sum_i m_i^* = Y(m^*), \tag{23}$$

$$X_i(m^*) = W_i - C_i(m^*), \tag{24}$$

where $C_i(m^*)$ is defined by (5).

Summing eq. (22) over all i yields

$$a(Y(m))X(m) + \sum_i b_i(Y(m)) = 1, \tag{25}$$

where

$$X(m) \equiv \sum_i X_i(m). \tag{26}$$

Summing eqs. (24) over all i and recalling (8), we have

$$X(m) = \sum_i W_i - Y(m). \tag{27}$$

The assumption that $U_i(X_i, Y)$ is strictly quasi-concave is equivalent to the assumption that $1/A(Y)$ is a strictly convex function and $B_i(Y)$ is a strictly concave function of Y.[2] Therefore if each $U_i(X_i, Y)$ is strictly quasi-concave, then so is the 'aggregate utility function',

$$U(X, Y) \equiv A(Y)X + \sum_i B_i(Y). \tag{28}$$

Now (25) is the first-order condition for maximizing (28) subject to the constraint (27). If preferences are strictly quasi-concave, therefore, eq. (27) will have at most one solution for $X(m)$ and $Y(m)$. If we also assume that the problem is non-trivial in the sense that there is some Pareto optimal allocation with positive total outputs of both public and private goods, then there is exactly one aggregate output vector (X^*, Y^*) that satisfies (25).

Having solved for $Y^* = \sum_i m_i^*$, we have next to solve for the individual m_i^*'s from the equation system (22) and (24). Let $a^* = a(Y^*)$ and $b_i^* = b_i(Y^*)$. Then this system of equations can be reduced to

$$a^* C_i(m^*) + \gamma(m_i^* - \bar{m}^*) = a^* W_i + b_i^* - \alpha_i. \tag{29}$$

Recalling (5) we notice that (29) is a quadratic function in the variables $m_i^* - \bar{m}^*$. This system of equations is simplified if we make the affine change of variables

$$q_i = m_i^* - \bar{m}^* + \frac{I-2}{Ia^*}. \tag{30}$$

Substituting from (30) into (29) and rearranging terms leads to

$$q_i^2 - \frac{1}{I}\sum_i q_i^2 = k_i \quad \text{for} \quad i = 1, \ldots, I, \tag{31}$$

[2]Quasi-concavity of $U(X, Y) = A(Y)X + B(Y)$ is seen to be equivalent to convexity of the function $h(Y) \equiv (1/A(Y))u - (B(Y)/A(Y))$ for all $u \geqq 0$. But $h(Y)$ is convex for all $u \geqq 0$ if and only if $(1/A(Y))$ is a convex function and $(B(Y)/A(Y))$ is a concave function of Y.

where

$$k_i = \frac{2}{\gamma}\left(\frac{I-2}{I}\right)\frac{1}{a^*}(a^*(W_i - \alpha_i Y) + b_i^* - \alpha_i). \tag{32}$$

(See appendix 1 for more details of these computations.) Summing (30) over i yields

$$\sum_i q_i = \frac{I-2}{a^*}. \tag{33}$$

The $I \times I$ system of eq. (31) is linear in the squares of the q_i's and is of rank $I-1$. The other equation, eq. (33), is linear in the q_i's.

Finally, we set $z_i = (a^*/(I-2))q_i$ and rewrite (31) and (33) in terms of the z_i's. This yields

$$A(z_1^2, \dots, z_I^2) = (k_1, \dots, k_I), \tag{34}$$

and

$$\sum_i z_i = 1, \tag{35}$$

where A is an $I \times I$ matrix for which the off-diagonal elements are all 1's and the diagonal elements are all $1-I$. The rows of the symmetric matrix A sum to $(0, \dots, 0)$. In fact, the rank of A is $I-1$, and its null space is spanned by the vector $(1, \dots, 1)$. Since the row space (and column space) of A is the orthogonal complement of the null space, $(k_1, \dots, k_I)$ is in the image (i.e., the column space) of A if and only if

$$(k_1, \dots, k_I)\cdot(1, \dots, 1) = 0, \quad \text{i.e.,} \quad \sum k_i = 0.$$

One uses (32) to check that the k_i's do indeed sum to zero. So (34) has a *line* of 'solutions' $(z_1^2, \dots, z_I^2)$ for each $(k_1, \dots, k_I)$ defined by (32). One then uses (35) to reduce this solution set to a finite number of points. Finally, one uses $z_i = (a^*/(I-2))q_i$ and (30) to find the unique message m^* which corresponds to each one of these solutions z^* of (34) and (35).

The k_i's in (32) contain all the exogenous data of the model. For example, if each citizen has the same wealth W_i and the same preferences, and if the tax shares are equalized so that each α_i equals $1/I$, then $k_i = 0$ for $i = 1, \dots, I$. So, in a sense, k represents the deviation from perfect symmetry. The solution to (34) in this special case where $k_1 = \dots = k_I = 0$ requires that

$$z_1^2 = z_2^2 = \dots = z_I^2. \tag{36}$$

Consider the case where $I=3$ and $k=0$. Then (35) and (36) are satisfied only at the symmetric solution $z_1=z_2=z_3=\frac{1}{3}$ and at the three asymmetric solutions in which one of the z_i's is -1 and the other two are $+1$.

More generally, for all I, (36) implies that for some $z>0$ and all i,

$$z_i = \pm z, \tag{37}$$

while (35) and (37) imply

$$1 = zN_+ - (I - N_+)z = (2N_+ - I)z, \tag{38}$$

where N_+ is the number of indices i for which $z_i>0$. For each choice of $N_+>N/2$ there are $\binom{I}{N_+}$ distinct solutions to (38) each of which corresponds to a different N_+ member subset of I having positive z_i's. Table 1 enumerates the solutions at $k_1=k_2=\ldots=k_I=0$ for various values of I. As we see, the number of solutions increases exponentially as I increases. In fact, for I odd, there are 2^{I-1} solutions.

Table 1

I	Number of solutions
3	4
4	5
5	16
6	22
7	64
n	$\sum_{n/2<j\leq n} \binom{n}{j}$

Although there are no simple algebraic expressions for the solutions of (34), (35) for general $(k_1,\ldots,k_n)$, we can tell a great deal about the number of solutions in general by using the tools of differential topology.

Consider the map $F:\sum_1 \to \sum_0$ where $\sum_a \equiv \{x \in R^n \mid \sum_i x_i = a\}$ and $F(x_1,\ldots,x_n)=(x_1^2,\ldots,x_n^2)A$, with A a matrix with 1's off the diagonal and $1-n$ in each diagonal location. The solutions to eqs. (34) and (35) are precisely the elements of the set $F^{-1}(k_1,\ldots,k_n)$. A vector $(k_1,\ldots,k_n)$ is said to be a *regular value* of F if $DF(x)$ is non-singular[3] for all $x \in F^{-1}(k)$ or if $F^{-1}(k)$ is empty. The *degree* of the map F at a regular value k is equal to

[3]Here $DF(x_i)$ is the Jacobian derivative of F at x_i. To evaluate it, choose global coordinate systems for the $(n-1)$-dimensional hyperplanes Σ_1 and Σ_0. Let $\tilde{F}: \boldsymbol{R}^{n-1} \to \boldsymbol{R}^{n-1}$ be F and $\tilde{x}_i$ be x_i in these coordinate systems. Then, one can use the $(n-1)\times(n-1)$ Jacobian matrix $((\partial \tilde{F}_h/\partial \tilde{x}_j(\tilde{x}_i)))_{h,j=1,\ldots,n-1}$ to represent $DF(x_i)$. By Sard's Theorem, most points in the range of any F are regular values in the sense that the non-regular values (i.e., 'critical values') form a set of measure zero in the range. See, for example, Milnor (1965).

$\sum_{x_i \in F^{-1}(k)} \operatorname{sign} \det \mathrm{D}F(x_i)$. If F is a mapping between compact manifolds without boundary, like a sphere or torus, then the degree of F at k turns out to be the same at all regular values k in the image manifold and is called the degree of the map F. [See Schwartz (1969) or Milnor (1965) for a complete discussion of degree theory. One can also define the degree of a map using homology theory or by an integral formula. These methods yield a degree theory for non-smooth maps.] In particular, if k is not in the image of F, i.e., F is not onto, then the degree of F at k is zero and so the degree of the map F is zero. As a result, degree theory is a powerful technique for showing that a smooth map between two compact manifolds is onto. One need only show that the degree is non-zero at one regular point in the image. Furthermore, it is clear from the definition of degree that the number of elements in the inverse image of F at any regular point must be at least as large as the degree of the map. So, a calculation at just *one* point can show that *every* point is in the image of F and can give a lower bound for the size of each $F^{-1}(k)$.[4]

We would like to apply this powerful technique to our map $F: \sum_1 \to \sum_0$. However, $\sum_1$ and $\sum_0$ are hyperplanes, not compact spaces. Some compactness must be added to F in order to make the degree theory work. One way of accomplishing this is to require that F be a 'proper map'. A map $G: X \to Y$ is *proper* if the inverse image of any compact set in Y is a compact set in X. (If G is continuous and X is compact, G is automatically proper.) If X and Y are affine spaces, like $\sum_a$ or even $\boldsymbol{R}^n$, then a continuous G is proper if the inverse image of any bounded set is bounded, i.e., if $|x_n| \to \infty$ in X, then $|G(x_n)|$ must $\to \infty$ in Y.

If $G: X \to Y$ is continuous and proper, then (1) the degree of G is well-defined, (2) G will be surjective if the degree of G is not zero, and (3) for all regular values k, the cardinality of $G^{-1}(k) \geqq$ absolute value of the degree of G[5]. [Balasko (1975) used the degree of proper mappings to derive results on the uniqueness of equilibrium.]

The analysis shows that the equations system (34) and (35) has at least the number of solutions stated in table 2 for any $(k_1, \ldots, k_n)$ such that $\sum_i k_i = 0$. Since eqs. (34) and (35) are the results of an affine change of variables from the Groves–Ledyard first-order conditions (22)–(24) [see (40) below], these

[4]Sometimes, one can even use the degree of a map to show that the map is one-to-one. For example, if the degree of F is 1 and if $\det \mathrm{D}F(x)$ never changes sign (i.e., F is 'sense-preserving'), then each point in the range must have exactly one pre-image. This is the idea behind Mas-Colell's (1979) proof of the Gale–Nikaido Theorem.

[5]To see why this statement is true, let $G: X \to Y$ be a continuous map between affine spaces X and Y. One can 'compactify' X and Y to $\tilde{X}$ and $\tilde{Y}$ by adding a point at infinity to both spaces. The new spaces $\tilde{X}$ and $\tilde{Y}$ can be considered as spheres. By requiring that G map$\{\infty\}$ to $\{\infty\}$, one defines an extension of G to a map $G: \tilde{X} \to \tilde{Y}$. The properness of G is exactly what one needs to show that $\tilde{G}$ is continuous everywhere, even at $\{\infty\}$. One can now apply all the techniques of degree theory to $\tilde{G}$ and hence to G.

Table 2

I	Degree of F
3	2
4	3
5	6
6	-10
7	-20
n	$-\sum_{j=0}^{[n/2-1]}(-1)^j\binom{n}{j}$

first-order conditions have at least as many solutions as are recorded in table 2.

Some solutions to these equations may not be Groves–Ledyard equilibria because they do not satisfy the economic non-negativity constraints of the original problem. To study this question, we need to invert our change of variables and see whether the vector $(m_1,\dots,m_I)$ that corresponds to a given solution $(z_1,\dots,z_I)$ of (33) and (34), allows positive consumptions for all consumers.

Since $z_i=(a^*/(I-2))q_i$, we see from (30) that

$$z_i=\left(\frac{a^*}{I-2}\right)(m_i-\bar{m}^*)+\frac{1}{I} \tag{39}$$

and

$$(m_i-\bar{m})^*=\left(\frac{I-2}{a^*}\right)\left(z_i-\frac{1}{I}\right). \tag{40}$$

Eq. (A.3) in appendix 1 shows that

$$C_i(m)\equiv\alpha_i Y^*+\frac{\gamma}{2}\frac{I}{I-2}\left[(m_i^*-\bar{m}^*)^2-\frac{1}{I}\sum_j(m_j^*-\bar{m}^*)^2\right]. \tag{41}$$

From (40) and (41) it follows that the solution $z_1,\dots,z_n$ implies

$$C_i=\alpha_i Y^*+\frac{\gamma}{2}\frac{(I-2)I}{a^{*2}}\left[\left(z_i-\frac{1}{I}\right)^2-\frac{1}{I}\sum_j\left(z_j-\frac{1}{I}\right)^2\right]. \tag{42}$$

It is clear from (42) that if the α_i's are chosen so that $\alpha_i Y^*<W_i$ for all i and if γ is chosen to be sufficiently small, then $C_i<W_i$ for C_i corresponding to any of the solutions $(z_1,\dots,z_n)$ of eqs. (34) and (35). We have argued before that very weak assumptions ensure that it is possible to choose the α_i's so that $\alpha_i Y^*<W_i$ for all i. Therefore it is always possible to choose parameters

$\alpha_1, \dots, \alpha_n$ and γ so that all of the solutions to (34) and (35) are Groves–Ledyard equilibria.

Appendix 1

In this appendix, we sketch the calculations involved in progressing from the system (29) to the system (31), (32). Given a message vector $m = (m_1, \dots, m_I)$, recall that

$$\bar{m} = \frac{1}{I}\sum_1^I m_h \quad \text{and} \quad \mu^i = \frac{1}{I-1}\sum_{h \neq i} m_h.$$

By adding and subtracting $m_i/(I-1)$ from the left-hand side, one computes easily that

$$m_i - \mu^i = \frac{I}{I-1}(m_i - \bar{m}). \tag{A.1}$$

First, let

$$p_i = m_i - \bar{m} = \frac{I-1}{I}(m_i - \mu^i).$$

Add and subtract $m_i - \bar{m}$ to the term in parentheses in (7). Then use (A.1) to compute that in (7)

$$\sigma^{i^2} = \frac{1}{I-2}\sum_{h \neq i}\left(p_h + \frac{1}{I-1}p_i\right)^2 = \frac{1}{I-2}\left(\sum_h p_h^2 - \frac{I}{I-1}p_i^2\right). \tag{A.2}$$

Plug (A.1) and (A.2) into (5), rearrange terms, and use (4) to find that

$$C_i(m) = \alpha_i Y + \frac{\gamma}{2}\frac{I}{I-2}\left(p_i^2 - \frac{1}{I}\sum_h p_h^2\right). \tag{A.3}$$

Plug (A.3) into (29) and rearrange terms again to find

$$p_i + \frac{a^*}{2}\frac{I}{I-2}\left(p_i^2 - \frac{1}{I}\sum_h p_h^2\right) = \frac{1}{\gamma}[a^*(W_i - \alpha_i Y) + b_i^* - \alpha_i]. \tag{A.4}$$

Finally, the change of variables (30),

$$q_i = p_i + \frac{I-2}{Ia^*},$$

changes the system (A.4) to the system (31), (32).

Appendix 2

In this appendix, we discuss the solution of the system (34),(35) for general $(k_1,\dots,k_I)$. Let $\Sigma_a=\{x\in \boldsymbol{R}^I \mid \sum_i x_i=a\}$ and let

$$F(x_1,\dots,x_I)=(x_1^2,\dots,x_I^2)A,$$

where A is the matrix with $1-I$ in each diagonal entry and 1 in each off-diagonal entry. Then, F maps Σ_1 to Σ_0 and a solution of (34),(35) is an element of $F^{-1}(k_1,\dots,k_I)$.

One approach is to 'decompose' F into $\psi\circ\phi$ where $\phi:\Sigma_1\to\boldsymbol{R}^I$ is the map

$$\phi(x_1,\dots,x_I)=(x_1^2,\dots,x_1^2),$$

and $\psi:\boldsymbol{R}^I\to\Sigma_0$ is the orthogonal projection. This decomposition works because the system (34) can be written as

$$\begin{bmatrix} q_1^2 \\ \vdots \\ q_I^2 \end{bmatrix}=\begin{bmatrix} K_1 \\ \vdots \\ K_I \end{bmatrix}+\frac{1}{I}\sum_1^I q_h^2\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

where $K_i=-k_i/I$. For $I=2$, one can show easily that $\psi\circ\phi$ is one-to-one and onto. Fig. 1 summarizes the geometry of this approach.

However, for $I=3$, ϕ is a map from a two-dimensional hyperplane into $\boldsymbol{R}^3$. One easily checks that the image of ϕ folds over itself around the points

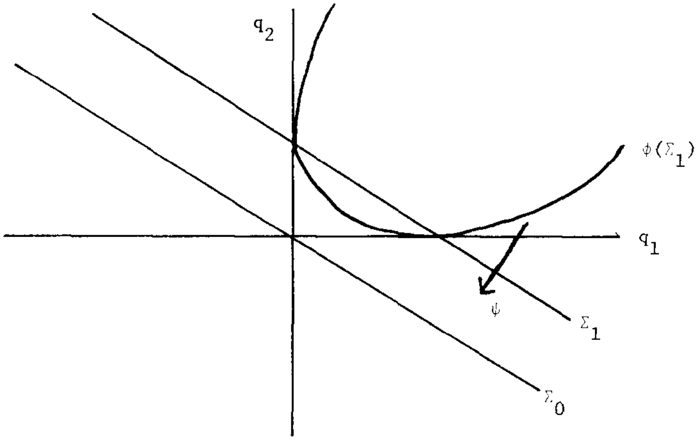

Fig. 1. $\psi\circ\phi$ for $I=2$.

$\phi(1,0,0)$, $\phi(0,1,0)$ and $\phi(0,0,1)$. These crossings of $\phi(\Sigma_1)$ turn out to be examples of the only generic singularity of a mapping from $\boldsymbol{R}^2 \to \boldsymbol{R}^3$, the Whitney Umbrella, as pictured in fig. 2. [See Martinet (1982) for more details on this singularity.] The occurrence of this singularity implies that ϕ and therefore F is not one-to-one. So we can expect multiple equilibria.

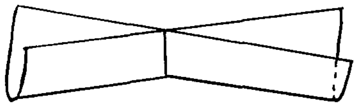

Fig. 2. Whitney's Umbrella.

We turn now to the more analytical approach described in the main part of the paper. We first show that F is proper. We then calculate the degree of F by calculating the degree at $k=(0,0,\ldots,0)$.

To show that F is proper, we need to show that the inverse image of any bounded set is bounded. We will use the Euclidean norm $|x|=(\sum x_i^2)^{\frac{1}{2}}$. Let

$$X_j = F_j(x_1,\ldots,x_I) = (1-I)x_j^2 + \sum_{h \neq j} x_h^2.$$

Then

$$X_j^2 = (1-I)^2 x_j^4 + \sum_{h \neq j} x_h^4 + 2 \sum_{h \neq j} x_h^2 x_j^2 - 2(I-1) \sum_{\substack{h<i \\ h \neq j \\ i \neq j}} x_h^2 x_i^2,$$

and

$$|F(x)|^2 = \sum_j X_j^2 = I\left[\sum_{i<j} (x_i^2 - x_j^2)^2\right]. \tag{A.5}$$

Suppose that

$$|F(x)|^2 \leqq b^2 \quad \text{and} \quad \sum x_h = 1. \tag{A.6}$$

We want to show that $|x|$ is bounded. Suppose there is an unbounded sequence $\{x^n\}$ which satisfies (A.6). Without loss of generality, we can assume that $x_1^n \to +\infty$. By (A.5) and (A.6),

$$({x_1^n}^2 - {x_j^n}^2)^2 \leqq b^2 \quad \text{for all } j. \tag{A.7}$$

Therefore, each sequence of numbers $\{x_j^n\}$ is also unbounded. By taking subsequences, we can assume that

$$\begin{aligned} &x_1^n,\ldots,x_h^n \to +\infty \quad \text{as} \quad n \to \infty, \\ &x_{h+1}^n,\ldots,x_I^n \to -\infty \quad \text{as} \quad n \to \infty. \end{aligned} \tag{A.8}$$

For $i=1,\dots,h$,

$$(x_1^n - x_i^n)^2 \leqq b^2/(x_1^n + x_i^n)^2 \to 0,$$

by (A.7) and (A.8). Similarly, for $i=h+1,\dots,I$,

$$(x_1^n - (-x_i^n))^2 \leqq b^2/[x_1^n + (-x_i^n)]^2 \to 0.$$

Choose N so that for $n>N$,

$$|x_1^n - x_i^n| < 1/2I \quad \text{for} \quad i \leqq h,$$

and

$$|x_1^n + x_i^n| < 1/2I \quad \text{for} \quad i > h.$$

Then, for $n>N$,

$$1 = \sum_i x_i^n = \sum_{i=1}^{h} (x_1^n + (x_i^n - x_1^n)) + \sum_{i=h+1}^{I} (-x_1^n + (x_i^n + x_1^n))$$

$$= (2h-I)x_1^n + \sum_{i=1}^{I} a_i^n \quad \text{where} \quad \left|\sum a_i^n\right| \leqq \sum |a_i^n| < \tfrac{1}{2}.$$

This implies that

$$|1-(2h-I)x_1^n| < \tfrac{1}{2} \quad \text{for all} \quad n>N.$$

This contradiction to $x_1^n \to \infty$ means that (A.6) defines a bounded set of x's, i.e., F is proper.

Since F is proper, the degree of F is well-defined and may be computed using any regular value. We will work with the value $k=(0,\dots,0)$ and will choose $(x_1,\dots,x_{I-1})$ as a coordinate system for both Σ_0 and Σ_1 in $\boldsymbol{R}^I$. In this coordinate system, the Jacobian matrix of DF is the $(I-1)\times(I-1)$ matrix:

$$\begin{bmatrix} -2(I-1)x_1 - 2x_I & 2x_2 - 2x_I & \dots & 2x_{I-1} - 2x_I \\ 2x_1 - 2x_I & -2(I-1)x_2 - 2x_I & \dots & 2x_{I-1} - 2x_I \\ \vdots & \vdots & & \vdots \\ 2x_1 - 2x_I & 2x_2 - 2x_I & & -2(I-1)x_{I-1} - 2x_I \end{bmatrix},$$

where $x_I = 1 - x_1 - x_2 - \dots - x_{I-1}$. To compute the determinant of this matrix, first subtract the first row from each of the other rows, then add x_1/x_j times column j to column 1 for $j>1$. The result will be an upper triangular matrix whose determinant (the product of the diagonal entries) is the same as that of our original Jacobian. Some simple algebra shows that

$$\det \mathrm{D}F(x)=(-1)^{I-1}[2^{I-1}I^{I-2}]\cdot x_1x_2\ldots x_I\cdot(1/x_1+\ldots+1/x_I). \tag{A.9}$$

The solutions of $F(z)=0$ must satisfy (35) and (36), i.e., $|z_i|$ must equal some non-zero constant a independent of i. If h of the z_i's are positive and $(I-h)$ are negative,

$$1=\sum_1^I z_i=ha+(I-h)(-a)=(2h-I)a,$$

or

$$a=1/(2h-I).$$

This implies that $h>I/2$. It also implies that

$$1/z_1+\ldots+1/z_I\neq 0;$$

so, $\det \mathrm{D}F(z)\neq 0$ in (A.9) for all z in $F^{-1}(0)$, i.e., 0 is a regular value of F.

For each I and each integer h such that $I/2<h<I$, there are exactly $\binom{I}{h}$ solutions of $F(z)=0$. For each of these solutions z (with I and h fixed), $\det \mathrm{D}F(z)$ will have the same sign by (A.9). If I is fixed and h changes by one, the sign of all the $\det \mathrm{D}F(z)$'s will also change. If $h=I$, this sign will be $(-1)^{I-1}$. It follows that the degree of F at 0

$$\equiv \sum_{z\in F^{-1}(0)} \operatorname{sign}\det \mathrm{D}F(z)$$

$$=(-1)^{I-1}\left[\binom{I}{I}-\binom{I}{I-1}+\binom{I}{I-2}-\ldots\pm\binom{I}{I^*}\right],$$

where I^* is the least integer strictly greater than $I/2$. These numbers are listed in table 2 for various values of I. Since they are all non-zero, F is surjective; their absolute value gives a lower bound for the cardinality of $F^{-1}(k)$ for each regular value k.

Let S denote the singular set of F in Σ_1, i.e., $S=\{z\in\Sigma_1 \mid \det \mathrm{D}F(z)=0\}$. Let T denote the component of $\Sigma_1\backslash S$ which contains the regular point $(1/I,1/I,\ldots,1/I)$. Then, the restriction

$$F|T: T\rightarrow F(T)$$

is a one-to-one mapping. For example, when $I=3$,

$$S=\{(x_1,x_2,x_3) \mid \Sigma x_i=1 \quad \text{and } x_1x_2+x_1x_3+x_2x_3=0\},$$

by (A.9). In our (x_1, x_2)-coordinates, S is the ellipse

$$x_1^2 + x_1 x_2 + x_2^2 - x_1 - x_2 = 0,$$

while $F(S)$ is a closed curve with 3 cusps in Σ_0. By following through the changes of coordinates, one notes that the ellipse S (and therefore the region T on which F is one-to-one) becomes larger as $a^* \to 0$. Since $a^* = A'(Y^*)/A(Y^*)$, $a^* = 0$ corresponds to the quasi-linear utility function $x_i + f_i(Y)$ that we studied earlier $[A(Y) \equiv 1]$ where the corresponsing F is globally one-to-one.

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