JOURNAL OF ALGEBRA 84, 503–553 (1983)

Canonical Elements in Local Cohomology Modules and the Direct Summand Conjecture

MELVIN HOCHSTER*

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

Communicated by D. A. Buchsbaum

Received August 27, 1982

1. INTRODUCTION

One of the objectives of this paper is to show that the usual homological consequences of the existence of big Cohen–Macaulay (henceforth, C–M) modules (e.g., the new intersection conjecture of Peskine–Szpiro and Roberts and the Evans–Griffith syzygy conjecture) follow from the direct summand conjecture when the residual characteristic of the local ring is positive. This gives a new and substantially more elementary proof of the standard homological conjectures in case the characteristic of the ring itself is positive, and reduces the general case of all these conjectures to one rather down-to-earth conjecture. Of course, this places the direct summand conjecture in a position of central importance, so that it now merits an all-out attack. Some partial results on this problem are given in Section 6. (The conjecture asserts that a regular Noetherian ring R is a direct summand (as an R-module) of every module-finite extension ring $S \supset R$.)

The other main objective of this paper is to formulate and develop a theory of certain "canonical elements" in the local cohomology of special modules of syzygies. (Neither the modules of syzygies nor the induced maps between them are canonical, but the identification between the canonical elements *is* independent of the choices.) In particular, a canonical element $\eta_R \in H^n_{\mathfrak{m}}(\mathrm{syz}^n K)$ is associated (see Section 3 for details) with each n-dimensional local ring $(R, \mathfrak{m}, K)$ (rings are commutative, associative, with identity; "local ring" means Noetherian ring with a unique maximal ideal). The conjecture that η_R is nonzero for all local rings R turns out to be equivalent to the direct summand conjecture: for a given R, an infinite family of cases of the latter implies the former.

* The author was supported in part by grants from the National Science Foundation and the John Simon Guggenheim Memorial Foundation.

0021-8693/83 $3.00

The canonical elements behave very functorially, so that the conjecture "$\eta_R \neq 0$ for all local R" lends itself to a large number of equivalent formulations. In fact, we formulate a condition equivalent to $\eta_R \neq 0$ ("property CE") in Section 2 without reference to local cohomology, and it is this form that we use to prove equivalence with the direct summand conjecture and utilize to deduce the usual homological conjectures. It is worth noting that $\eta_R \neq 0$ whenever R has a big C–M module. (This is the essential content of Theorem (3.8).)

The reader should be aware that, roughly speaking, the conjectures under consideration here are known for rings containing a field and in dimension $\leqslant 2$, and open otherwise. See [14, 18, 25].

To emphasize the very elementary nature of the arguments in the early part of the paper, we have avoided all unnecessary machinery, including local cohomology. This is slightly awkward occasionally, but seems worthwhile.

In Section 3 we formulate and study the theory of canonical elements. In Section 4 we discuss some connections with canonical modules pointed out to be author by Joseph Lipman. In Section 6 we return to an investigation of the direct summand conjecture. In particular, we give a new proof of it in char. $p > 0$, and in mixed characteristic we show that it reduces to the case of a formal power series ring over a complete *unramified* discrete valuation ring.

The author is indebted to E. Graham Evans, who suggested a conjecture closely related to (3.16) several years ago, and to Joseph Lipman, for many valuable conversations concerning the material in this paper.

2. Property CE, the Direct Summand Conjecture, and the Deduction of the Homological Conjectures

If $\mathbf{x}$ is the sequence $x_1,\ldots,x_n \in R$ and M is an R-module, $K_*(\mathbf{x}; M)$ denotes the (homological) Koszul complex of M with respect to $x_1,\ldots,x_n$. If t is a positive integer, $\mathbf{x}^t = x_1^t,\ldots,x_n^t$. If $s \geqslant t$, there is a map $K_*(\mathbf{x}^s; M) \to K_*(\mathbf{x}^t; M)$ which multiplies the free generator indexed by $i_1 \cdots i_r$ by $(x_{i_1} \cdots x_{i_r})^{s-t}$.

We first want to define what it means for a local ring to "have property CE" or to "satisfy CE." (This will later turn out to mean that the canonical element η_R associated with R is not 0.)

(2.1) Definition. A local ring $(R, \mathfrak{m}, K)$ of dimension n *satisfies* CE (or *has property* CE) if for every projective resolution

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to K \to 0$$

of the residue class field K and for every system of parameters $x_1,\dots,x_n$ for R, if ϕ is any map of complexes $K_*(\mathbf{x};R)\to P_*$ which lifts the quotient surjection $R/(x_1,\dots,x_n)\to K$, then $\phi_n: K_n(\mathbf{x};R)\to P_n$ is *nonzero*.

Of course, since the projective resolution is exact and $K_*(\mathbf{x};R)$ is free, there do exist such maps ϕ. We also note that projectives over local rings are always free [23]. Moreover:

(2.2) *Remarks.* (1) $K_n(\mathbf{x};R)=R$.

(2) The condition is independent of the resolution, since given two resolutions P_*, Q_* there are maps $P_*\to Q_*$, $Q_*\to P_*$ which lift the identity map on K.

(3) Hence, it is enough to consider a minimal resolution of K.

(4) One need only assume that the induced map $R/(x_1,\dots,x_n)R\to K$ is nonzero (for one can multiply by a unit to adjust this otherwise).

(5) If Q_* is a projective resolution

$$\to Q_j\to\cdots\to Q_0\to 0$$

of M (i.e., $H_j(Q_*)=0$, $j\geqslant 1$, while $H_0(Q_*)\cong M$), let $\operatorname{syz}^i_R(M;Q_*)$ denote $\operatorname{Coker}(Q_{i+1}\to Q_i)$, $i\geqslant 0$, which we identify with $\operatorname{Ker}(Q_{i-1}\to Q_{i-2})$ if $i\geqslant 2$. (We also abbreviate this to $\operatorname{syz}^i_R M$, $\operatorname{syz}^i(M,Q_*)$, or $\operatorname{syz}^i M$, if the meaning is clear from the context.) Then given a map $\phi: K_*(\mathbf{x};R)\to P_*$ as in the definition of "R satisfies CE," there is an induced map $\phi_*: R\to \operatorname{syz}^n(K;P_*)$ ($R=K_n(\mathbf{x};R)$). Evidently, if $\phi_n=0$, then $\phi_*=0$, while if $\phi_*=0$ we may choose ϕ_n to be zero.

Thus, R satisfies CE if and only if for some (equivalently, every) choice of P_*, for every system of parameters $x_1,\dots,x_n$ and for every choice of ϕ, the induced map

$$\phi_*: R\ (\text{or } K_n(\mathbf{x};R))\to \operatorname{syz}^n(K;P_*)$$

is nonzero.

(6) The choice of ϕ is unique up to homotopy. It follows easily that ϕ_* can be altered only by a map of R to $\operatorname{syz}^n(K;P_*)$ which extends to $R^n=K_{n-1}(\mathbf{x};R)$ and R is mapped into R^n via $r\mapsto r(x_1,-x_2,\dots,(-1)^{n+1}x_n)$. It follows that $\phi_*(1)$ can be altered precisely by adding an element of $(x_1,\dots,x_n)\operatorname{syz}^n(K;P_*)$.

Thus, R satisfies CE if and only if for some (equivalently, every) choice of P_*, for every choice of $x_1,\dots,x_n$, and for some (equivalently, every) choice of $\phi: K_*(\mathbf{x};R)\to P_*$ which induces the quotient surjection $R/(x_1,\dots,x_n)\to K$, the induced map

$$\bar{\phi}_*: K_n(x;R)\to \operatorname{syz}^n K/(x_1,\dots,x_n)\operatorname{syz}^n K$$

is nonzero.

(7) If ϕ_n is one induced map $K_n(x; R) \to P_n$, then the map $K_*(\mathbf{x}^t; R) \to K_*(\mathbf{x}; R)$ described earlier yields a choice of $\psi: K_*(\mathbf{x}^t; R) \to P_*$ and hence a map $K_n(\mathbf{x}^t; R) \to P_n$. This map turns out to be $(x_1^{t-1} \cdots x_n^{t-1})\,\phi_n$. Combining this observation with (6), we see that R satisfies CE if and only if for some (equivalently, every) choice of P_*, for every choice of system of parameters $x_1, \ldots, x_n$, and for some (equivalently, every) choice of ϕ, for all positive integers t

$$x_1^{t-1} \cdots x_n^{t-1} \phi_n(1) \notin (x_1^t, \ldots, x_n^t)\,\mathrm{syz}^n K.$$

(8) If the condition given in the last sentense of (7) holds for one system of parameters $x_1, \ldots, x_n$, then it holds for *every* system of paremeters. To see this note that if $\mathbf{y} = y_1, \ldots, y_n$ and $\mathbf{z} = z_1, \ldots, z_n$ are two systems of parameters and $(z_1, \ldots, z_n) R \subset (y_1, \ldots, y_n) R$, say $z_i = \sum_{j=1}^n a_{ij} y_j$, $1 \leqslant i \leqslant n$, then there is a map $K_*(\mathbf{z}; R) \to K_*(\mathbf{y}; R)$ such that the matrix of the induced map on K_r's is $\Lambda^n(a_{ij})$; hence if we can find an admissible map ϕ from $K_*(\mathbf{y}; R)$ to a free resolution of K such that $\phi_n = 0$, we can compose with $K_*(\mathbf{z}; R) \to K_*(\mathbf{y}; R)$ to obtain such a map for $\mathbf{z}$.

Hence, if $\phi_n \neq 0$ for all choices of ϕ for a family of systems of parameters $x_{1t}, \ldots, x_{nt}$, $t = 1, 2, 3, \ldots$, such that the ideals $(x_{1t}, \ldots, x_{nt})$ are cofinal with th powers of $\mathfrak{m}$, then $\phi_n \neq 0$ for all systems of parameters and choices of ϕ. In particular, we may choose $x_{it} = x_i^t$.

Yet another characterization of satisfying CE that we shall need in the sequel is

(2.3) PROPOSITION. *$(R, \mathfrak{m})$ satisfies* CE *if and only if for every system of parameters $x_1, \ldots, x_n$, for every free complex F_**

$$\cdots \to F_i \to \cdots \to F_0 \to 0$$

with finitely generated F_i, and for every map $\phi: K_*(\mathbf{x}; R) \to F_*$ such that $\phi_0(R)$ has a nonzero image in $H_0(F_*) \otimes K$, where $K = R/\mathfrak{m}$, the map $\phi_n: K_n(\mathbf{x}; R) \to F_n$ is nonzero.

Proof. $\Rightarrow$ The hypothesis implies that we may choose an augmentation $F_0 \to K \to 0$ so that we still have a complex and the map $R/(x_1, \ldots, x_n) \to K$ induced by $K_*(\mathbf{x}; R) \to F_*$ is nonzero. But then we may map $F_* \to K \to 0$ to projective resolution $P_* \to K \to 0$ such that the induced map on K is the identity and the composition then gives a map $K_*(\mathbf{x}; R) \to P_*$ which induces a nonzero map $R/(x_1, \ldots, x_n) R \to K$. But then if R satisfies CE, we have $(F_n \to P_n) \circ \phi_n \neq 0$, whence $\phi_n \neq 0$, as required.

$\Leftarrow$ is obvious. Q.E.D.

(2.4) COROLLARY. *Let* $(R, m) \to^h (S, n)$ *be a local homomorphism of local rings and* $x_1, \ldots, x_n$ *a system of parameters for* R *such that* $h(x_1), \ldots, h(x_n)$ *is a system of parameters for* S. *Suppose that* S *satisfies* CE. *Then* R *satisfies* CE.

Proof. Given a counterexample $\phi: K_*(\mathbf{x}; R) \to F_*$ to the statement of Proposition 2.3, apply $\otimes S$ to obtain a counterexample over S, Q.E.D.

We next motivate the study of rings which satisfy CE by proving that the new intersection conjecture holds for such rings. Later we shall see that virtually all the other usual consequences of the existence of big C–M modules (other than the existence itself) also follow.

We first recall the new intersection conjecture of Peskine–Szpiro [26] and Roberts [27], in its simplest form.

(2.5) *Conjecture* (new intersection conjecture). *Let* (R, m) *be a local ring and let* F_* *be a finite complex of finitely generated free* R*-modules*:

$$0 \to F_d \to \cdots \to F_i \to \cdots \to F_0 \to 0.$$

Call the length of F_*, d. *Suppose that the homology modules* $H_i(F_*)$ *all have finite length* (i.e., *are killed by a power of* m) *and that* $H_*(F_*) \neq 0$. *Then* $\dim R \leqslant d$.

This conjecture was proved for local rings of positive prime characteristic $p > 0$ independently in [26, 27], and it is observed in [15] that it holds whenever R has a big C–M module. It is important, since it implies the original Peskine–Szpiro intersection conjecture [25], Auslander's zerodivisor conjecture [1, 2] and an affirmative answer to Bass' question [3]. See also [14, 18, 22] for more information.

We now give a purely elementary proof that if R satisfies CE, then the new intersection conjecture holds for R. We then show how to deduce that every local ring of positive residual characteristic satisfies CE from the direct summand conjecture (which is known [13] for rings containing a field). This yields a new proof of the homological conjectures in char $p > 0$; moreover, the general case reduces to poving the direct summand conjecture.

For application to the syzygy problem studied by Evans and Griffith [6], we prove a slightly improved version of the new intersection conjecture. First note that if $H_0(F_*) = 0$, we can shorten the complex by splitting a piece off the right-hand end, and so instead of assuming that $H_*(F_*) \neq 0$, one might as well assume that $H_0(F_*) \neq 0$. We can then weaken the condition that $H_0(F_*)$ have finite length as follows:

(2.5°) *Conjecture* (Improved new intersection conjecture). *Let* (R, m) *be a local ring and let* F_* *be a finite complex of finitely generated free modules*

$$0 \to F_d \to \cdots \to F_0 \to 0$$

such that $H_i(F_*)$ has finite length for $i > 0$ and $H_0(F_*)$ has a (nonzero) *minimal generator z such that Rz has finite length. Then* $\dim R \leqslant d$.

(2.6) THEOREM. *Let (R, m) be a local ring which satisfies* CE. *Then the improved new intersection theorem holds for R.*

Proof. Let F_* be a complex of length $d < \dim R = n$ as described in (2.5°), and let $\mathbf{x} = x_1,\ldots, x_n$ be a system of parameters for R. Choose a map $\phi_0 : R \to F_0$ such that the image of 1 in F_0 maps to the *minimal* generator z of $H_0(F_*)$ such that R_z has finite length. It follows that $\phi_0(1) \notin mF_0$, i.e., $\phi_0(1)$ is apart of a free basis for F_0. We shall show that for sufficiently large t, ϕ_0 lifts to a map of complexes

$$\phi : K_*(\mathbf{x}^t; R) \to F_*.$$

In fact, we shall use induction on i to prove that we can define ϕ out to stage $K_i(\mathbf{x}^t; R)$ for all i, possibly enlarging t. Let $M = \operatorname{Coker}(F_1 \to F_0)$. Let $K_j(t) = K_j(\mathbf{x}^t; R)$. Since Rz has finite length, we have $\operatorname{Ann} Rz \supset (x_1^t,\ldots, x_n^t)R$ for large t, so that for sufficiently large t we have a commutative diagram

$$\begin{array}{ccc} F_0 & \longrightarrow & M \\ \uparrow{\scriptstyle \phi_0} & & \uparrow \\ K_0(t) & \longrightarrow & R/(x_1^t,\ldots, x_n^t) \\ \| & & \\ R & & \end{array}$$

Working inductively, we suppose that for some $i \geqslant 0$, we have already defined $\phi_0,\ldots, \phi_i$ as indicated in the diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & F_{i+1} & \longrightarrow & F_i & \longrightarrow \cdots \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \uparrow & & \uparrow{\scriptstyle \phi_i} & & \uparrow{\scriptstyle \phi_0} & & \uparrow & & \\ \cdots & \longrightarrow & K_{i+1}(t) & \longrightarrow & K_i(t) & \longrightarrow \cdots \longrightarrow & K_0(t) & \longrightarrow & R/(x_1^t,\ldots, x_n^t) & \longrightarrow & 0 \end{array}$$

so that the diagram commutes. The problem is then to define ϕ_{i+1} so as to make the leftmost square commute. While we cannot necessarily do this for the value of t already selected, we can do it for $t + s$ sufficiently large. To see this, let $Z = \operatorname{Ker}(F_i \to F_{i-1})$ and $B = \operatorname{Im}(F_{i+1} \to F_i)$. Since $H_i(F_*) = Z/B$ has finite length, we can choose a positive integer c such that $(x_1,\ldots, x_n)^c Z \subset B$. By the Artin–Rees lemma, we can choose a positive integer s such that $(x_1,\ldots, x_n)^s F_i \cap Z \subset (x_1,\ldots, x_n)^c Z \subset B$. By combining the map of complexes to stage i with the map of Koszul complexes, we obtain a diagram

$$\begin{array}{ccccccccccc}
\longrightarrow & F_{i+1} & \xrightarrow{\delta_{i+1}} & F_i & \xrightarrow{\delta_i} & \cdots \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & & \uparrow{\scriptstyle \phi_i} & & & \uparrow{\scriptstyle \phi_0} & & \uparrow & & \\
 & & & K_i(t) & \longrightarrow & \cdots \longrightarrow & K_0(t) & \longrightarrow & R/(x_1^t,\ldots,x_n^t) & \longrightarrow & 0 \\
 & & & \uparrow{\scriptstyle \mu_i} & & & \mu_0=\uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \\
\longrightarrow & K_{i+1}(t+s) & \xrightarrow{d_{i+1}} & K_i(t+s) & \xrightarrow{d_i} & \cdots \longrightarrow & K_0(t+s) & \longrightarrow & R/(x_1^{t+s},\ldots,x_n^{t+s}) & \longrightarrow & 0.
\end{array}$$

We can complete the first step of the proof by constructing a map $K_{i+1}(t+s) \to F_{i+1}$ which makes the diagram commute. But it follows from the definition of the map μ of Koszul complexes that $\mu K_i(t+s) \subset (x_1,\ldots,x_n)^{si} K_i(t)$ whence the image of any free generator U_j of $K_{i+1}(t+s)$ under $\phi_i \mu_i d_{i+1}$ lies in $(x_1,\ldots,x_n)^s F_i$. Moreover, $\delta_i \phi_i \mu_i d_{i+1} U_j = \phi_{i-1}\mu_{i-1}(d_i d_{i+1} U_j) = 0$, whence $(\phi_i \mu_i)\, d_{i+1} U_j \in (x_1,\ldots,x_n)^s F_i \cap Z \subset B$, and we can choose $f_j \in F_{i+1}$ such that $\delta_{i+1} f_j = (\phi_i \mu_i)(d_{i+1} U_j)$. If we choose such an f_j for each free generator U_j of $K_{i+1}(t+s)$, the map which sends each U_j to the corresponding f_j makes the diagram commute.

Thus, we eventually obtain a map ϕ: $K_*(\mathbf{x}^t; R) \to F_*$ for some suitably large t. Since $F_n = 0$, $\phi_n = 0$, and R cannot satisfy CE. This contradicts the assumption. Q.E.D.

The improved new intersection theorem is proved for local rings R which have a big C–M module in [6], although it is not explicitly stated there. The next result shows that if every local ring satisfies CE, then the syzygy problem of Evans–Griffith can be settled affirmatively. In particular, the direct summand conjecture implies an affirmative answer for the syzygy problem.

(2.6°) COROLLARY. *Let R be a Cohen–Macaulay domain and let M be a non-free* kth *syzygy with* $\operatorname{pd} M < \infty$. *Suppose that all local rings of homomorphic image domains of R satisfy* CE. *Then M has torsion-free rank at least k.*

Proof. Precisely as in [6], we note that we can localize a counterexample so as to get a new counterexample such that the new M is locally free on the punctured spectrum of the new local ring. Assume that we are in this situation. Consider a minimal generator $w \in M$ and let I be the order ideal of w, i.e.,

$$I = O_M(w) = \{f(w) : f \in \operatorname{Hom}_R(M, R)\}.$$

As in [6], we note that if M is a counterexample of smallest rank k, we must have $\operatorname{ht} I \leqslant k-1$. Hence, we can choose $P \supset I$ with $\operatorname{ht} P \leqslant k-1$, and we can then tensor a minimal free resolution of M, say

$$0 \to F_d \to \cdots \to F_0 \to M \to 0$$

with $A = R/P$. We thus obtain an A-free complex with augmentation $\bar{M}$ (where $\bar{N}$ denotes N/PN), namely,

$$0 \to \bar{F}_d \to \cdots \to \bar{F}_0 \to \bar{M} \to 0.$$

Here, $d = \operatorname{pd}_R M \leqslant \dim R - k < \dim R - (k-1) = \dim A$. As in [6], we note that z, the image of w in $\bar{M}$, generates a nonzero submodule of finite length (in [6] it is shown even that the image of w in M/IM generates a submodule of finite length) and since $w \in M - \mathfrak{m}M$ we know that z is a minimal generator of $\bar{M}$. But then $d < \dim A$ contradicts the hypothesis that A satisfies CE, for we may apply Theorem (2.6). The contradiction shows that a counterexample cannot exist. Q.E.D.

This is virtually the same argument given in [6]; the difference is that Evans–Griffith use big C–M modules to establish the improved new intersection theorem (which they do not isolate explicitly) rather than talking about property CE.

We next observe

(2.7) THEOREM. *If a local ring $(R, \mathfrak{m}, K)$ has a big Cohen–Macaulay module M, then R satisfies* CE.

Proof. Recall that M is a (not necessarily finitely generated) R-module such that some system of parameters $x_1, \ldots, x_n$ is an M-sequence and $(x_1, \ldots, x_n)M \neq M$ (equivalently, $\mathfrak{m}M \neq M$). Then, for all t, $x_1^t, \ldots, x_n^t$ is an M-sequence. Suppose we have a map ϕ from $K_*(\mathbf{x}^t; R)$ to a minimal projective resolution F^* of K such that $\phi_0 = \mathrm{id}_R$ and $\phi_n = 0$. Since $K_*(\mathbf{x}^t; M)$ is acyclic, we may lift any map $\psi_0 : F_0 \to K(\mathbf{x}^t; M)$ to a map $\psi : F_* \to K_*(\mathbf{x}^t; M)$. Choose ψ_0 such that $\psi_0(R) \not\subset \mathfrak{m}M$. Then $\theta = \psi \circ \phi$ is a map $K_*(\mathbf{x}^t; R) \to K_*(\mathbf{x}^t; M)$ such that $\theta_0(R) \not\subset \mathfrak{m}M$ while $\theta_n = 0$. Let I be the identity map on $K_*(\mathbf{x}^t; R)$. Then $\theta' = I \otimes \theta_0$ is another such map with $\theta'_0 = \theta_0$, whence, since $K_*(\mathbf{x}^t; R)$ is free and $K_*(\mathbf{x}^t; M)$ is acyclic, θ and θ' are homotopic. Thus, $\theta'_n - \theta_n = h \circ d$, where h, d are indicated in the diagram

$$\begin{array}{ccccccc}
 & & & & M & & \\
 & & & & \wr\| & & \\
0 & \longrightarrow & K_n(\mathbf{x}^t; M) & & & & \\
 & & \uparrow & \nwarrow^{h} & & & \\
 & & \theta'_n \uparrow & \uparrow \theta_n & & & \\
0 & \longrightarrow & K_n(\mathbf{x}^t; R) & \xrightarrow{d} & K_n(\mathbf{x}^t; R) & & \\
 & & \wr\| & & \wr\| & & \\
 & & R & \xrightarrow{[x_1^t \cdots \pm x_n^t]} & R^n & &
\end{array}$$

If we identify $K_n(\mathbf{x}^t; M)$ with M, $\theta'_n(K_n(\mathbf{x}^t; R))$ is identified with $\psi_0(R)$, while $hd(K_n(\mathbf{x}^t; R)) \subset (x_1^t,...,x_n^t)M$, which contradicts $\psi_0(R) \not\subset M$. Q.E.D.

We now want to prove

(2.8) THEOREM. *If the direct summand conjecture is true, then every local ring satisfies* CE.

Later, we shall make more precise statements which may be useful even if the direct summand conjecture fails.

In proving (2.8), we may assume that R has residual characteristic $p > 0$; in case the residual characteristic is 0, the implication holds in a formal sense, since in that case R is known to have a big C–M module.

We next note that by Corollary (2.4), in order to prove (2.8), it suffices to consider the case where R is complete, since we can map $R \to \hat{R}$; moreover, we may kill a prime of maximum coheight, and so reduce to the case where R is a complete local domain. We do not need to go quite this far, but henceforth we assume the following:

(1) R is a finite module over a formal power series ring, which we take to be $A = K[[x_1,..., x_n]]$ if R contains a field (in which case we may take K to be a coefficient field), while $A = V[[x_2,..., x_n]]$ if R does not contain a field, where V is a discrete valuation ring whose fraction field has characteristic 0, whose residue class field K has char $p > 0$, and is the residue class field of R as well, and where x_1 denotes a generator of the maximal ideal of V.

(2) R is reduced (has no nonzero nilpotents).

(3) For every i, x_i is not a zerodivisor in R.

(4) In the case where R has mixed characteristic, we assume also that R is a domain.

Then, if char $R = p > 0$, let

$$R^\infty = \varinjlim (R \xrightarrow{F} R \xrightarrow{F} \cdots \to R \xrightarrow{F} R \xrightarrow{F} \cdots),$$

which may be viewed as the result of adjoining all (p^e)th roots of elements of R.

If R has mixed characteristic, we define R^∞ to be a ring which satisfies the following conditions:

(i) R^∞ is integral over R.

(ii) $R \subset R^\infty \subset \Omega$, where Ω is an algebraic closure of the fraction field of R.

(iii) R^∞ contains all (p^e)th roots of the elements x_i, $1 \leqslant i \leqslant n$.

(iv) If (x_i^∞) denotes $\bigcup_e x_i^{1/p^e}R^\infty$, for some consistent choice of (p^e)th roots of x_i, then $R^\infty/(x_i^\infty)$ is perfect, i.e., F is an automorphism.

(v) For all $i \geqslant 2$, x_1, x_i is a regular sequence in R^∞.

In fact, we note that in order that S satisfy (i)–(v), where $R \subset S \subset \Omega$, it suffices that

(α) S be integral over R.

(β) If $\Theta \in \Omega$ and $\theta^p \in S$, then $\theta \in S$.

(γ) S be integrally closed in its fraction field.

Note that (β) $\Rightarrow$ (iii) and that F is surjective on $R^\infty/(x_i^\infty)$. On the other hand, if $u^p \in (x_i^\infty)$, say $u^p = (x_i^{1/p^e})\theta$, then $u = (x_i^{1/p^{e+1}})\theta^{1/p}$ for suitable choices. Thus, F is injective mod(x_i^∞). Moreover, (γ) $\Rightarrow$ (v), for S is a direct limit of normal, module-finite extensions of R, and x_1, x_i, $i \geqslant 2$, is a regular sequence in each of these.

Thus, we may choose R^∞ to be the entire integral closure of R in Ω, or the smallest normal subring of Ω closed under extraction of pth roots. (Smaller choices may also be possible.)

Henceforth, let x_{ie} denote a (p^e)th root of x_i, $e = 0, 1, 2,..$. (thus $x_{i0} = x_i$); assume these have been chosen so that $(x_{i,e+1})p = x_{ie}$. We shall retain the convention that (x_i^∞) denotes $\bigcup_{e>0} x_{ie}R^\infty$.

(2.9) THEOREM. *If* char $R = p > 0$, *then* R *has* CE.

If R *is a complete local domain of mixed characteristic,* A *is as in the list of four conditions following the statement of Theorem* (2.8), *and for every* e *the regular local ring* $A_e = A[x_{ie},..., x_{ne}]$ *is a direct summand of each module-finite extension algebra* T *of* A_e *such that* $A_e \subset T \subset R^\infty$, *then* R *satisfies* CE.

Before proceeding to the proof of this theorem we introduce some complexes of ideals. Let $J_1,..., J_n$ be ideals of a ring S. For each i we have a complex

$$0 \to J_i \xrightarrow{\alpha_i} S \to 0 \qquad (\alpha_i \text{ is inclusion})$$

which we denote $K_*(J_i; S)$. If $\mathbf{J} = J_1,..., J_n$, we write

$$K_*(\mathbf{J}; S) = \bigotimes_{i=1}^{n} K_*(J_i; S).$$

There is a natural inclusion $\gamma_i\colon K_*(J_i; S) \to K_*(1; S)$ given by

$$\begin{array}{ccccc} 0 & \to & S & \xrightarrow{1} & S \to 0 \\ & & \downarrow & & \uparrow \text{id} \\ 0 & \to & J_i & \longrightarrow & S \to 0 \end{array}$$

which induces a map $\gamma: K_*(\mathbf{J}; S) \to K_*(1,..., 1; S)$. We denote by $K_*(\cdot, \mathbf{J}; S)$ the complex which is the image of γ. Written out, we have

$$0 \to J_1 \cdots J_n \to \bigoplus_{i_1 < \cdots < i_{n-1}} J_{i_1} \cdots J_{i_{n-1}}$$

$$\to \cdots \to \bigoplus_{i_1 < i_2} J_{i_1} J_{i_2} \to \bigoplus_i J_i \to S \to 0.$$

There is a similar complex $K_*(\cap; J:S)$ in which $J_{i_1} \cdots J_{i_r}$ is replaced by $J_{i_1} \cap \cdots \cap J_{i_r}$. In fact, let $\mathbf{A} = A_1,..., A_n$ be subgroups of an Abelian group B, and let $K_*(\cap; \mathbf{A}; B)$ denote the complex

$$0 \to A_1 \cap \cdots \cap A_n \to \bigoplus_{i_1 < \cdots < i_{n-1}} (A_{i_1} \cap \cdots \cap A_{i_{n-1}})$$

$$\to \cdots \to \bigoplus_{i_1 < i_2} (A_{i_1} \cap A_{i_2}) \to \bigoplus_i A_i \to B \to 0,$$

where the maps are direct sums of inclusion maps with signs chosen as in the Koszul complex. If $B \subset B'$ and $A_i \subset A_i' \subset B'$ for every i, we have an induced map $K_*(\cap; \mathbf{A}; B) \to K_*(\cap; \mathbf{A}', B')$. In particular, $K_*(\cap; A; B)$ may be viewed as a subcomplex of $K_*(\cap; B,..., B; B)$, which is the same as the usual Koszul complex $K_*(1,..., 1; B) = K_*(1,..., 1; \mathbb{Z}) \otimes_{\mathbb{Z}} B$. The following result is similar to one in [7]:

(2.10) LEMMA. *Let $A_1,..., A_n$ be subgroups of an Abelian group B, and suppose that for all integers q with $2 \leqslant q \leqslant n$, and for every group C which is an intersection of a subset of $A_{q+2},..., A_n$ (including $C = B$),*

$$(A_{q+1} \cap C) \cap (A_1 \cap C + \cdots + A_q \cap C) = \sum_{i=1}^{n} (A_{q+1} \cap C) \cap (A_i \cap C).$$

Then $K_*(\cap; \mathbf{A}; B)$ is acyclic.

In particular, if $A_1,..., A_n$ belong to a family of subgroups of B which is closed under finite addition and intersection and, such that, within this family, intersection distributes over addition, then $K_*(\cap; \mathbf{A}, B)$ is acyclic.

Proof. This is easily checked if $n \leqslant 2$. If $n = m + 1$, $m \geqslant 2$, then we note that the distributivity condition is inherited by the sequence $A_1,..., A_m$ and also by the sequence $A_1 \cap A_n,..., A_m \cap A_n$. The inclusions $A_i \cap A_n \hookrightarrow A_i$, $1 \leqslant i \leqslant m$ induce a map of complexes

$$K_*(\cap; A_1,..., A_m; B)$$

$$\uparrow$$

$$K_*(\cap; A_1 \cap A_n,..., A_m \cap A_n; A_n).$$

By induction on n, we say assume that both rows are acyclic. If we insert the augmentations we get an exact total complex. Without the augmentations, the total complex is precisely $K_*(\cap; A_1,..., A_n; B)$, and so the homology H_i vanishes, $i \geqslant 2$. It remains to show that $H_1 = 0$. If not, choose a cycle $c = (a_1,..., a_n) \in \bigoplus A_i$, *not* a boundary, and such that the largest q with $a_{q+1} \neq 0$ is minimum among such cycles. Thus $c = (a_1,..., a_{q+1}, 0,..., 0)$, and $a_1 + \cdots + a_{q+1} = 0$, i.e., $-a_{q+1} = a_1 + \cdots + a_q \in A_{q+1} \cap (A_1 + \cdots + A_q) = \sum_{i=1}^{q} A_{q+1} \cap A_i$; say $-a_{q+1} = a_1' + \cdots + a_q'$, where $a' \in A_{q+1} \cap A_i$, $1 \leqslant i \leqslant q$. Let $d \in \bigoplus_{i_1 < i_2} A_{i_1} \cap A_{i_2}$ be such that the (i_1, i_2)-component of d is 0 unless $(i_1, i_2) = (i, q+1)$, $1 \leqslant i \leqslant q$, in which case it is a_i'. Then if d^* is the image of d in $\bigoplus A_t$, we know that the qth component of d^* is $\pm a_{q+1}$ and the ith component is 0 for $i > q+1$. Then $c \mp d^*$ is a cycle, not a boundary, with a 0 in the $q+1$ component as well, which contradicts the minimality of q. Hence, contrary to our assumption, $H_1 = 0$. Q.E.D.

(2.11) PROPOSITION. *Let T be a ring of characteristic $p > 0$ such that the Frobenius F is an automorphism, i.e., T is perfect. Let $\mathscr{F}$ be the set of ideals of T which are stable under F^{-1}. Then:*

(1) *$\mathscr{F}$ is closed under finite addition, multiplication, and intersection.*

(2) *For any $x \in T$, $(x^\infty) \in \mathscr{F}$.*

(3) *If $J_1,..., J_r \in \mathscr{F}$, then $J_1 \cdots J_r = J_1 \cap \cdots \cap J_r$.*

(4) *If $J_1,..., J_r, J \in \mathscr{F}$, then $J \cap (J_1 + \cdots + J_r) = J \cap J_1 + \cdots + J \cap J_r$.*

(5) *If $\mathbf{J} = J_1,..., J_n \in \mathscr{F}$, then the natural inclusion $K(\cdot, \mathbf{J}, T) \to K(\cap; \mathbf{J}; T)$ is an isomorphism, and both complexes are acyclic.*

Proof. Conditions (1) and (2) are clear, while (3) reduces at once, by induction, to the case $r = 2$. But if $u \in J_1 \cap J_2$ and J_1, J_2 are F^1-stable, then $u^{1/p} \in J_1$ and $u^{1/p} \in J_2$, whence $u = (u^{1/p})(u^{1/p})^{p-1} \in J_1 J_2$.

Condition (4) is immediate, since by (3), we may replace the intersections by products. Condition (3) implies the equality of the two complexes, while (4) implies that $K_*(\cap; \mathbf{J}; T)$ is acyclic, by virtue of Lemma (2.10). Q.E.D.

(2.12) *Remarks.* If at least one of $J_1, J_2 \subset T$ is flat, say J_2, then when we apply $\otimes_T J_2$ to $0 \mapsto J_1 \hookrightarrow T$ we obtain $0 \to J_1 \otimes J_2 \hookrightarrow J_2$, whence the obvious surjection $J_1 \otimes J_2 \twoheadrightarrow J_1 J_2$ is an isomorphism. It follows that if all but at most one of $J_1,..., J_r$ are flat, then $J_1 \otimes \cdots \otimes J_r \to J_1 \cdots J_r$ is an isomorphism. Now, if x is a nonzerodivisor in T, $(x^\infty) = \bigcup_e (x^{1/p^e})$ is T-flat. Hence, we obtain

(2.13) PROPOSITION. *If T is a perfect algebra of* char $p > 0$ *and $\mathbf{J} = J_1,..., J_n$ is a sequence of F^{-1}-stable ideals, at most one of which is not flat, then*

$$K_*(\mathbf{J}; T) \cong K_*(\cdot; \mathbf{J}; T) \cong K_*(\cap; \mathbf{J}; T),$$

and each of these is acyclic. In particular, if $x_1,\dots,x_n$ *are nonzero-divisors in* T, *then*

$$K_*((x_1^\infty),\dots,(x_n^\infty);T)$$

is acyclic, and is a flat resolution of $T/((x_1^\infty),\dots,(x_n^\infty))$.

(2.14) *Remarks.* If $J_1,\dots,J_n$ are flat ideals of an algebra T such that $J_{q+1}\cap(J_1+\cdots+J_q)=J_{q+1}(J_1+\cdots+J_q)$, $1\leqslant q\leqslant n-1$, then $K_*(\mathbf{J};T)$ is a flat resolution of $T/\sum J_i$. To see this, proceed by induction on n. Assuming, inductively, that $K_*(J_1,\dots,J_q;T)$ is a flat resolution of T/I_q, where $I_q=J_1+\cdots+J_q$, we need only show that $K_*(J_1,\dots,J_{q+1};T)$, which is the total complex of the double complex $K_*(J_1,\dots,J_q;T)\otimes(0\to J_{q+1}\to T\to 0)$, is acyclic. Since the first (resp. the second) factor is a flat resolution of T/I_q (resp. of T/J_{q+1}), the homology is $\operatorname{Tor}_*^T(T/I_q,\ T/J_{q+1})$. Since J_{q+1} is flat, Tor dim $T/J_{q+1}\leqslant 1$, and we need only show that $\operatorname{Tor}_1^T(T/I_q,T/J_{q+1})=0$. But this is $I_q\cap J_{q+1}/I_qJ_{q+1}$. Q.E.D.

This gives another proof that $K_*((x_1^\infty),\dots,(x_n^\infty);T)$ is acyclic in Proposition (2.13).

Before returning to the proof of Theorem (2.9), we observe

(2.15) PROPOSITION. *Let* R *be a complete local domain of mixed characteristic and let* $A=V[[x_2,\dots,x_n]]$ *and* R^∞ *be as in Theorem* (2.9), *where* x *is a regular parameter for* V. *Let* $\{x_{ie}\}_e$ *be a consistent system of* (p^e)th *roots for* x_i, *as earlier, and let* $(x_i^\infty)=\bigcup_e x_{ie}R^\infty$. *Then*

$$K_*((x_1^\infty),\dots,(x_n^\infty);R^\infty)$$

is acyclic, and is a flat resolution of $R^\infty/((x_1^\infty)+\cdots+(x_n^\infty))$.

Proof. As in Remarks (2.14) we prove by induction on q that the complexes $K^*((x_1^\infty),\dots,(x_q^\infty);R^\infty)$ are all acyclic. For $q=1$ this is clear, while for $q=2$, we make use of condition (v) on R^∞: x_1,x_i is a regular sequence, $i\geqslant 2$, so that $(x_i^\infty)\cap(x_1^\infty)=(x_i^\infty)(x_1^\infty)$. It remains to show that if $2\leqslant q\leqslant n-1$, then $(x_{q+1}^\infty)\cap((x_1^\infty)+\cdots+(x_q^\infty))=(x_{q+1}^\infty)((x_1^\infty)+\cdots+(x_q^\infty))$. Let $T=R^\infty/(x_1^\infty)$. By condition (iv) on R^∞, T is perfect. Let $^-$ denote reduction modulo (x_1^∞). Then $\bar{x}_2,\dots,\bar{x}_n$ are nonzero-divisors in T. Thus, since $(\bar{x}_{q+1}^\infty)$ and $\sum_{i=1}^q(\bar{x}_i^\infty)$ are F^{-1}-stable, we have $(\bar{x}_{q+1}^\infty)\cap\sum_{i=2}^q(\bar{x}_i^\infty)=\bar{x}_{q+1}\sum_{i=2}^q(\bar{x}_i^\infty)$. Let $I_q=(x_{q+1}^\infty)\cap\sum_{i=1}^q(x_i^\infty)$. Working modulo (x_1^∞), $I_q\subset(\bar{x}_{q+1}^\infty)\cap\sum_{i=2}^q(\bar{x}_i^\infty)\Rightarrow I_q\subset(x_1^\infty)+(x_{q+1}^\infty)\sum_{i=1}^q(x_i^\infty)$. But the element of (x_1^∞) required to represent a given element of I_q will be in (x_{q+1}^∞), i.e., in $(x_1^\infty)\cap(x_{q+1}^\infty)=(x_{q+1}^\infty)(x_1^\infty)$, and so $I_q\subset(x_{q+1}^\infty)(x_1^\infty)+(x_{q+1}^\infty)\sum_{i=1}^q(x_i^\infty)=(x_{q+1}^\infty)\sum_{i=1}^q(x_i^\infty)\subset I_q$, and so $I_q=(x_{q+1}^\infty)\sum_{i=1}^q(x_i^\infty)$. Q.E.D.

(2.16) *Remarks.* By virtue of (2.13) and (2.15) we now have that $K_*((x_1^\infty),\ldots,(x_n^\infty);R^\infty)$ is acyclic in both the case char $R = p$ and the case of mixed characteristic. Notice that in the first case we may view $R^\infty/((x_1^\infty)+\cdots+(x_n^\infty)) \cong K^\infty$. In either case we denote the augmentation by L. We have a natural map $K \to L$.

When x_i is a nonzero-divisor in S we have an isomorphism of complexes

$$K_*(x_i;S) \to K_*((x_i);S)$$

via

$$\begin{array}{ccccccc} 0 & \longrightarrow & S & \longrightarrow & S & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x_i} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & (x_i) & \longrightarrow & S & \longrightarrow & 0 \end{array}$$

and so we have an isomorphism of complexes $K_*(x_1,\ldots,x_n;S) \to K_*((x_1),\ldots,(x_n);S)$ when $x_1,\ldots,x_n$ are nonzero-devisors in S. The free generator of $S \cong K_n(\mathbf{x};S)$ maps to $x_1\cdots x_n$ in $K_n((x_1),\ldots,(x_n);S) = (x_1\cdots x_n)$. This yields a composite map of complexes

$$\begin{array}{c} K_*(x_1,\ldots,x_n;R) \\ \downarrow \\ K_*(x_1,\ldots,x_n;R^\infty) \\ \downarrow \\ K_*(x_1R^\infty,\ldots,x_nR^\infty;R^\infty) \\ \downarrow \\ K_*((x_1^\infty),\ldots,(x_n^\infty);R^\infty) \end{array}$$

such that the free generator in $K_n(\mathbf{x};R)$ maps to $x_1\cdots x_n \in (x_1^\infty)\cdots(x_n^\infty) \cong K_n((x_1^\infty),\ldots,(x_n^\infty);R^\infty)$. Call this composite map ${}_1\theta$. If we compose with the usual map $K_*(\mathbf{x}^t;R) \to K_*(\mathbf{x};R)$ we obtain a map ${}_t\theta$: $K_*(\mathbf{x}^t;R) \to K_*((x_1^\infty),\ldots,(x_n^\infty);R)$ such that the free generator of $K_n(\mathbf{x}^t;R)$ maps to $x_1^{t-1}\cdots x_n^{t-1}(x_1\cdots x_n) = (x_1\cdots x_n)^t$. (${}_t\theta$ may also be obtained by tensoring the n maps of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \xrightarrow{x_i^t} & R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x_i^t} & & \downarrow & & \\ 0 & \longrightarrow & (x_i^\infty) & \hookrightarrow & R^\infty & \longrightarrow & 0 \end{array}$$

indicated above.)

(2.17) *Proof of Theorem* (2.9). Let $x_1,\dots,x_n$ be the special system of parameters already described. If we have a map ϕ from $K_*(x_1^t,\dots,x_n^t;R)$ to a minimal free resolution F_* of K which is id_R in degree 0 but such that $\phi_n = 0$, then since F_* is free and $K_*((x_1^\infty),\dots,(x_n^\infty);R^\infty)$ is acyclic there is a map of complexes

$$\psi\colon F_* \to K_*((x_1^\infty),\dots,(x_n^\infty);R^\infty)$$

which lifts the map $K \subset\!\!\to L$ of augmentations. Then $\theta = \psi\phi$ is a map

$$K_*(\mathbf{x}^t;R) \to K_*((x_1^\infty),\dots,(x_n^\infty);R^\infty)$$

which induces

$$R/(x_1^t),\dots,x_n^t) \twoheadrightarrow K \subset\!\!\to L$$

on the augmentations, and such that $\theta_n = 0$. But we have another such map ${}_t\theta$, and since $K_*(\mathbf{x}^t;R)$ is free and $K_*((x_1^\infty),\dots,(x_n^\infty);R^\infty)$ is acyclic, θ and ${}_t\theta$ are homotopic. After making obvious identifications, we have the following diagram near the nth spot:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (x_1^\infty)\cdots(x_n^\infty) & \longrightarrow & & & \\ & & \uparrow\theta \quad \uparrow{}_t\theta_n & \searrow^{h} & & & \\ 0 & \longrightarrow & R & \xrightarrow{[x_1^t \cdots \pm x_n^t]} & R^n & \longrightarrow & \end{array}$$

where $\theta_n = 0$, ${}_t\theta_n(1) = x_1^t \cdots x_n^t$, and $hd = {}_t\theta_n - \theta_n$. But then $x_1^t \cdots x_n^t = dh(1) \in (\prod_j (x_j^\infty))(x_1^t,\dots,x_n^t)\,R^\infty$, whence, for sufficiently large e,

$$x_1^t \cdots x_n^t = \sum_j r_j x_j^t (x_{1e} \cdots x_{ne}), \qquad r_j \in R^\infty. \tag{$\neq$}$$

The symbol A_e has already been defined in the mixed characteristic case. In char $p > 0$, let K be a coefficient field, let $A = K[[x_1,\dots,x_n]] \subset R$, and let $A_e = A[x_{1e},\dots,x_{ne}]$.

In either case, let S be the ring generated over A_e by the elements r_j. We abbreviate $y_j = x_{je}$. Thus, $y_1,\dots,y_n$ is a regular system of parameters for A_e, and if we let $q = t_{p^e}$, Eq. ($\neq$) becomes

$$y_1^q \cdots y_n^q = \sum_j r_j y_j^q\, y_1 \cdots y_n, \qquad r_j \in S$$

or

$$y_1^{q-1} \cdots y_n^{q-1} = \sum_j r_j y_j^q, \qquad r_j \in S.$$

In char $p > 0$, we know that the regular ring A_e is a direct summand of S ([13]; but see Section 6 for a new proof), while in the mixed characteristic case, we have *assumed* that A_e is a direct summand of S. In either case we may apply an A_e-linear retraction $S \to A_e$ to both sides to obtain

$$y_1^{q-1} \cdots y_n^{q-1} = \sum_j a_j y_j^q, \qquad a_j \in A_e$$

which is well known to be impossible (cf. [13]). (In fact, $y_1^{q-1} \cdots y_n^{q-1}$ generates the socle in the 0-dimensional Gorenstein ring $A_e/(y_1^q, \ldots, y_n^q)$.)

Q.E.D.

(2.18) *Remarks.* This gives a new proof of the homological conjectures in char $p > 0$ as well as a reduction of them to the direct summand conjecture in residual char p. The fact that local rings have CE in char p can be used to prove this holds in equal characteristic 0, by the same "equational" techniques used to prove the existence of big C–M modules; see [14]. However, the *direct* implication

$$\text{(direct summand conjecture)} \Rightarrow \text{(all local rings have CE)}$$

does not seem to work in characteristic 0 (although it is true in a formal sense, since both are known). Of course, the homological conjectures we are considering are already known in the equicharacteristic case: see [14, 25–28].

Later we shall give yet another proof of property CE in char $p > 0$.

We next want to show that the conjecture that all local rings have CE implies the Eisenbud–Evans–Bruns principal ideal theorem. This theorem was first proved for local rings which possess a big C–M module by Eisenbud and Evans in [5]. Bruns [4] then gave a more elementary proof which handles the general case. Nonetheless, we include here a modification of the Eisenbud–Evans proof which utilizes property CE instead of big C–M modules, expressly for the purpose of illustrating further how property CE can be used to replace possession of a big C–M module in a proof. As in the proof of (2.10) we write $O_E(x)$ for $\{f(x): f \in \mathrm{Hom}_R(E, R)\}$, the order ideal (the notation $\mathrm{Tr}(x)$ is also used for this by some authors). The result is

(2.19) THEOREM (Eisenbud–Evans–Bruns). *Let (R, m, K) be a local ring, E a finitely generated R-module, and let rank E denote*

$$\max_P \{\dim_{\kappa(P)} \kappa(P) \otimes E\}$$

where P runs through all minimal primes of R, and $\kappa(P) = R_P/PR_P$. Let $x \in E$. Then

$$\mathrm{ht}\, O_E(x) \leqslant \mathrm{rank}\, E.$$

What we shall prove here is

(2.20) THEOREM. *Let (R, m, K) be a local ring such that for each minimal prime P of R, R/P has* CE. *Then the Eisenbud–Evans–Bruns principal ideal theorem holds for R.*

Proof. Exactly as in [5], we may replace R by R/P for a suitable minimal prime P and by modifying E we may assume that $O_E(x)$ is primary to m. Thus, we may assume that R is an integral domain, say, of dimension n. Rank E is now the torsion-free rank of E. Write E as $\operatorname{Coker}(R^q \to^A R^p)$, where A is a $q \times p$ matrix. Let $(y_1,..., y_p) \in m(R^p)$ represent x in $E = R^p/\operatorname{Im}(R^q)$. Since $O_E(x)$ is primary to m, we can find n homomorphisms $\phi_1,..., \phi_n \in \operatorname{Hom}(E, R)$ such that $\phi_1(x),..., \phi_n(x)$ is a system of parameters for R. Let $\hat{\phi}_i$ be the composite map $R^p \twoheadrightarrow E \to^{\phi_i} R$, and suppose B is the $n \times p$ matrix whose ith row is the transpose of the matrix of $\hat{\phi}_i$. The row space of B is then isomorphic to a submodule of $\operatorname{Hom}(E, R)$, which has the same torsion-free rank as E. Assume this rank is less than n: we then have $\bigwedge^n B = 0$. Let $y = [y_i]$, a $p \times 1$ matrix, and let $Z = [\phi_i(x)]$, an $n \times 1$ matrix. Since $y_1,..., y_p \in m$, and since $BY = Z$ (for the ith row of B, transposed, represents $\hat{\phi}_i$), we have the following commutative diagram, in which the rows are the Koszul complexes on $y_1,..., y_p$ (with a slightly different augmentation) and $\Phi_1(x),..., \Phi_n(x)$, respectively:

$$\begin{array}{ccccccccccccc}
\cdots \to \bigwedge^n R^p & \to \cdots \to & \bigwedge^i R^p & \to \cdots \to & R^p & \xrightarrow{Y} & R & \longrightarrow & R/m & \longrightarrow & 0 \\
\uparrow{\scriptstyle \bigwedge^n B} & & \uparrow{\scriptstyle \bigwedge^i B} & & \uparrow{\scriptstyle B} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \\
0 \to \bigwedge^n R^n & \to \cdots \to & \bigwedge^n R^n & \to \cdots \to & R^n & \xrightarrow{Z} & R & \longrightarrow & R/(\phi_1(x),..., \phi_n(x)) & \longrightarrow & 0
\end{array}$$

Since $\phi_1(x),..., \phi_n(x)$ is a system of parameters and $\bigwedge^n B = 0$, we have contradicted the assumption that R has CE (cf. Proposition (2.3)). Q.E.D.

Foxby [8, 9] has forms of the new intersection theorem for flat complexes which he deduces from big C–M modules. The author does not know whether these follow from the conjecture that every local ring has CE.

(2.21) *Remark.* It seems to this writer that the hypothesis needed for Theorem 1.1 of [5] is that "R/P has C–M modules for each minimal prime P of R", rather than that "R has C–M modules."

3. Canonical Elements in Local Cohomology Modules and the Functoriality of Property CE

Let R be a Noetherian ring and M an R-module. The reader should consult part (5) of the Remarks (2.2) for notation involving syz. If P^* is a

projective resolution of M (an acyclic projective left complex together with an isomorphism $H_0(P_*) \cong M$), we shall write P_*/d, where d is a nonnegative integer, for the exact sequence

$$0 \to \operatorname{syz}^d(P^*) \to P_{d-1} \to \cdots \to P_0 \to M \to 0.$$

Now, any exact sequence $\mathscr{E}$ of arbitrary R-modules

$$0 \to Q \to M_{d-1} \to \cdots \to M_0 \to M \to 0$$

determines an element (which we denote by $\varepsilon_{\mathscr{E}}$ or $\varepsilon(\mathscr{E})$) in $\operatorname{Ext}^d_R(M, Q)$. If I is an ideal of R and $M = R/I$, then recalling that the local cohomology $H^*_I(\ )$ may be defined by

$$H^*_I(M) = \varinjlim_t \operatorname{Ext}^*_R(R/I^t, M)$$

we see that we have a natural transformation of functors $\operatorname{Ext}^*_R(R/I, \ ) \to H^*_I(\ )$, so that $\varepsilon(\mathscr{E})$ has an image, which we denote $\eta_{\mathscr{E}}$ or $\eta(\mathscr{E})$, in $H^d_I(Q)$.

(3.1) *Remark.* If P_* is a projective resolution of R/I, and $\mathscr{E}$ is any exact sequence $0 \to Q \to M_{d-1} \to \cdots \to M_0 \to R/I \to 0$, then we can fill in a map of complexes

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & 0 & \longrightarrow & Q & \longrightarrow & M_{d-1} & \longrightarrow \cdots \longrightarrow & M_0 & \longrightarrow & R/I & \longrightarrow 0 \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \mathrm{id}} & \\
\cdots \longrightarrow & P_{d+1} & \longrightarrow & P_d & \longrightarrow & P_{d-1} & \longrightarrow \cdots \longrightarrow & P_0 & \longrightarrow & R/I & \longrightarrow 0
\end{array} \qquad (\#)$$

so we obtain a map ϕ: $\operatorname{syz}^d P_* \to Q$. For any module N, $\operatorname{Ext}^d_R(R/I, N)$ may be viewed as

$$\operatorname{Hom}_R(\operatorname{syz}^d_R(P_*), N)/\operatorname{Im}(\operatorname{Hom}_R(P_{d-1}, N))$$

and then $\varepsilon_{\mathscr{E}}$ is the class of ϕ in $\operatorname{Ext}^d_R(R/I, Q)$.

(3.2) DEFINITION. Let R be a Noetherian ring, $I \subset R$ an ideal, and d a nonnegative integer. Let P_* be a projective resolution of R/I. Let $\mathscr{E} = P_*/d$, i.e.,

$$0 \to \operatorname{syz}^d P_* \to P_{d-1} \to \cdots \to P_0 \to R/I \to 0.$$

Then we refer to $\varepsilon(\mathscr{E})$ as the *canonical element* in $\operatorname{Est}^d_R(R/I, \operatorname{syz}^d P_*)$, and $n(\mathscr{E})$ as the *canonical element* in $H^d_R(R/I, \operatorname{syz}^d P_*)$.

The following basic result summarizes some trivial but important facts which will be used repeatedly, and justifies the use of the term "canonical element."

(3.3) PROPOSITION. (a) *In the situation of the diagram labelled* ($\neq$)

$$\phi_*: \operatorname{Ext}_R^d(R/I, \operatorname{syz}^d(P_*)) \to \operatorname{Ext}_R^d(R/I, Q)$$

takes $\eta(P_*/d)$ *to* $\eta(\mathscr{E})$, *and, hence,*

$$\phi_*: H_I^d(\operatorname{syz}^d(F_*)) \to H_I^d(Q)$$

takes $\eta(F_*/d)$ *to* $\eta(\mathscr{E})$.

(b) *If* P_*, P'_* *are two projective resolutions of* R/I, *then any map* ϕ: $\operatorname{syz}^d(P_*) \to \operatorname{syz}^d(P'_*)$ *induced by a map of complexes which lifts* $\mathrm{id}_{R/I}$ *takes* $\varepsilon(P_*/d)$ *to* $\varepsilon(P'_*/d)$ *and* $\eta(P_*/d)$ *to* $\eta(P'_*/d)$.

(c) *In the situation of* (b), *there always exist such maps* ϕ: $\operatorname{syz}^d(P_*) \to \operatorname{syz}^d(P'_*)$ *and* ψ: $\operatorname{syz}^d(P'_*) \to \operatorname{syz}^d(P_*)$: *hence* $\operatorname{Ann}_R \varepsilon(P_*/d) = \operatorname{Ann}_R \varepsilon(P'_*/d)$ *and* $\operatorname{Ann}_R \eta(P_*/D) = \operatorname{Ann}_R \eta(P'_*/d)$.

Given the ideal I, we can construct the elements $\varepsilon(P_*/d)$, $\eta(P_*/d)$ in $\operatorname{Ext}_R^d(R/I, \operatorname{syz}^d(P_*))$ and in $H_I^d(\operatorname{syz}^d(P_*))$, respectively. Neither P_*, nor $\operatorname{syz}^d P_*$, nor the induced maps between the local cohomology modules are canonical. But the elements $\varepsilon(P_*/d)$, $\eta(P_*/d)$ are mapped to their counterparts when we change P_*, independent of how the maps of complexes are chosen. This justifies the terminology "canonical element" in $\operatorname{Ext}_R^d(R/I, \operatorname{syz}^d(P_*))$ (resp. in $H_I^d(\operatorname{syz}^d(P_*))$).

We shall identify the various $\varepsilon(P_*/d)$ (resp. $\eta(P_*/d)$) in these noncanonical modules obtained from various P_* by means of the noncanonical maps ϕ_*: despite all the choices, the identification is "canonical," by Proposition (3.3).

We write ε_I^d (resp. η_I^d) for the canonical element in $\operatorname{Ext}_R^d(R/I, \operatorname{syz}^d(R/I))$ (resp. in $H_I^d(\operatorname{syz}^d(R/I))$ (we are making tacit use of our identifications to suppress reference to P_*), provided $I \neq R$ (if $I = R$, ε and η are 0). *In practice, d will almost always be height I.* When the value of d or I is clear, the corresponding super- or subscript is omitted. *When* $(R, \mathfrak{m}, K)$ *is local and* $\dim R = d$, *we write* ε *or* ε_R (*resp.* η *or* η_R) *for* $\varepsilon_{\mathfrak{m}}^d$ (*resp.* $\eta_{\mathfrak{m}}^d$). This case is the focus of our interest. This notation should not cause confusion: the ε_I, η_I notations are used only for proper ideals I.

We next prove an elementary but *very* important fact concerning change of rings:

(3.4) THEOREM. *Let* h: $R \to S$ *be a homomorphism of Noetherian rings. Let* $I \subset R$, $J \subset S$ *be proper ideals such that* $h(I) \subset J$ *and* $\operatorname{Rad} J = \operatorname{Rad} IS$

(where Rad J, *the radical of* J, *consists of all elements nilpotent modulo* J*). Consider a commutative diagram*

$$\begin{array}{ccccccccccc} \mathscr{F} & 0 & \longrightarrow & V & \longrightarrow & N_{d-1} & \longrightarrow \cdots \longrightarrow & N_0 & \longrightarrow & S/J & \longrightarrow 0 \\ \uparrow & & & \uparrow{\scriptstyle \phi_d} & & \uparrow{\scriptstyle \phi_{d-1}} & & \uparrow{\scriptstyle \phi_0} & & \uparrow{\scriptstyle \alpha} & \\ \mathscr{E} & 0 & \longrightarrow & U & \longrightarrow & M_{d-1} & \longrightarrow \cdots \longrightarrow & M_0 & \longrightarrow & R/I & \longrightarrow 0 \end{array}$$

where α *is induced by* h, $\mathscr{E}$ *(resp.* $\mathscr{F}$*) is an exact sequence of* R*-modules (resp.* S*-modules) and the vertical arrows give a map of complexes over* R.

Then $\phi_*(\eta_{\mathscr{E}}) = \eta_{\mathscr{F}}$, *where* $\phi^*: H^d_I(U) \to H^d_I(V) \cong H^d_J(V)$ *is induced by* ϕ.

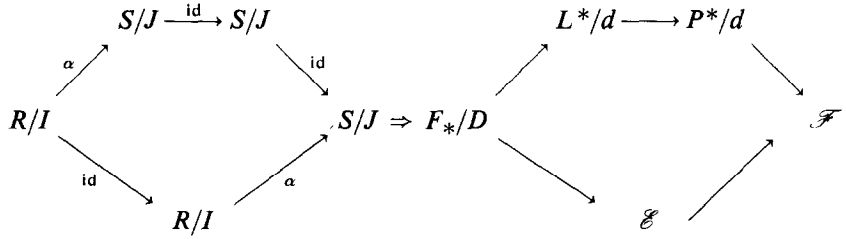
Proof. Note first that

$$H^d_J({}_SV) \cong H^d_{IS}({}_SV) \cong H^d_I({}_RV),$$

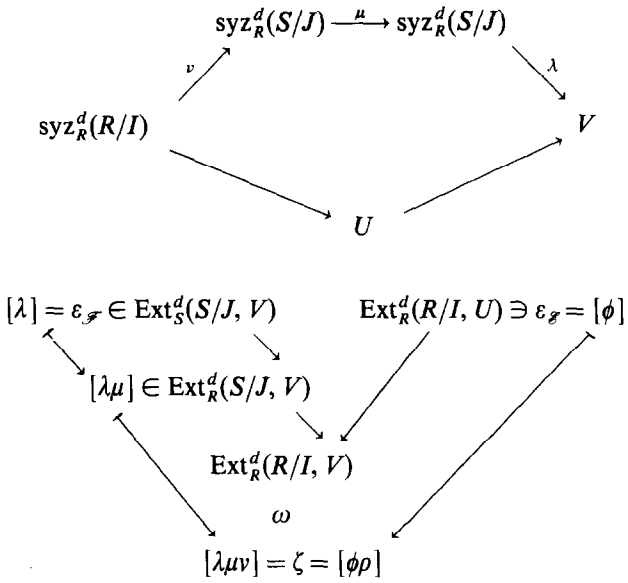
where the subscript indicates over which ring we are regarding V as a module. Thus, $\phi: U \to {}_RV$ induces

$$\phi^*: H^d_I(U) \to H^d_I({}_RV) \cong H^d_J({}_SV).$$

Let F_*, L_* be R-projective resolutions of R/I, S/J, respectively, and let P_* be an S-projective resolution of S/J. Then the left-hand commutative diagram below lifts to a diagram, *not necessarily commutative*, of complexes, shown on the right. Although the diagram of complexes need not commute, the two induced maps $\mathrm{syz}^d(F_*) \to V$ differ by a restriction of a map $F_{d-1} \to V$, since both come from liftings of $\alpha: R/I \to S/J$ to maps of complexes $F_* \to \mathscr{F}$.



Thus, they represent the same element ζ of $\mathrm{Ext}^d_R(R/I, V)$. Explicitly, in the lower diagram below we have $[\lambda\mu\nu] = [\phi\rho]$ (where $[\ ]$ denotes class in Ext^d_R). Hence, $\varepsilon_{\mathscr{E}}$, $\varepsilon_{\mathscr{F}}$ both map to ζ as shown on the right:



It follows that $\eta_{\mathscr{F}} \in H_J^d({}_S V)$ and $\eta_{\mathscr{G}} \in H_I^d(U)$ have the same image in $H_I^d({}_R V) \cong H_J^d({}_S V)$. Q.E.D.

(3.5) *Remark.* We cannot compare $\varepsilon_{\mathscr{G}}$ and $\varepsilon_{\mathscr{F}}$ directly (without introducing the auxiliary module $\mathrm{Ext}_R^d(R/I, V)$). The tremendous advantage of local cohomology is that the *direct limit* of the maps $\mathrm{Ext}_S^d(S/J^t, V) \to \mathrm{Ext}_R^d(R/I^t, V)$ is an isomorphism so that no auxiliary module is necessary. Of course, if $R = S$ and $I = J$ a much simpler argument shows that $\phi_*(\varepsilon_{\mathscr{G}}) = \varepsilon_{\mathscr{F}}$.

(3.6) COROLLARY. *Let R, I, S, J and α be as in Theorem* (3.4). *If $\phi\colon \mathrm{syz}_R^d(R/I) \to \mathrm{syz}_J^d(S/J)$ is any lifting of α, then $\phi_*(\eta_I^d) = \eta_J^d$.*

Thus, the canonical element η_I^d behaves in a very functorial way.

We next give an interpretation of what it means if $\eta_I^d \neq 0$. We shall use the notation θ_M for the natural map $\mathrm{Ext}_R^d(R/I, M) \to H_I^d(M)$.

(3.7) THEOREM. *Let R be a Noetherian ring, I a proper ideal, and d a nonnegative integer. The following conditions on R, I, d are equivalent:*

(1) $\eta_I^d \neq 0$.

(2) *For some R-module M (not necesarily finitely generated), $\theta_M \neq 0$.*

(3) *For some (equivalently, every) choice of $N = \mathrm{syz}^d(R/I)$, $\theta_N \neq 0$.*

Proof. By definition, $\eta_I^d = \theta_N(\varepsilon_I^d)$, independent of the choice of $N = \mathrm{syz}^d(R/I)$. Hence, (1) $\Rightarrow$ (3) with "every" $\Rightarrow$ (3) with "some" $\Rightarrow$ (2). It remains to show that (2) $\Rightarrow$ (1). Let $\delta \in \mathrm{Ext}_R^d(R/I, M)$ be such that $\theta_M(\delta) \neq 0$,

and let P_* be a projective resolution of R/I. Then δ is represented by a map $\phi\colon N = \operatorname{syz}^d P_* \to M$, and under the induced map $\phi_*\colon \operatorname{Ext}^d(R/I, N) \to \operatorname{Ext}^d(R/I, M)$, $\phi_*(\varepsilon_I) = \delta$. Thus, we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Ext}^d_R(R/I, N) & \xrightarrow{\theta_N} & H^d_I(N) \\ \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} \\ \operatorname{Ext}^d_R(R/I, M) & \xrightarrow{\theta_M} & H^d_I(M) \end{array}$$

and $0 \neq \theta_M(\delta) = \theta_M\phi_*(\varepsilon_I) = \phi_*(\theta_N(\varepsilon_I)) = \phi_*(\eta_I)$, which implies that $\eta I \neq 0$. Q.E.D.

If R is a ring, $x_1, \ldots, x_d \in R$, M is an R-module, and $x_1, \ldots, x_d$ is a regular sequence on M, then it follows easily from the long exact sequence for Ext and induction on d that $\operatorname{Ext}^i(R/(x_1^t, \ldots, x_d^t), M) = 0$, $i < d$, while

$$\operatorname{Ext}^d(R/(x_1^t, \ldots, x_d^t), M) \cong M/(x_1^t, \ldots, x_d^t)M.$$

If $s \leqslant t$, we have, in fact, a commutative diagram

$$\begin{array}{ccc} \operatorname{Ext}^d(R/(\mathbf{x}^s), M) & \cong & M/(x^s)M \\ \downarrow & & \\ \operatorname{Ext}^d(R/(\mathbf{x}^t), M) & \cong & M/(\mathbf{x}^t)M \end{array}$$

Thus, one obtains the usual identification $H^d_{(\mathbf{x})}(M) \cong \varinjlim_t M/(\mathbf{x}^t)M$ (cf. [11]), which is valid even when $x_1, \ldots, x_d$ is not a regular sequence on M. Moreover, we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Ext}^d(R/(\mathbf{x}), M) & \xrightarrow{\theta_M} & H^d_{(\mathbf{x})}(M) \\ \wr\| & & \wr\| \\ M/(x_1, \ldots, x_d)M & \xrightarrow{\theta'} & \varinjlim_t M/(\mathbf{x}^t)M \end{array}$$

where θ^t is the same map as in the direct limit system. It is an easy exercise to show that when $x_1, \ldots, x_d$ is a regular sequence, each of the maps $M/(\mathbf{x}^s)M \to M/(\mathbf{x}^t)M$ is injective. Hence, θ_M is injective, whence $\eta_{(\mathbf{x})} \neq 0$. We have proved

(3.8) THEOREM. *Let $x_1, \ldots, x_d \in R$, a Noetherian ring, and suppose there exists an R-module M such that $x_1, \ldots, x_d$ is a regular sequence on M (which includes the requirement that $(x_1, \ldots, x_d)M \neq M$). Let $I = (x_1, \ldots, x_d)R$. Then $\eta_I \neq 0$. (Note that M need not be finitely generated.)*

Note that in the above discussion we viewed $H^d_{(\mathbf{x})}(M)$ as $\varinjlim_t \operatorname{Ext}^d_R(R/(\mathbf{x}^t); M)$ rather than as $\varinjlim_t \operatorname{Ext}^d_R(R/(\mathbf{x})^t, M)$. This makes no difference since the two sequences of ideals $(\mathbf{x}^t)$, $(\mathbf{x})^t$ both begin with $(\mathbf{x})$, and each is cofinal in the other.

When $x_1,..., x_d$ is an R-sequence, we can make a reasonably understandable calculation of the canonical element: choose the Koszul complex $K_*(\mathbf{x}; R)$ as a resolution of $R/(\mathbf{x})$. Thus, $\operatorname{syz}^d R/(\mathbf{x}) \cong R$, and $\operatorname{Ext}^d(R/(x_1,..., x_d), R) \cong R/(x_1,..., x_d)$ so that $\varepsilon_{(\mathbf{x})}$ is the identity in $R/(\mathbf{x})$. Thus, $\eta_{(\mathbf{x})}$ is the image of $1 + (\mathbf{x}) \in R/(\mathbf{x})$ in $H^d_{(\mathbf{x})}(R) \cong \varinjlim_t R/(x_1^t,..., x_d^t)$. This identification depends on the choice of $x_1,..., x_d$. We state this formally.

(3.9) THEOREM. *Let R be a Noetherian ring, let $x_1,..., x_d$ be an R-sequence, and identify $\operatorname{syz}^d R/(\mathbf{x}) = \operatorname{syz}^d(K_*(\mathbf{x}; R)) = R$. Then the canonical element $\eta_{(\mathbf{x})}$ in $H^d_{(\mathbf{x})}(\operatorname{syz}^d R/(x))$ may be identified with the image of $1 + (\mathbf{x}) \in R/(x)$ in $\varinjlim_t R/(\mathbf{x}^t) \cong H^d_{(\mathbf{x})}(\operatorname{syz}^d R/(\mathbf{x}))$.*

Now let R be a Noetherian ring, J an ideal of R, $x_1,..., x_d \in J$, and suppose $\operatorname{Rad}(x_1,..., x_d) = \operatorname{Rad} J$. Let $X_1,..., X_d$ be indeterminates over $\mathbb{Z}$, the integers, let $A = \mathbb{Z}[X_1,..., X_d]$, and map $A \to R$ such that $X_i \mapsto x_i$, $1 \leqslant i \leqslant d$: call this map h. Let P_* be a projective resolution of R/J over R, with $P_0 = R$ (for simplicity). Then we can map the free A-complex $K_*(\mathbf{X}; A)$ to the acyclic R-complex P_* so as to lift $h: A \to R$, and we obtain a diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \operatorname{syz}^d R/J & \longrightarrow \cdots \longrightarrow & P_i & \longrightarrow \cdots \longrightarrow & R & \longrightarrow & R/J & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \varphi} & & \uparrow & & \uparrow{\scriptstyle h} & & \uparrow & & \\
0 & \longrightarrow & K_d(\mathbf{X}; A) & \longrightarrow \cdots \longrightarrow & K_i(\mathbf{X}; A) & \longrightarrow \cdots \longrightarrow & A & \longrightarrow & A/(\mathbf{X}) & \longrightarrow & 0
\end{array}$$

By Corollary (3.6), we may view η^d_J as the image of $\eta^d(\mathbf{X})$ under ϕ_*: $H^d_{(X)}(A) \to H^d_J(\operatorname{syz}^d R/J) = H^d_{(\mathbf{x})}(\operatorname{syz}^d R/J)$. If we identify $H^d_{(X)}(A)$ with $\varinjlim_t (A/(\mathbf{X}^t))$ and $H^d_{(\mathbf{x})}(\operatorname{syz}^d R/J)$ with $\varinjlim_t (\operatorname{syz}^d(R/J)/(\mathbf{x}^t)\ \operatorname{syz}^d(R/J)$, and if we identify $\eta_{(\mathbf{X})}$ as the image of $1 + (\mathbf{X}) \in A/(X)$ in $\varinjlim_t A/(\mathbf{X}^t)$, then we see

(3.10) THEOREM. *With notation as above, let $N = \operatorname{syz}^d R/J$. Then η^d_J may be identified with the image of*

$$\phi(1) + (\mathbf{x})N \in N/(\mathbf{x})N$$

in $\varinjlim_t N/(\mathbf{x}^t)N$. Hence $\eta^d_J \neq 0$ if and only if for all integers $t \geqslant 0$,

$$(x_1^{t-1} \cdots x_d^{t-1})\, \phi(1) \notin (x_1^t,..., x_d^t)N.$$

(3.11) *Remark.* In the above situation, we may replace $K_*(\mathbf{X}; A)$ by $K_*(\mathbf{x}; R)$: h is replaced by id_R, and ϕ is the map from $R = K_*(\mathbf{x}; R)$ to

$N = \operatorname{syz}^d P_*$. $\phi(1_A)$ is replaced by $\phi(1_R)$. All this is immediate from the fact that if Q is an A-module and P is an R-module, $\operatorname{Hom}_A(Q, P) \cong \operatorname{Hom}_R(R \otimes Q, P)$, whence $\operatorname{Hom}_A(K_*(\mathbf{X}; A), P_*)$ (as complexes) $\cong \operatorname{Hom}_R(K_*(x; R), P_*)$ (as complexes). Moreover, we may replace the assumption that $P_0 = R$, $(A \to P_0) = h$, by the assumption that $R \to P_0$ lifts $R/(x) \to R/J$. This is worth stating formally:

(3.12) THEOREM. *Let R be a Noetherian ring, J a proper ideal, and $x_1, \ldots, x_d \in J$ such that $\operatorname{Rad}(\mathbf{x}) = \operatorname{Rad} J$. Map $K_*(\mathbf{x}; R)$ to P_*, a projective resolution of R/J, such that $R = K_0(\mathbf{x}; R) \to P_0$ lifts $R/(\mathbf{x}) \to R/J$. Let ϕ be the induced map of $R = K_*^d(\mathbf{x}; R)$ to $\operatorname{syz}^d P_* = N$. Then η_J^d may be identified with the image of $\phi(1) + (\mathbf{x})N \in N/(\mathbf{x})N$ in $\varinjlim_t N/(\mathbf{x}^t)N \in H_J^d(N)$.*

Hence, $\eta_J^d \neq 0$ if and only if for every integer $t \geqslant 0$,

$$(x_1 \cdots x_d)^{t-1} \phi(1) \notin (x_1^t, \ldots, x_d^t)N.$$

(3.13) *Remark.* Let B be any ring such that $A = \mathbb{Z}[X_1, \ldots, X_d] \to^h R$ factors $A \to B \to^k R$, and let $y_1, \ldots, y_d$ be the images of $X_1, \ldots, X_d$ in B. Then we may lift k to a map $K_*(\mathbf{y}; B) \to P_*$ and consider $\psi: B \to \operatorname{syz}^d P_*$ induced by this lifting, where $B = K_d(\mathbf{y}; B)$. Evidently, $\psi(1) + (\mathbf{x})N$ represents η_J, for given any B-module E, and any R-module P, $\operatorname{Hom}_B(E, P) \cong \operatorname{Hom}_R(R \otimes_B E, P)$, and it follows that the maps from $K_*(\mathbf{y}; B)$ to P_* are the same as those from $K_*(\mathbf{x}; R)$ to P_*.

Before turning our attention to maximal ideals of local rings, we note some trivial but useful facts about η_I in the general case.

(3.14) PROPOSITION. *Let R be a Noetherian ring and I a proper ideal. Let $d \geqslant 0$ be an integer.*

(1) *I kills ε_I^d and η_I^d.*

(2) *If $J \subset I$, $\operatorname{Rad} J = \operatorname{Rad} I$ and $\eta_I^d \neq 0$, then $\eta_J^d \neq 0$. Equivalently, if $\eta_J^d = 0$, then $\eta_I^d = 0$.*

(3) *If $H_I^d(M) \neq 0$ for some R-module M, then there exists a positive integer t such that for all $J \subset I^t$ with $\operatorname{Rad} J = \operatorname{Rad} I$, $\eta_J \neq 0$.*

Proof. (1) $\varepsilon_I^d \in \operatorname{Ext}^d(R/I, \operatorname{syz}^d R/I)$, which is killed by I, and η_I is the image of ε_J^d under the natural map to $H_I^d(\operatorname{syz}^d R/I)$.

(2) For every M, the map $\operatorname{Ext}_R^d(R/I, M) \to H_I^d(M)$ factors $\operatorname{Ext}_R^d(R/I, M) \to \operatorname{Ext}_R^d(R/J, M) \to H_I^d(M)$. Since the powers of I and the powers of J are each cofinal in the other,

$$\operatorname{Ext}_R^d(R/J, M) \to H_J^d(M) \cong H_I^d(M)$$

is zero for all M implies that

$$\operatorname{Ext}_R^d(R/I, M) \to H_I^d(M)$$

is zero for all M.

(3) If $H_I^d(M) \neq 0$, we choose t such that $\operatorname{Ext}_R^d(R/I^t, M) \to H_{I^t}(M) \cong H_I(M)$ is nonzero, for $H_I(M) = \varinjlim \operatorname{Ext}_R^d(R/I, M)$. It follows that $\eta_{I^t} \neq 0$, and the rest follows from part (2). Q.E.D.

We now focus attention on the local case. Let $(R, \mathfrak{m}, K)$ be a (Noetherian) local ring, with $\dim R = n$. Let $x_1, \ldots, x_n$ be a system of parameters. It is understood that ε_R, η_R denote $\varepsilon_{\mathfrak{m}}^n, \eta_{\mathfrak{m}}^n$, i.e., $d = n$. Since $\mathfrak{m} = \operatorname{Rad}(x_1, \ldots, x_n)$, the theory of Theorems (3.10) and (3.12) may be applied with $J = \mathfrak{m}$. We have at once

(3.15) THEOREM. *A local ring $(R, \mathfrak{m}, K)$ has property* CE *if and only if* $\eta_R \neq 0$.

Proof. Compare Remarks (2.2)(7) with Theorem (3.12). Q.E.D.

Note that, knowing this, we can view Theorem (3.8) as a generalization of Theorem (2.7). Theorem (2.8) can now be seen to assert that if the direct summand conjecture is true, then for every local ring R, $\eta_R \neq 0$.

(3.16) *Conjecture* (Canonical element conjecture). *For every local ring* R, $\eta_R \neq 0$.

It is easy to see that this is, in fact, equivalent to the direct summand conjecture (Theorem (2.8) gives one implication while Corollary (3.22) gives the other, which is easier).

Of course, we know from the existence of big C–M modules that $\eta_R \neq 0$ if R contains a field, and we have given another proof in Section 2 for the case where $\operatorname{char} R = p > 0$. But we can now give a different, very simple proof for this case based entirely on the functorial behavior of η_R (and, of course, the Frobenius).

(3.17) THEOREM. *Let $(R, \mathfrak{m}, K)$ be a local ring of positive prime characteristic $p > 0$. Then $\eta_R \neq 0$.*

Proof (By functorial properties of η_R). Since $H_{\mathfrak{m}}^R(R) \neq 0$ (where $n = \dim R$), we can choose t such that $\eta_{\mathfrak{m}^t} \neq 0$, by (3.14). Now apply Corollary (3.6) with $h = F^{p^e}$, $I = \mathfrak{m}$, $S = R$, and $J = F^{p^e}(\mathfrak{m})S \subset \mathfrak{m}^{p^e}$. It follows that $\eta_{\mathfrak{m}}$ maps to η_J. But for large e, $J \subset \mathfrak{m}^t$, and $\eta_J \neq 0$. Thus, $\eta_{\mathfrak{m}} \neq 0$. Q.E.D.

We next observe

(3.18) PROPOSITION. *Let $(R, m) \to (S, n)$ be a local homomorphism of local rings which takes a system of parameters for R into a system of parameters for S.*

(a) *If $\eta_S \neq 0$, then $\eta_R \neq 0$.*

(b) *If S is R-flat and $mS = n$, then $\eta_S \neq 0$ if and only if $\eta_R \neq 0$.*

(c) *In particular, $\eta_R \neq 0 \Leftrightarrow \eta_{\hat{R}} \neq 0$, where $\hat{R}$ is the m-adic completion of R.*

Proof. Let $\dim R$ $(= \dim S) = n$. Then (a) holds simply because, by Corollary (3.6), η_R maps to η_S.

To prove (b), choose a map of $K_*(\mathbf{x}; R)$ to P^*/n for some projective resolution P^* of K; the problem is to show that $\eta_R \neq 0 \Rightarrow \eta_S \neq 0$. Thus, assume that for the induced map $R \to^{\phi} \operatorname{syz}^n P_*$, $(x_1 \cdots x_n)^{t-1} \phi(1) \notin (x_1^t, \ldots, x_n^t) \operatorname{syz}^n P_*$. Now apply $\otimes_R S$. Since S is faithfully flat over R, these facts are preserved: snce $mS = n$, $P_* \otimes_R S$ is a projective resolution of $S/mS = S/n$. (The faitfhful flatness yields $\operatorname{syz}_S^n(P_* \otimes_R S) \cong (\operatorname{syz}_R^n P_*) \otimes_R S$.)

(c) is immediate from (b). Q.E.D.

(3.19) *Remark.* It follows that the conjecture that $\eta_R \neq 0$ for all local rings R reduces to the complete case. Many other reductions are possible: one can enlarge the residue class field so that it is algebraically closed, one can kill a prime ideal of maximum coheight, one can normalize. Thus if $\eta_R \neq 0$ when R is any complete local (even normal) domain, then $\eta_R \neq 0$ for all R. In all of the operations mentioned, one maps $R \to S$, S local, so that a system of parameters for R is still a system of parameters for S. A complete local domain is always a finite module over a regular local ring. Hence, the following proposition has some interest:

(3.20) THEOREM. *Let $(A, \mathfrak{p}) \subset (R, m)$ be a local inclusion of a regular local ring A in a local ring R of dimension n. Let $x_1, \ldots, x_n$ be a regular system of parameters for A and assume that $x_1, \ldots, x_n$ is a system of parameters for R. Map $K_*(\mathbf{x}; A) \to P_*$, where P_* is a projective resolution of R/m so that $A = K_0(\mathbf{x}; A) \to P_0$ lifts $A/(\mathbf{x}) \to R/m$, and let $\phi: A = K_n(\mathbf{x}; A) \to N = \operatorname{syz}^n P_*$ be the map induced by the lifting.*

Then $\eta_R \neq 0$ if and only if $A \to^{\phi} N$ is pure as a map of A-modules, (i.e., if and only if for all A-modules E, $\phi \otimes \mathrm{id}_E: E \to N \otimes E$ is injective).

If R is a finite module over A, then this is equivalent to the assertion that $A \to^{\phi} \operatorname{syz}^n P_*$ splits, i.e., that ϕ is injective and $\phi(A)$ is a direct summand of $\operatorname{syz}^n P_*$ as an A-module.

Proof. We first note that in [20], Proposition 6.11, p. 140], the fact that S is an R-module plays no role. Thus $A \to N$ is pure if and only if $\phi_1: E \to$

$N \otimes E$, where $E = \varinjlim_t A/(x_1^t,\ldots,x_n^t)$ $(= H^n_{(\mathbf{x})}(A) =$ the injective hull of I over $A)$, does not kill the image of 1. But this simply says that

$$x_1^{t-1} \cdots x_n^{t-1}\phi(1) \notin (x_1^t,\ldots,x_n^t)N,$$

all t, and the result follows from Theorem (3.12) and Remark (3.13).

In case R is a finite module over A, P_* may be chosen to consist of finitely generated free modules, so that $\operatorname{syz}^n P_*$ is a finite A-module. In this case, purity is equivalent to splitting (see, for example, [21, Corollary 5.2]). If P_* is not finite, one may split it into a finite resolution and an exact sequence of free modules which may be ignored. Q.E.D.

(3.21) *Remark.* Given a specific complete local domain R, it ought to be possible to test whether η_R is 0 or not: resolve $R/\mathfrak{m}$ to $n = \dim R$ steps, represent R as a finite module over a regular ring A with regular system of parameters $x_1,\ldots,x_n$, map $K_*(\mathbf{x};A)$ to the resolution P_* of $R/\mathfrak{m}$, and then see whether $\phi: A \to \operatorname{syz}^n P_*$ splits. One simply needs to understand ϕ as a map of A-modules. The point is, this seems to be a much more concrete question than whether R has a big C–M module, the quantification required is not so elaborate.

(3.22) COROLLARY. *Let A be a regular local ring and suppose $A \subset R$, where R is a local ring module-finite over A. If $\eta_R \neq 0$, then the map $A \to R$ splits as a map of A-modules.*

Proof. The map $A \to N$, where $N = \operatorname{syz}^N R/\mathfrak{m}$ is as in Theorem (3.20), splits as a map of a A-modules. Since N is an R-module, there is an induced map $R = A \otimes_A R \to N$ as R-modules, and $A \to N$ factors $A \to R \to^{\beta} N$. If $\psi: N \to A$ splits $A \to N$, then $\psi\beta$ splits $A \to R$. Q.E.D.

This gives one of several possible proofs that the canonical element conjecture implies the direct summand conjecture: a much easier fact than the converse, Theorem (2.8), which was one of the main results of Section 2.

(3.23) *Remark.* The question of whether $\eta_R = 0$ can also be translated into a problem of solving equations in R such that certain of the variables turn out to be a system of parameters. Suppose we seek a local ring R which is a counterexample, i.e., such that $\eta_R = 0$. Then over R there will exist a map of free complexes:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & R^{m_n} & \xrightarrow{Y_n} & \cdots & \longrightarrow & R^{m_i} & \xrightarrow{Y_i} & \cdots & \longrightarrow & R^{m_2} & \xrightarrow{Y_2} R^{m_1} & \xrightarrow{Y_1} R \\
 & & \uparrow{\scriptstyle 0 = Y'_n} & & & & \uparrow{\scriptstyle Y'_i} & & & & \uparrow{\scriptstyle Y'_2} & \uparrow{\scriptstyle Y'_1} & \uparrow{\scriptstyle \mathrm{id} = Y'_0} \\
0 & \longrightarrow & R & \xrightarrow{X_n} & \cdots & \longrightarrow & R^{\binom{n}{i}} & \xrightarrow{X_i} & \cdots & \longrightarrow & R^{\binom{n}{2}} & \xrightarrow{X_2} R^n & \xrightarrow{X_2} R
\end{array}$$

where the bottom row is the Koszul complex of a system of parameters $x_1,\ldots,x_n$ for R, Y_i is an $m_i \times m_{i-1}$ matrix over R, $1 \leqslant i \leqslant n$, and Y_i' is an $\binom{n}{i} \times m_i$ matrix over R, $0 \leqslant i \leqslant n$ $(m_0 = 1)$; moreover, we impose the conditions that $Y_n' = 0$, $Y_0' = [1]$, and that some fixed power of the entries of Y, say the tth, is in $(x_1,\ldots,x_n)R$ (so that Y_1 will have entries in the maximal ideal of R). Of course, X_i is a fixed $\binom{n}{i} \times \binom{n}{i-1}$ matrix each of whose entries is either a certain $\pm x_v$, or 0: the matrices of $K_*(\mathbf{x}; R)$.

Hence, if we view all the matrices Y_i, Y_i' as having unknown entries, and the x_i as unknown as well, then finding an n-dimensional local ring R with $\eta_R = 0$ is equivalent to finding integers $m_1,\ldots,m_n$ $(m_0 = 1)$ and t such that the matrix equations

(i) $Y_{i+1}Y_i = 0$, $1 \leqslant i \leqslant n$,

(ii) $Y_i Y_i' = Y_{i-1}' X_i$, $1 \leqslant i \leqslant n$,

(iii) $Y_1' = [1]$,

(iv) $Y_n' = 0_{1 \times m_n}$,

(v) $y_j^t = \sum_{i=1}^n y_{ji}^* x_i$, where

$$Y_1 = \begin{bmatrix} y_1 \\ \vdots \\ y_{m_1} \end{bmatrix}$$

(where Y_i, Y_i' have the sizes specified above, and the X's are determined by the x_i as in the Koszul complex) have a solution (for the entries) in a local ring R such that $x_1,\ldots,x_n$ have values which are a system of parameters. Of course, we can get rid of the matrices, and simply write a system of polynomial equations with coefficients in $\mathbb{Z}$ in $x_1,\ldots,x_n$, the entries of the matrices Y_i, Y_i', and the coefficients y_{ji}^* needed in (v).

Thus, the conjecture $\eta_R \neq 0$, all local R, admits reduction from equal characteristic 0 to equal characteristic p, by the method of [14], and, the proofs that $\eta_R \neq 0$ in char $p > 0$ given here that do not depend on big C–M modules also yield proofs in equal characteristic 0. In fact, by virtue of Proposition (3.18)(a) we get slightly more, just as in the big C–M modules case.

(3.24) THEOREM. *Let R be a local ring such that no prime integer p is a part of a system of parameters. Then $\eta_R \neq 0$.*

Proof. The hypothesis implies that we may choose a prime $q \subset R$ of coheight $\dim R$ such that $S = R/q$ is equicharacteristic. Then $\eta_S \neq 0 \Rightarrow \eta_R \neq 0$. Q.E.D.

4. Canonical Modules and Canonical Elements

In this section we shall give an interpretation of "satisfying CE" in terms of canonical modules, which was pointed out to the author by Joseph Lipman. Later, we shall exploit this point of view in studying the question, when can one conclude that R satisfies CE from the fact that R/xR, where x is a nonzero-divisor, satisfies CE. See Section 5.

We first recall some basic facts about canonical modules. Let (R, m, K) be local, $\dim R = n$, and let $E = E_R(K)$ be an injective hull for K over R: we assume that we have a fixed injection $K \to E$ as well. Let $\check{}$ denote the functor $\mathrm{Hom}_R(\ , E)$. A finitely generated R-module Ω is called a canonical module (or dualizing module) for R if $\Omega\check{} \cong H^n_m(R)$. If it exists, Ω is unique up to nonunique isomorphism. If R is module-finite over a Gorenstein local ring S, where $\dim S = q$, then we may take $\Omega = \mathrm{Ext}^{q-n}(R, S)$, so that R does have a canonical module. In particular, every complete local ring has a canonical module.

When R is C–M, local duality gives an isomorphism

$$\mathrm{Ext}^i_R(M, \Omega)\check{} \cong H^{n-i}_m(M).$$

When R is not necessarily C–M, this still holds for $i = 0$, i.e.,

$$\mathrm{Hom}_R(M, \Omega)\check{} \cong H^n_m(M).$$

(Consider a presentation $R^s \to R^r \to M \to 0$ and use the fact that $(\check{}) \circ \mathrm{Hom}_R(\ , \Omega)$ and H^n_m are both right exact functors and give the same result when applied to R.) The reader is referred to [11] for more details.

Let $R^\#$ denote the ring $\mathrm{Hom}_R(\Omega, \Omega)$. Scalar multiplication on Ω gives an obvious ring homomorphism $R \to R^\#$, so that $R^\#$ is a module-finite R-algebra. Although it is not obvious that $R^\#$ is commutative, we shall soon prove this. Moreover, under mild assumptions, $R \cong R^\#$. In fact

(4.1) Proposition. *Let (R, m) be a local ring of dimension n which is a homomorphic image of a Gorenstein ring, so that R has a canonical module Ω. Then*:

(a) $H^n_m(\Omega) \cong (R^\#)\check{}$.

(b) *The following conditions are equivalent*:

(i) *R is S_2 and for every minimal prime P of R, $\dim R/P = \dim R$.*

(ii) *$R \to R^\#$ is an isomorphism.*

(iii) *The map $\delta: H^n_m(\Omega) \to E$ which is the composite*

$$H^n_m(\Omega) \cong (R^\#)\check{} \to R\check{} = E$$

is an isomorphism.

Proof. (a) is immediate from local duality: $H^n_m(\Omega) \cong \operatorname{Ext}^0_R(\Omega, \Omega)^{\vee} = \operatorname{Hom}(\Omega, \Omega)^{\vee} = (R^{\#})^{\vee}$. The equivalence of (b)(ii) and (iii) is clear, since the map in (iii), up to composition with isomorphisms, is simply the dual of $R \to R^{\#}$, and $^{\vee}$ is faithfully exact. It remains to prove the equivalence of (i) with (ii) (or (iii)). This will follow from the discussion and Lemma (4.2).

Let $P_1, \ldots, P_r$ be the minimal primes of R such that $\dim R/P_i = \dim R$, and let $S = R - \bigcup_i P_i$. Thus, $y \in \mathfrak{m} \cap S$ if and only if y is part of a system of parameters for R, i.e., if and only if $\dim R/yR < \dim R$. Let $Q = \operatorname{Ker}(R \to S^{-1}R) = \{x \in R$: x is killed by some parameter$\}$. Let $\bar{R} = R/Q$. There will exist a single parameter y such that $yQ = 0$, so that, as a module, $\dim Q < n$. Q is the largest ideal of R whose dimension is $< n$. Note that in $\bar{R}$, every parameter is a nonzerodivisor. (This is equivalent to asserting that $\bar{R}$ is S_1 *and* every minimal prime has coheight equal to the dimension.) Then

(4.2) LEMMA. *With the same hypotheses as in* (4.1) *and the preceding discussion*:

(a) $H^n_{\mathfrak{m}}(R) \to H^n_{\bar{\mathfrak{m}}}(\bar{R})$ *is an isomorphism.*

(b) $\Omega_R \cong \Omega_{\bar{R}}$.

(c) *If* $x \in R$ *is a parameter,* x *is not a zero-divisor on* Ω_R.

(d) $\operatorname{Ker}(R \to \operatorname{Hom}_R(\Omega, \Omega)) = Q$. *Thus,* $R \to \operatorname{Hom}_R(\Omega, \Omega)$ *is injective if and only if* $R = \bar{R}$.

(e) *There is an injection* $\operatorname{Hom}_R(\Omega, \Omega) \to S^{-1}R \cong S^{-1}\bar{R}$ *as rings. Hence,* $\operatorname{Hom}_R(\Omega, \Omega)$ *is a commutative semilocal ring module-finite over* R.

Proof. (a) This part is immediate from the exact sequence for local cohomology: consider the short exact sequence $0 \to Q \to R \to \bar{R} \to 0$, and use the fact that, since $\dim Q < \dim R$, $H^n_m(Q) = 0$.

(b) This follows from (a) and the fact that the dual of an $\bar{R}$-module into the injective hull $E_R(K)$ of K over R is the same as its dual into $E_{\bar{R}}(K)$.

(c) Let x be a parameter. Then $0 \to \bar{R} \to^{x} \bar{R} \to \bar{R}/x\bar{R} \to 0$ is exact. Since R is a homomorphic image of a Gorenstein ring T, so is $\bar{R}$, say $\bar{R} = T/I$, where I will have all minimal primes of the same height, say, d. Then from the short exact sequence we get

$$0 \longrightarrow \operatorname{Ext}^d_T(\bar{R}, T) \xrightarrow{x} \operatorname{Ext}^d_T(\bar{R}, T) \longrightarrow \operatorname{Ext}^{d+1}_T(\bar{R}/x\bar{R}, T)$$

or

$$0 \longrightarrow \Omega_{\bar{R}} \xrightarrow{x} \Omega_{\bar{R}} \longrightarrow \Omega_{\bar{R}/x\bar{R}}$$

so that x is not a zerodivisor on $\Omega_{\bar{R}} = \Omega_R$, and $\Omega_{\bar{R}}/x\Omega_{\bar{R}}$ injects into $\Omega_{\bar{R}/x\bar{R}}$. This establishes (c), and also shows that $\Omega_R \to S^{-1}\Omega_R$ is injective and, hence, that

$$\operatorname{Hom}_R(\Omega, \Omega) \to S^{-1} \operatorname{Hom}_R(\Omega, \Omega)$$

is injective. But

$$S^{-1} \operatorname{Hom}_R(\Omega, \Omega) \cong \operatorname{Hom}_{S^{-1}R}(S^{-1}\Omega, S^{-1}\Omega),$$

where $S^{-1}R = S^{-1}\bar{R}$ is a zero-dimensional semilocal ring, say $S^{-1}R = \prod_{i=0}^{h} A_i$, where each A_i is Artinian local, and $S^{-1}\Omega$ is the product of the canonical modules of the A_i. But then

$$S^{-1}\bar{R} = S^{-1}R \to \operatorname{Hom}_{S^{-1}R}(S^{-1}\Omega, S^{-1}\Omega)$$

is an isomorphism, since this is true for each factor. Thus, $\operatorname{Hom}_R(\Omega, \Omega)$ injects into $S^{-1}R = S^{-1}\bar{R}$, as claimed in (e). But, since it is module-finite over R, it must be semilocal.

(d) This is immediate from the facts that Ω is an $\bar{R}$-module (so that Q is in the kernel) and that the map becomes an isomorphism after localizing at S. Q.E.D.

We now return to the proof of Proposition (4.1).

We must show (i) $\Leftrightarrow$ (ii). From either (i) or (ii) it follows that $R = \bar{R}$ and so henceforth we assume this. We next claim that if x_1, x_2 is part of a system of parameters for R, then x_1, x_2 is a regular sequence on Ω, and hence on $R^{\#} = \operatorname{Hom}_R(\Omega, \Omega)$. To see this, note that we already have an embedding $\Omega/x_1\,\Omega \hookrightarrow \Omega_{R/x_1R}$. Since the image of x_2 in R/x_1R is part of a system of parameters, x_2 is a nonzero-divisor on Ω_{R/x_1R}, and hence also on $\Omega/x_1\Omega$. The calculation of both Ω and $R^{\#} = \operatorname{Hom}_R(\Omega, \Omega)$ commutes with localization, and it follows that both Ω and $R^{\#}$ are S_2. It is clear now that (ii) $\Rightarrow$ (i).

It remains only to show that if R satisfies (i), then $R \to R^{\#}$ is an isomorphism. We already know the map is injective and that $R^{\#} \hookrightarrow S^{-1}R$. If $\operatorname{ht} P \leqslant 2$, then R_P is C–M and $R_P \to (R_P)^{\#}$ is an isomorphism. It follows that if $C = \operatorname{Coker}(R \to R^{\#})$, then if $C \neq 0$, $\operatorname{ht} \operatorname{Ann} C \geqslant 3$, and so, since R is S_2, $\operatorname{Ann} C$ contains an R-sequence of length at least 2. Since $\operatorname{Ext}_R^j(C, R)$ is first nonzero at the depth of R on $\operatorname{Ann} C$, $\operatorname{Ext}_R^1(C, R) = 0$, which implies $R^{\#} = R \oplus C$. This is impossible, since $R^{\#} \to S^{-1}R$ is injective and C is killed upon localization at S. The only possibility is that $C = 0$. Q.E.D.

Next we want to observe that for every module M we have a pairing

$$\operatorname{Hom}_R(M, \Omega) \otimes H^n_{\mathfrak{m}}(M) \to E \qquad (\#)$$

induced by composing the Yoneda pairing

$$\mathrm{Hom}_R(M, \Omega) \otimes H^n_{\mathfrak{m}}(M) \to H^n_{\mathfrak{m}}(\Omega)$$

with the map $\delta: H^n_{\mathfrak{m}}(\Omega) \to E$. This is a perfect pairing which induces functorially the isomorphism

$$H^n_{\mathfrak{m}}(M) \cong \mathrm{Hom}_R(M, \Omega)^{\vee}$$

discussed earlier. The key point is that when $M = R$, $(\neq)$ becomes

$$\Omega \otimes H^n_{\mathfrak{m}}(R) \xrightarrow{\cong} H^n_{\mathfrak{m}}(\Omega) \xrightarrow{\delta} E$$

and we have

$$\begin{array}{ccccc} \Omega \otimes H^n_{\mathfrak{m}}(R) & \cong & H^n_{\mathfrak{m}}(\Omega) & \xrightarrow{\delta} & E \\ \| & & \| & & \| \\ \Omega \otimes \Omega^{\vee} & \longrightarrow & (R^{\#})^{\vee} & \longrightarrow & R^{\vee} \end{array}$$

(Recall that $R^{\#} = \mathrm{Hom}_R(\Omega, \Omega)$. The standard pairing $\Omega \otimes \Omega^{\vee} \to E$ is the dual of $R \to \mathrm{Hom}(\Omega, \Omega)$.)

Joseph Lipman pointed out to the author that the property $\eta_R \neq 0$ is equivalent to $\theta_\Omega \neq 0$.

(4.3) THEOREM. *Let $(R, \mathfrak{m}, K)$ be a local ring of dimension n which is a homomorphic image of a Gorenstein ring. Let $\gamma: R \to H^n_{\mathfrak{m}}(\mathrm{syz}^n K)$ map $1 \in R$ to η_R. Let $^{\vee}$ denote $\mathrm{Hom}_R(\ , E)$, as above, and let $\hat{}$ denote $\mathfrak{m}$-adic completion. Then there is a commutative diagram*

$$\begin{array}{ccccccc} \mathrm{Hom}_R(\mathrm{syz}^n K, \Omega) & \longrightarrow & \mathrm{Hom}_R(\mathrm{syz}^n K, \Omega)^{\wedge} & \overset{\alpha}{\cong} & H^n_{\mathfrak{m}}(\mathrm{syz}^n K) & \xrightarrow{\gamma^{\vee}} & E \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \hat{\beta}} & & & & \uparrow{\scriptstyle \delta} \\ \mathrm{Ext}^n_R(K, \Omega) & \longrightarrow & \mathrm{Ext}^n_R(K, \Omega)^{\wedge} & & \longrightarrow & & H^n_{\mathfrak{m}}(\Omega) \end{array}$$

where $\beta(f) = f_*(\varepsilon_R)$, $\hat{\beta}$ is the completion of β, α is induced by local duality, and the composite map $\mathrm{Ext}^n_R(K, \Omega) \to H^n_{\mathfrak{m}}(\Omega)$ is θ_Ω.

Hence, $\eta_R \neq 0$ if and only if $\theta_\Omega \neq 0$ if and only if $\delta\theta_\Omega \neq 0$.

Moreover, since $\mathrm{Ext}^n_R(K, \Omega)$ is a K-vector space $\mathrm{Im}(\delta\theta_\Omega) \subset K$, the unique copy of K in E, and if $\lambda_R: \mathrm{Ext}^n_R(K, \Omega) \to K$ is $\delta\theta_\Omega$ with its range restricted, then $\eta_R \neq 0$ if and only if $\lambda_R \neq 0$.

Proof. It is easy to see that we may replace R by $\hat{R}$ here. Hence, we assume $R = \hat{R}$ and consider the diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(\mathrm{syz}^n K, \Omega) \overset{\alpha}{\cong} H^n_m(\mathrm{syz}^n K) & \xrightarrow{\tilde{\delta}} & E \\ \downarrow \beta & & \uparrow \delta \\ \mathrm{Ext}^n_R(K, \Omega) & \xrightarrow{\theta_\Omega} & H^n_m(\Omega) \end{array}$$

We first check that it commutes. Let $f \in \mathrm{Hom}_R(\mathrm{syz}^n K, \Omega)$. Then $\beta(f) = f_*(\varepsilon_R) \Rightarrow \theta_\Omega(\beta(f)) = f_*(\eta_R)$, so that $(\delta\theta_\Omega\beta)(f) = \delta(f_*(\eta_R))$. On the other hand, α is induced by

$$\mathrm{Hom}_R(\mathrm{syz}^n K, \Omega) \otimes H^n(\mathrm{syz}^n K) \longrightarrow H^n(\Omega) \xrightarrow{\delta} E,$$

where $f \otimes \lambda \mapsto \delta(f_*(\lambda))$; here, f_* is the map f induces from $H^n_m(\mathrm{syz}^n K) \to H^n_m(\Omega)$. Thus $\tilde{\gamma}\,\alpha(f) = \tilde{\gamma} \circ (\lambda \mapsto \delta(f_*(\lambda)) = \delta(f_*(\gamma(1)) = \delta f_*(\eta_R)$, as needed.

Now, $\delta\theta_\Omega \neq 0 \Rightarrow \theta_\Omega \neq 0 \Rightarrow \eta_R \neq 0$ (since $\eta_R \neq 0 \Leftrightarrow \theta_n \neq 0$ for some M) and it will suffice to show that $\eta_R \neq 0 \Rightarrow \delta\theta_\Omega \neq 0$. Suppose, to the contrary, that $\delta\theta_\Omega = 0$. By the commutativity of the diagram, $\tilde{\gamma}\,\alpha = \delta\theta_\Omega\beta = 0 \Rightarrow \tilde{\gamma} = 0$ (since α is an isomorphism) $\Rightarrow \gamma = 0 \Rightarrow \eta_R = 0$, a contradiction. The last statement in the theorem follows immediately. Q.E.D.

5. Comparison of M_R and $M_{R/xR}$

Throughout this section (R, m, K) denotes a local ring of dimension n which is a homomorphis image of a Gorenstein local ring S of dimension q and $x \in m$ denotes a nonzero-divisor in R. There is little or no loss of generality in assuming that R is a homomorphic image of a Gorenstein ring, since η_R vanishes if and only if $\eta_{\hat{R}}$ vanishes. We shall let $\bar{}$ denote the result of applying $\otimes_R R/xR$. Thus $\bar{R} = R/xR$. We want to study the question, under what conditions can we conclude that $\eta_R \neq 0$ given that $\eta_{R/xR} \neq 0$?

Let $\Omega_R = \mathrm{Ext}^{q-n}_S(R, S)$, while $\Omega_{\bar{R}} = \mathrm{Ext}^{q-(n-1)}_S(\bar{R}, S)$, which are dual to $H^n_m(R)$, $H^{n-1}_m(\bar{R})$, respectively.

(Note that $H^{n-1}_{m\bar{R}}(\bar{R}) \cong H^{n-1}_m(\bar{R})$.) In certain good cases $\Omega_{\bar{R}} \cong \Omega_R/x\Omega_R = \bar{\Omega}_R$, but not in general. In fact, the short exact sequence

$$0 \to R \xrightarrow{x} R \to \bar{R} \to 0$$

yields

$$0 \to \Omega_R \xrightarrow{x} \Omega_R \to \Omega_{\bar{R}} \to \mathrm{Ext}^{q-n+1}_S(R, S) \xrightarrow{x} \mathrm{Ext}^{q-n+1}_S(R, S).$$

Let us write $\Omega'_R = \mathrm{Ext}^{q-n+1}_S(R, S)$: this is dual to the second highest local cohomology module $H^{n-1}_m(R)$ of R. Summarizing, we have

(5.1) LEMMA. *$\Omega_{\bar{R}} \cong \bar{\Omega}_R$ if and only if x is not a zero-divisor on Ω'_R: equivalently, $\Omega_{\bar{R}} \cong \bar{\Omega}_R$ if and only if $H^{n-1}_m(R)$ is x-divisible.*

In the general case, let $C = \operatorname{Ann}_{\Omega'_R} x$. Then we have short exact sequences

$$0 \to \Omega_R \xrightarrow{x} \Omega_R \to \bar{\Omega}_R \to 0 \qquad \text{and} \qquad 0 \to \bar{\Omega}_R \to \Omega_{\bar{R}} \to C \to 0.$$

Let $\Omega = \Omega_R$ (so that $\bar{\Omega} = \Omega/x\Omega$), $\mathscr{E}^i(M) = \operatorname{Ext}^i_R(K, M)$, $\bar{\mathscr{E}}^i(M) = \operatorname{Ext}^i_{\bar{R}}(K, M)$ and let E_R, $E_{\bar{R}}$ be the injective hulls of K over R, $\bar{R}$, respectively. We view $E_{\bar{R}}$ as $\operatorname{Ann}_{E_R} x$. We then have the commutative diagram

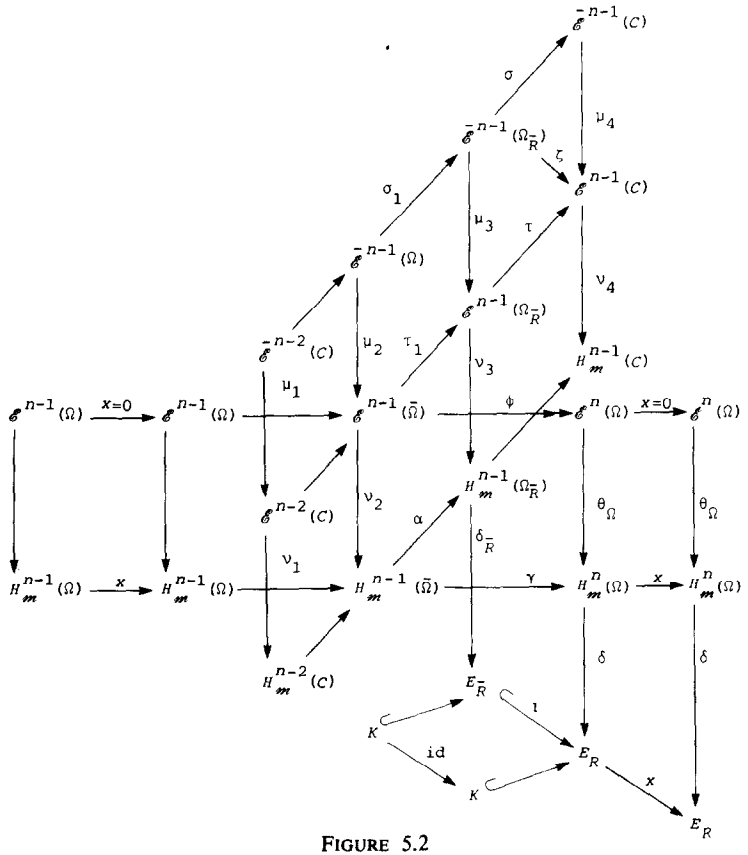
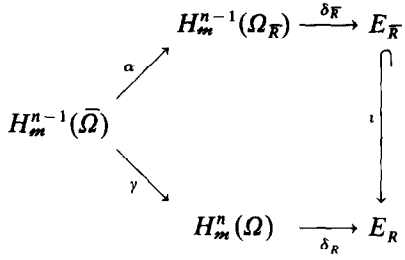


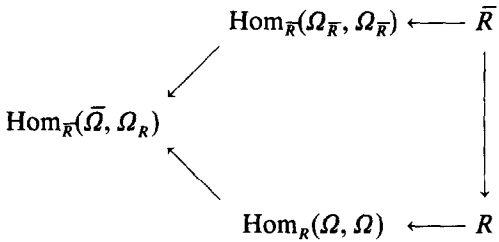
FIGURE 5.2

Here, $\bar{\mathscr{E}}^i \to \mathscr{E}^i$ is the map of Ext's for change of rings and $\mathscr{E}^i \to H^i_m$ is the obvious map. These maps induce all the vertical arrows except $\delta_{\bar{R}}$, δ, which were discussed in detail in Section 4. The horizontal arrows are long exact sequences arising from $0 \to \Omega \to^x \Omega \to \bar{\Omega} \to 0$, while the rising arrows come from long exact sequences derived from $0 \to \bar{\Omega} \to \Omega_{\bar{R}} \to C \to 0$.

The commutativity of this diagram is not quite obvious: one must check that $\delta\gamma = \iota\delta_R\alpha$. The point is that the diagram



is the dual, into E_R, of the diagram



Here, one uses the facts that $\mathrm{Hom}_R(\ , E_R)$ on $\bar{R}$-modules is isomorphic with $\mathrm{Hom}_{\bar{R}}(\ , E_{\bar{R}})$ and local duality over R and $\bar{R}$ for the highest local cohomology using Ω, $\Omega_{\bar{R}}$, respectively (i.e., $H^n_{\mathfrak{m}}(M) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(M, \Omega), E)$ while $H^{n-1}_{\mathfrak{m}}(N) \cong \mathrm{Hom}_{\bar{R}}(\mathrm{Hom}_{\bar{R}}(N, \Omega_R), E_{\bar{R}})$ for an $\bar{R}$-module N).

(5.3) THEOREM. *Let $(R, \mathfrak{m}, K)$ be a local ring which is a homomorphic image of a Gorenstein ring with $\dim R = n$. Let x be a nonzero-divisor in R and let $\bar{R}$, Ω, $\bar{\Omega}$, $\Omega_{\bar{R}}$, and C be as above. Let $\zeta\colon \mathrm{Ext}^{n-1}_{\bar{R}}(K, \Omega_{\bar{R}}) \to \mathrm{Ext}^{n-1}_R(K, C)$ be the map $\mu_4\sigma$ (or $\tau\mu_3$) indicated in Fig. 5.2. If $\zeta = 0$ and $\eta_{\bar{R}} \neq 0$, then $\eta_R \neq 0$.*

In particular, if $C = 0$ and $\eta_{\bar{R}} \neq 0$, then $\eta_R \neq 0$.

Proof. We refer to Fig. 5.2 throughout the argument. Suppose $\zeta = 0$. Then $\tau\mu_3 = 0$, so that $\mathrm{Im}\,\mu_3 \subset \mathrm{Im}\,\tau_1$ $(= \mathrm{Ker}\,\tau)$. Now $\eta_{\bar{R}} \neq 0$ implies that the composite map $\delta_R\eta\mu_3$ maps onto the copy of K in $E_{\bar{R}}$, and hence $\iota\delta_R\eta\mu_3$ maps onto the copy of K in E_R. Choose $z \in \mathrm{Ext}^{n-1}_{\bar{R}}(K, \bar{R})$ such that $\iota\delta_R\eta\mu_3(z)$ generates the copy of K in E_R. Since $\mathrm{Im}\,\mu_3 \subset \mathrm{Im}\,\tau_1$, we can choose $w \in \mathrm{Ext}^{n-1}_R(K, \bar{\Omega})$ such that $\tau_1(w) = \mu_3(z)$. Then $\iota\delta_{\bar{R}}\nu_3\tau_1(w)$ generates the copy of K in E_R, and hence, from the commutativity of the diagram, so does $\delta\theta_\Omega\phi(w)$. Hence $\delta\theta_\Omega$ (and θ_Ω) are nonzero and $\eta_R \neq 0$. Q.E.D.

(5.4) COROLLARY. *Let R be a local ring which is a homomorphic image of a Gorenstein ring, of mixed characteristic $p > 0$, such that p is not a zero-divisor in R. If p is also not a zero-divisor on Ω'_R, then $\eta_R \neq 0$.*

Proof. By Lemma (5.1) with $x = p$ we have $C = 0$, and $\eta_{\bar{R}} \neq 0$ since $\bar{R} = R/pR$ has characteristic p. The result is then immediate from Theorem (5.3).

(5.5) *Remarks.* (a) The general question of whether $\eta_R \neq 0$ for local rings of mixed characteristic p reduces to the case where R is a complete local domain and hence module-finite over a regular local ring A. In this case Ω'_R is simply $\operatorname{Ext}^1_A(R, A)$.

(b) If depth $R > 0$ and depth $\Omega'_R > 0$, then we can choose $x \in R$ not a zero-divisor such that x is also not a zero-divisor on Ω'_R. Hence, if there is a local domain R with $\eta_R = 0$ and we choose any such domain of smallest dimension, Ω'_R will have depth 0.

(c) It seems to be difficult to understand the significance of the condition that

$$\zeta\colon \operatorname{Ext}^{n-1}_{\bar{R}}(K, \Omega_{\bar{R}}) \to \operatorname{Ext}^{n-1}_R(K, C)$$

be zero when $C \neq 0$.

If R_P is C–M for all primes P of height $\leqslant i$, let us say that R is C–M$_i$ (this is weaker than assuming that R is S_i).

(5.6) PROPOSITION. *Let i be $\leqslant n$. If R is* C–M$_i$, *then* $\operatorname{Ann}_R C$ *has height* $\geqslant i + 1$. *Hence,* $\dim C \leqslant \dim R - (i + 1)$.

Proof. If P were a prime of height $\leqslant i$ containing $\operatorname{Ann}_R C$, then C_P would be nonzero even though R_P is C–M, a contradiction (the sequence $0 \to \bar{\Omega} \to \Omega_{\bar{R}} \to C \to 0$ may be localized to give the corresponding sequence for R_P).

Q.E.D.

(5.7) *Remark.* If R is C–M$_2$, then $\dim C \leqslant n - 3$, where $n = \dim R$, and this implies

$$H^{n-2}_m(C) = H^{n-1}_m(C) = 0.$$

This yields some simplication in Fig. 1; the map

$$\alpha\colon H^{n-1}_m(\bar{\Omega}) \to H^{n-1}_m(\Omega_{\bar{R}})$$

is then an isomorphism.

6. The Direct Summand Conjecture

In this section we study the direct summand conjecture in detail. We shall show that the conjecture reduces to the unramified case. In fact

(6.1) Theorem. *If A is a complete local domain, let T_A denote the integral closure of A in an algebraic closure of its fraction field. Then the following statements are equivalent:*

(1) *If A is a regular Noetherian ring and R is a module-finite extension, then A is a direct summand of R as an A-module.*

(2) *If A is a complete unramified regular local ring with algebraically closed residue class field and R is a module-finite extension domain of A, then A is a direct summand of R.*

(3) *If A is a complete unramified regular local ring, then A is a direct summand of T_A.*

(4) *If A is a complete unramified regular local ring, then* $\operatorname{Hom}_A(T_A, A) \neq 0$.

(5) *If A is a complete unramified regular local ring with maximal ideal $\mathfrak{m}$, then $H^n_{\mathfrak{m}}(T_A) \neq 0$.*

(6) *For every local ring R, $\eta_R \neq 0$.*

(7) *For every complete local domain R, $\eta_R \neq 0$.*

(8) *If $x_1, \ldots, x_n$ is a system of parameters for a local ring R, then there do not exist integers $b > a \geqslant 0$ and elements $y_1, \ldots, y_n \in R$ such that*

$$(x_1 \cdots x_n)^a = \sum_{i=1}^{n} y_i x_i^b.$$

Before proving this theorem we should make several observations. We could have added to the list a version (1°) of (1) in which A is local ((1) $\Rightarrow$ (1°) $\Rightarrow$ (2) is obvious). All of the statements are known to be true in the equicharacteristic case and so we could have restricted attention to the mixed characteristic case. We could have fixed the residual characteristic p and also the dimension n of A (or R), using the local version (1°) in place of (1): the statements are equivalent for fixed p, n. This will be clear from the proof. In (8), it is easy to see that it suffices to consider the case where $a = t$, $b = t + 1$. Moreover, it turns out that it suffices to consider the case where $x_1 = p$, the residual characteristic. We refer to (8) in the mixed characteristic case with $x_1 = p$ as (8°).

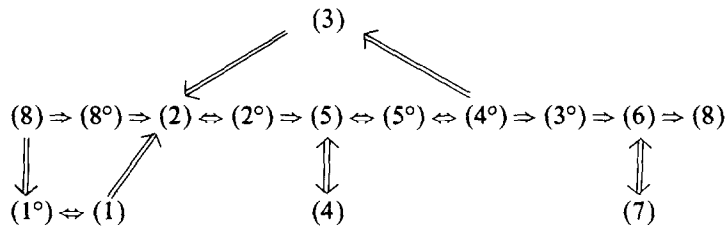
Proof of Theorem (6.1). We first note that $A \to R$ splits if and only if $\operatorname{Hom}_A(R, A) \to \operatorname{Hom}_A(A, A)$ is onto. This yields the implication (1°) $\Rightarrow$ (1), and so (1°) $\Leftrightarrow$ (1). Moreover, we may apply $\otimes_A B$, where B is a regular ring

faithfully flat over A, and it will suffice to show that $B \to B \otimes R$ splits instead of showing that $A \to R$ splits. This permits reduction to the case where A is complete, with an algebraically closed residue class field. Moreover, if we map R further to, say R', and if $A \to R'$ splits, then A to R splits. This permits us to kill a minimal prime of R disjoint from $A - \{0\}$ and so reduce to the case where R is a domain. Consider the statement:

(2°) *If A is a complete unramified regular local ring and R is a module-finite extension of A, then A is a direct summand of R.*

Then we have shown (2) $\Rightarrow$ (2°), while (2°) $\Rightarrow$ (2), obviously.

Let (3°), (4°), and (5°) denote the strengthened versions of (3), (4) and (5), respectively, in which the hypothesis is weakened slightly: A is assumed to have the form $V[[x_2,\ldots,x_n]]$, where V is a complete discrete valuation ring, but not necessarily unramified (V might be ramified). We shall complete the proof by showing that



from which it is easy to see that all the statements occurring are equivalent (note that (8) occurs twice in the top row). It is shown in [13] that if A is a regular local ring with regular system of parameters $x_1,\ldots,x_n$ and R is a module-finite extension, then $A \subset\!\!\to R$ splits if and only if for every positive integer t,

$$(x_1 \cdots x_n)^t = \sum_{i=1}^{n} y_i x_i^{t+1}$$

has *no* solution for the y_i in R. We may assume that we are in the mixed characteristic case. Now, (8) $\Rightarrow$ (8°) is obvious while (8°) $\Rightarrow$ (2) and (8) $\Rightarrow$ (1°) follow from the result in [13] just mentioned. Statements (1) $\Leftrightarrow$ (1°) and (1) $\Rightarrow$ (2) $\Rightarrow$ (2°) have already been proved.

To see that (2°) $\Rightarrow$ (5) we view T_A as the direct limit of all the module-finite extensions R with $A \subset R \subset T_A$. The map $H^n_m(A) \to H^n_m(T_A)$ arises as the direct limit of the maps $H^n_m(A) \to H^n_m(R)$ and since $A \to R$ splits for each such R, we have that each $H^n_m(A) \to H^n_m(R)$ is injective. It follows that $H^n_m(A) \to H^n_m(T_A)$ is injective. Since $H^n_m(A) \neq 0$, we have that $H^n_m(T_A) \neq 0$.

To prove (5) $\Leftrightarrow$ (5°), let $B = V[[x_2,\ldots,x_n]]$, where V is a complete, possibly ramified, DVR. Then V is module-finite over an *unramified*

complete DVR V_0. Let $A = V_0[[x_2,..., x_n]] \subset B$. Then B is module-finite over A, and we may view T_B as T_A as well. Moreover, $m_A B$ is primary to m_B, whence

$$H^n_{m_B}(T_B) \cong H^n_{m_A}(T_B) \cong H^n_{m_A}(T_A)$$

and we are done.

(4) $\Leftrightarrow$ (5) and (4°) $\Leftrightarrow$ (5°) Both are proved by the same argument: let E be the injective hull of the residue class field K of A and then note that

$$H^n_m(T_A) \cong \mathrm{Hom}_A(\mathrm{Hom}_A(T_A, A), E)$$

by local duality [11] and that $\mathrm{Hom}_A(\ , E)$ is a faithfully exact functor on A-modules.

(4°) $\Rightarrow$ (3°) This is one of the most interesting implications. Choose $\phi: T_A \to A$ with $\phi \neq 0$. Let $a_1,..., a_m$ be generators if $\mathrm{Im}\,\phi$. Since A is a UFD we can find the greatest common divisor a of $a_1,..., a_m$. Replacing ϕ by $(1/a)\phi$, we see that we may assume that a is a unit, so that $\mathrm{Im}\,\phi$ is not contained in a proper principal ideal of A. In particular, it follows that ϕ takes on a value which is not divisible by $x_1 = x$, where x generates the maximal ideal of V (here, $A = V[[x_2,..., x_n]]$). Let $r \in T_A$ be such that x does not divide $\phi(r)$. Replacing ϕ once more by the map $(t \mapsto \phi(rt))$ we see that we may assume without loss of generality that x does not divide $\phi(1)$.

Let q be an integer $\geqslant 2$. Then A has a unique continuous V-endomorphism α_0 which maps $x_i \mapsto x_i^q$, $q \geqslant 1$. This map extends to an automorphism of the ring

$$A^\infty = \bigcup_e v[[x_2^{1/q^e},..., x_n^{1/q^e}]]$$

which sends $x_i^{q^e} \to w_i^{q^{e+1}}$, $e \in \mathbb{Z}$. A^∞ is integral over A, so that the algebraic closure L of the fraction field of A^∞ may also be thought of as an algebraic closure of the fraction field of A. The automorphism of A^∞ extends uniquely to its fraction field and then, nonuniquely, to an automorphism of L. Restricting it to the integral closure of A^∞ (or, equivalently, A) in L we get an automorphism α of T_A which extends $\alpha_0: A \to A$.

Now, since x does not divide $\phi(1)$ we can choose an integer e so large that

$$\phi(1) \notin (x, x_2^Q,..., x_n^Q)A,$$

where $Q = q^e$. Let $\beta = \alpha^e$. Let $A_0 = \beta(A) = V[[x_2^Q,..., x_n^Q]]$. Then A is a free A_0-module and $\phi(1)$ is part of a free basis, by Nakayama's lemma. Hence, there is an A_0-linear map $\psi: A \to A_0$ such that $\psi(\phi(1)) = 1$, and so $\sigma = \psi \circ \phi$: $T_A \to A_0$ is an A_0-linear retraction of T_A to A_0. Now since A is module-finite over A_0, we may identify $T_A \cong T_{A_0}$. Thus, A_0 is a direct summand, as an A_0-

module, of T_{A_0}. But $A_0 = V[[x_2^Q,..., x_n^Q]] \cong A$. It follows that A is a direct summand of T_A. Thus $(4°) \Rightarrow (3°)$, as stated.

$(3°) \Rightarrow (6)$ This was proved, essentially in Section 2: $\eta_R \neq 0$ is equivalent, by Theorem (3.15), to R having property CE and what we need is basically the assertion in the second paragraph of Theorem (2.9) (whose proof is completed in (2.17)), together with the observation that the algebras T discussed there may all be thought of as lying between A and T_A.

We alredy know $(6) \Leftrightarrow (7)$; see Proposition (3.18) and Remark (3.19). To see that $(6) \Rightarrow (8)$ let $x_1,..., x_n$ be a s.o.p. for R and suppose

$$(x_1 \cdots x_n)^a = \sum_{i=1}^{n} y_i x_i^b, \qquad b > a.$$

Multiplyging by $u = (x_1 \cdots x_n)^{b-a-1}$ and replacing y_i by $y_i u$ we see that we may assume $a = b - 1$. Consider the standard map of $K_*(\mathbf{x}^b; R) \to K_*(\mathbf{x}, R)$. In degrees $n, n-1$ we have the diagram

$$\begin{array}{ccc} R & \xrightarrow{d=[\pm x_1 \cdots \pm x_n]} & R^n \\ {\scriptstyle \mu_n = \cdot (x_1 \cdots x_n)^{b-1}} \uparrow & \nearrow & \uparrow {\scriptstyle \mu_{n-1}} \\ R & \xrightarrow[d'=[\pm x_1^b \cdots \pm x_n^b]]{} & R^n \end{array}$$

The equation $(x_1 \cdots x_n)^{b-1} = \sum_{i=1}^n y_i x_i^b$ gives us precisely what we need to construct a map $h: K_{n-1}(\mathbf{x}^b; R) \to K_n(\mathbf{x}; R)$ such that $hd' = \mu_r$, for we can use the y_i with suitable signs as the entries of the matrix of h. We are then free to replace μ_{n-1} by $\mu_{n-1} - h$ and μ_n by 0, i.e., we can fill in the map of Koszul complexes so that the last map $R \to R$ is 0. We can then map $K_*(\mathbf{x}; R)$ to a resolution of K, the residue class field, and composing we get a map of $K_*(\mathbf{x}^t; R)$ to the resolution of K such that the map in degree n is 0. This means that R does not satisfy CE, i.e., that $\eta_R = 0$, a contradiction. Thus, $(6) \Rightarrow (8)$. It now follows that all the statements whose numbers occur in the first row of the diagram of implications are equivalent, and now we are done, because the only remaining implications $(3°) \Rightarrow (3) \Rightarrow (2)$ are obvious (the last because each module-finite extension domain of A lies between A and T_A). Q.E.D.

(6.2) *Remark.* The idea of the proof of the implication $(4°) \Rightarrow (3°)$ also gives a new proof of the direct summand conjecture in characteristic p, where it is a theorem. By the remarks at the beginning of the proof of (6.1), we can assume that $A = K[[x_1,..., x_n]]$, where K is an algebraically closed (or at least perfect) field, and we can assume that R is a domain, module-finite over A. Everything is now simpler than in the proof of $(4°) \Rightarrow (3°)$. Take any A-linear map $\phi: R \to A$ such that $\phi(1) \neq 0$. Instead of using the

artificially constructed endomorphism α of the enlargement T_A of R to carry out the proof, we can use the Frobenius endomorphism F of R instead. For a suitably large e, $\phi(1)$ will be part of a free basis for A over $F^e(A)$ and it will follow that there is an $F^e(A)$-linear retraction of R to $F^e(A)$, which we may restrict to $F^e(R)$. Thus $F^e(A)$ is a direct summand of $F^e(R)$ as an $F^e(A)$-module. But R, as an A-algebra, is isomorphic with $F^e(R)$ as an $F^e(A)$-algebra, and so A is a direct summand of R as an A-module.

A bit more generally, we have

(6.3) THEOREM. *Let A be a complete local regular ring of the form $V[[x_2,..., x_n]]$, where V is a discrete valuation ring, and let R be a module-finite extension domain of A. Let $q_2,..., q_n$ be integers $\geqslant 2$. Let R^∞ be a domain integral over R which has a V-endomorphism α whose restriction to A is the unique continuous V-endomorphism satisfying $\alpha(x_i) = x_i^{q_i}$, $2 \leqslant i \leqslant n$. Suppose also that* $\mathrm{Hom}_A(R^\infty, A) \neq 0$ *(equivalently, that $H^n_{\mathfrak{m}_A}(R^\infty) \neq 0$).*

Then A is a direct summand of R^∞ and, hence, of R as an A-module.

In Remark (6.2) we are using $R^\infty = R$ and $\alpha = F$. In the proof of $(4°) \Rightarrow (3°)$ we need $R^\infty = T_A$ and α was constructed.

(6.4) *Remark.* Let $A = V[[x_2,..., x_n]]$, let $A \subset R \subset T_A$ with R module-finite over A and let $q_2,..., q_n$ be fixed integers $\geqslant 2$. Let $\alpha: A \to A$ be the unique continuous V-endomorphism of A such that $\alpha(x_i) = x_i^{q_i}$, $2 \leqslant i \leqslant n$. Extend α to an endomorphism, which we also call α, of T_A as in the proof of $(4°) \Rightarrow (3°)$ in (6.1). $\alpha: T_A \to T_A$ is then actually an automorphism. Then there is an obvious "minimal" choise for R^∞, to wit, the subring of T_A generated by R, $\alpha(R)$, $\alpha^2(R)$,..., $\alpha^k(R)$,.... The problem is to show that for such an R^∞, $\mathrm{Hom}_A(R^\infty, A) \neq 0$; or, equivalently, that $H^n_{\mathfrak{m}}(R^\infty) \neq 0$.

(6.5) *Remark.* It is worth noting that if there is a counterexample to the direct summand conjecture (and we can assume mixed characteristic here), then there is one in which A is pointed étale extension of $V[x_2,..., x_n]_{\mathfrak{m}}$, where V is a complete unramified discrete valuation ring with maximal ideal pV and $\mathfrak{m} = (p, x_2,..., x_n)$. The same ideas as in [14] (mainly, Artin approximation, but applied to the equations defining the algebra structure of R as well as to the equation

$$(x_1 \cdots x_n)^t = \sum_{i=1}^{n} y_i x_i^{t+1})$$

suffice to pass from a counterexample where $A = V[[x_2,..., x_n]]$ to one with A as described. Thus, the full difficulty of the problem exists for algebras essentially of finite type over a discrete valuation ring.

(6.6) *Remark.* There are several results which assert that if A is regular and R is the integral closure of A in a finite field extention of the fraction

field of A of *sufficiently small degree* in some sense then $A \subset\!\!\!\rightarrow R$ splits. Of course, this is only interesting when the degree d of the field extension is not invertible in A: if d is invertible, we can use $1/d$ times field trace to get a retraction.

In particular, the case where $d = 2$ is handled in [19] where it is shown that if A satisfies R_2, S_3 and is locally factorial and R is an integral extension such that the extension of function fields is quadratic, then $A \subset\!\!\!\rightarrow R$ splits. Many cases where $d = p$ or p^2, where p is the residual characteristic of A are done in [14].

(6.7) *Remark.* The direct summand conjecture is equivalent to the conjecture that if $x_1,..., x_n, y_1,..., y_n$ are elements of a Noetherian ring R such that

$$(x_1 \cdots x_n)^t = \sum_{i=1}^{n} y_i x_i^{t+1}$$

for some positive integer t and $(x_1,..., x_n)R = I$ is a proper ideal, then $\operatorname{ht} I \leqslant n - 1$. (Clearly, $\operatorname{ht} I \leqslant n$.) For if there were a counterexample we could localize at a minimal prime of I of height n and then $x_1,..., x_n$ would map to a s.o.p. for the local ring obtained.

The equation above can be multigraded over $\mathbb{Z}^n$ by giving x_i multidegree $e_i = (0, 0,..., 0, 1, 0,..., 0)$ (the "1" is in the ith spot) and y_i multidegree $(t, t,..., t, -1, t,..., t) = \sum_{j=1}^{n} te_j - (t+1)e_i$. We shall use this multigrading in studying certain local cohomology modules below.

(6.8) *Remark.* Let $V = \mathbb{Z}_{(p)}$, where p is prime, let t be a positive integer, and let

$$R = V[X_1,..., X_n, Y_1,..., Y_n, Z_1,..., Z_r]_{\mathfrak{m}}/(F_t),$$

where $\mathfrak{m} = (p, X_1,..., X_n, Y_1,..., Y_n, Z_1,..., Z_n)$ and

$$F_t = X_1^t \cdots X_n^t - \sum_{i=1}^{n} Y_i X_i^{t+1}.$$

If $\mathscr{L}$ is a locally free coherent sheaf on the punctured spectrum W of R and M is an R-module which represents $\mathscr{L}$, we can define an integer

$$\theta(\mathscr{L}) = l(\operatorname{Tor}_{2j}^R(M, R((X))) - l(\operatorname{Tor}_{2j-1}^R(M, R((X)))$$

for j large: the Tor's are independent of the choice of M and have finite length. θ is an additive map from locally free coherent sheaves on W to $\mathbb{Z}$ and vanishes on the trivial sheaf: see [16]. The author conjectures that θ is

identically zero. If this were true, the direct summand conjecture would follow. The reader is referred to [16] for details.

It may be possible to reduce the problem to showing that θ vanishes on multigraded bundles in the case where $r=0$ (so that there are no Z's). However, there remain some technical problems in carrying this program through.

We conclude this section with an investigation of the local cohomology of the ring

$$R_{n,t}=\mathbb{Z}[X_1,\ldots,X_n,Y_1,\ldots,Y_n]/(F_{n,t}),$$

where $F_{n,t}=(X_1\cdots X_n)^t-\sum_{i=1}^n Y_iX_i^{t+1}$. Let x_i, y_i, respectively, denote the images of X_i, Y_i in the quotient and multigrade the ring $R_{n,t}$ by $\mathbb{Z}^n$ so that x_i, y_i have the same multidegrees described in Remark (6.7). Let

$$H=H_{n,t}=H^n_{(x_1,\ldots,x_n)}(R_{n,t}).$$

Then H is also multigraded by $\mathbb{Z}^n$. If $c=(c_1,\ldots,c_n)\in\mathbb{Z}^n$ let H_c (or $H_{R,t,c}$) denote the component of H in multidegree c. Our objective is to compute the Abelian groups $H_{n,t,c}$. The reason for our interest is that if all of these groups vanish, then the direct summand conjecture in mixed characteristic follows. Somewhat weaker statements would be enough. We shall show directly from our calculations that $H_{n,t,c}=0$ if $n\leqslant 2$, which gives another proof of the direct summand conjecture in dimension $\leqslant 2$. The case where $n=3$ looks difficult to settle.

Let $U_1,\ldots,U_n$ be new indeterminates. View the polynomial ring $\mathbb{Z}[U]=\mathbb{Z}[U_1,\ldots,U_n]$ as an Abelian group. Let I_n be the ideal in $\mathbb{Z}[U]$ generated by $\sum_{i=1}^n U_i-1$, but thought of as a subgroup. Let $G_{n,t,c}$ denote the subgroup of $\mathbb{Z}[U]$ spanned by all monomials $U_1^{a_1}\cdots U_n^{a_n}$ such that for some choice of i, $1\leqslant i\leqslant n$,

$$a_i\geqslant t\left(\sum_{j\neq i}a_j\right)-c_i.$$

We refer to the monomials in $G_{n,t,c}$ as "(t,c)-unbalanced."

(6.9) THEOREM. $H_{n,t,c}\cong\mathbb{Z}[U_1,\ldots,U_n]/(I_n+G_{n,t,c})$.

Proof. $H^n_X(R)$ may be identified with

$$\operatorname{Coker}\left(\oplus_{i=1}^n R_{w_i}\to R_{x_1\cdots x_n}\right),$$

where $w_i=x_1\cdots\hat{x}_i\cdots x_n$ and the map

$$R_{w_i}\to R_{x_1\cdots x_n}$$

is, up to sign, the inclusion map. Thus,

$$H^n_{(X)}(R) \cong R_{x_1 \cdots x_n} \Big/ \sum_{i=1}^{n} \operatorname{Im}(R_{w_i}),$$

where both the modules in the quotient on the right-hand side are multigraded in the obvious way. Let $x = x_1 \cdots x_n$, and $X = X_1 \cdots X_n$. Let $W_i = X_1 \cdots \hat{X}_i \cdots X_n$, $U_i = Y_i X_i / W_i^t$ and $u_i = y_i x_i / w_i^t$. U_i and u_i have multidegree $(0,\ldots,0)$ and in $\mathbb{Z}[X, Y]_X$, up to a unit, $(X_1 \cdots X_n)^t - \sum_i Y_i X_i^{t+1}$ is $1 - \sum_i U_i$. Now

$$\mathbb{Z}[X, Y]_X \cong \mathbb{Z}[X, U]_X,$$

where the X's and U's are indeterminates and so we have an obvious isomorphism

$$\begin{aligned} R_x &\cong \left(\mathbb{Z}[U] \Big/ \left(1 - \sum_i U_i\right)\right)[X]_X \\ &= (\mathbb{Z}[U]/I_n)[X]_X \\ &\cong \bigoplus_{c \in \mathbb{Z}^n} \mathbb{Z}[U]/I_n \end{aligned}$$

as an Abelian group, and this identifies

$$[R_x]_c \cong \mathbb{Z}[U]/I_n$$

as Abelian groups. In fact, $[R_x]_c = (\mathbb{Z}[U]/I_n)\, x_1^{c_1} \cdots x_n^{c_n}$. To complete the argument it will suffice to show that the image of $[R_{w_i}]_c$ is $[R_x]_c$, after identifying $[R_x]_c$ with $\mathbb{Z}[U]/I_n$, is spanned by the monomials $U_1^{a_1} \cdots U_n^{a_n}$ such that

$$a_i \geqslant t\left(\sum_{j \neq i} a_j\right) - c_i.$$

But $[R_{w_i}]_c$ is spanned by all monomials

$$x_1^{b_1} \cdots x_n^{b_n} y_1^{a_1} \cdots y_n^{a_n}$$

such that b_i and all the a_j, $1 \leqslant j \leqslant n$, are nonnegative and for each j, $1 \leqslant j \leqslant n$,

$$b_j + t \sum_{j \neq i} a_j - a_i = c_i.$$

For given $a_1,\dots,a_n$ there will be at most one choice for $b_1,\dots,b_n$ such that these equations hold, and there will exist such a choice if and only if the value of b_i forced by the equation with $j=i$ is nonnegative, i.e., if and only if

$$c_i - \left(t\sum_{j\neq i} a_j - a_i\right) \geqslant 0 \qquad \text{or} \qquad a_i \geqslant t\left(\sum_{j\neq i} a_j\right) - c_i,$$

and if this is true then $x^{b_1}\cdots x_n^{b_n}y_1^{a_1}\cdots y_n^{a_n}$ maps to $U_1^{a_1}\cdots U_n^{a_n}$. Q.E.D.

(6.10) COROLLARY. *$H_{n,t,c}$ is a divisible Abelian group for all n, t, c.*

Proof. It suffices to show that each of the generators $u_1^{a_1}\cdots u_n^{a_n}$ is a multiple of each prime p. Fix p and choose e so large that for every i,

$$U_i^{p^e}(U_1^{a_1}\cdots U_n^{a_n}) \in G_{n,t,c}$$

(i.e., such that for all i, $p^e + a_i \geqslant t(\sum_{j\neq i} a_j) - c_i$). Now

$$\left(\left(\sum_{i=1}^{n} U_i\right) - 1\right)^{p^e} U_1^{a_1}\cdots U_n^{a_n} \in I_n$$

and can be rewritten

$$\sum_{i=1}^{n} U_i^{p^e}U_1^{a_1}\cdots U_n^{a_n} + (-1)^{p^e}U_1^{a_1}\cdots U_n^{a_1} + pW$$

for a certain W, because all the binomial coefficients in the remaining terms are divisible by p. But then $U_1^{a_1}\cdots U_n^{a_n} \equiv \pm pW$ modulo $I_n + G_{n,t,c}$, as required. Q.E.D.

(6.11) COROLLARY. *$H_{n,t,c} = 0$ for $n \leqslant 2$.*

Proof. For $n=1$ we have that $U_1^a - 1 \in I_1 = (U_1 - 1)$ for all a, and $a \geqslant t(0) - c_1$ for all sufficiently large a, so that $1 \equiv U_1 \equiv U_1^2 \equiv \cdots \equiv U_1^a \equiv 0$ for all sufficiently large a.

For $n=2$ we note that modulo $I_2 = (U_1 + U_2 - 1)$ every element is in the span of the powers of U_1. Now $U_1^{a_1}U_2^0$ will lie in $G_{n,t,c}$ for all sufficiently large a_1, which shows that $H_{2,t,c}$ is finitely generated as an Abelian group. Since it is also divisible, it must be zero. Q.E.D.

Partially order $\mathbb{Z}^n$ so that $(c_1,\dots,c_n) \leqslant (d_1,\dots,d_n)$ precisely if $c_i \leqslant d_i$, $1 \leqslant i \leqslant n$. Then

(6.12) PROPOSITION. *If $n \geqslant m$, $t \geqslant s$, and $d \geqslant c$ there is a surjection*

$$H_{n,t,c} \twoheadrightarrow H_{m,s,d}.$$

Proof. If $m = n$ this is clear because $G_{n,t,c} \subset G_{n,s,d}$. It suffices then to prove that there is a surjection $H_{n,t,c} \twoheadrightarrow H_{m,t,c}$ for $n \geqslant m$. This follows from the fact that ring homomorphism

$$\mathbb{Z}[U_1,..., U_n] \twoheadrightarrow \mathbb{Z}[U_1,..., U_m]$$

which fixes $\mathbb{Z}[U_1,..., U_m]$ and kills U_j, $j > m$, maps I_n to I_m and each monomial generator of $G_{n,t,c}$ either to 0 or else to a monomial generator of $G_{m,t,c}$, according as the generator does or does not contain a positive power of U_j for some $j > m$. Q.E.D.

Define $H_n(t)$, $t \geqslant 0$, to be $H_{n,t,c(t)}$, where $c(t) = (-t, -t,..., -t)$. Thus, if $G_n(t)$ is the span of all the monomials

$$U_1^{a_1} \cdots U_n^{a_n}$$

such that for some i, $1 \leqslant i \leqslant n$,

$$a_i \geqslant t\left(1 + \sum_{j \neq i} a_j\right),$$

then $H_n(t) = \mathbb{Z}[U_1,..., U_n]/(I_n + G_n(t))$. Obviously, for any s, c we can choose a positive integer t such that $t \geqslant s$ and $c(t) \leqslant c$ and so Proposition (6.12) yields

(6.13) PROPOSITION. *For every positive integer n and for all choices of $s \geqslant 0$ and $c \in \mathbb{Z}^n$, for all sufficiently large $t \geqslant 0$, $H_{n,s,c}$ is a homomorphic image of $H_n(t)$.*

We now want to make the connection between the study of the Abelian groups $H_{n,t,c}$ and the direct summand conjecture. First recall that

$$R_{n,t} = \mathbb{Z}[X_1,..., X_n, Y_1,..., Y_n]/(F_{n,t}),$$

where $F_{n,t} = (X_1 \cdots X_n)^t - \sum_{i=1}^n Y_i X_i^{t+1}$ and so for any commutative ring D, $D \otimes R_{n,t} \cong D[X_1,..., X_n, Y_1,..., Y_n]/(F_{n,t})$. (All tensor products $\otimes$ with no base ring specified are to be taken over $\mathbb{Z}$.) We then have

(6.14) PROPOSITION. *Let D be a commutative ring. For a fixed integer $n \geqslant 1$ the following conditions are equivalent:*

(1) $H^n_{(X_1,\ldots,X_n)}(D \otimes R_{n,t}) = 0$ *for all t.*

(2) $D \otimes H_{n,t,c} = 0$ *for all t and c.*

(3) $D \otimes H_n(t) = 0$ *for all t.*

(4) *The image of the monomial* 1 *in* $D \otimes H_n(t)$ *is* 0 *for all t.*

If these equivalent conditions hold for a given value of n, then the direct summand conjecture is true for all module-finite extensions of a regular ring R such that R is a D-algebra and $\dim R \leqslant n$.

Proof. That (1) and (2) are equivalent is immediate from the right exactness of $H^n_{(x)}$ and the calculation for $H^n_{(x)}(R_{n,t})$ carried out earlier. Condition (2) $\Leftrightarrow$ (3) is clear because first, the $H_n(t)$ are a subset of $H_{n,t,c}$ second, every $H_{n,t,c}$ is a homomorphis image of some $H_n(t')$, and third, $D \otimes$ is right exact.

(3) $\Rightarrow$ (4) This is obvious. To see that (4) $\Rightarrow$ (3) fix n, t, and a monomial $\mu = U_1^{a_1} \cdots U_n^{a_n}$ whose image in $D \otimes H_n(t)$ we want to prove is 0. Choose $t' \geqslant t(1 + \sum_j a_j)$. Since the image of 1 is 0 in $D \otimes H_n(t')$ we can write

$$1 = \sum_\lambda d_\lambda \lambda + (P)\left(\sum U_i - 1\right),$$

where λ runs through a finite set of monomials in $G_n(t')$, every $d_\lambda \in D$, and $P \in D[U_1, \ldots, U_n]$. Multiply the equation through by μ. The key point is that each $\lambda\mu$ is in $G_n(t)$ (which demonstrates that μ maps to 0 in $D \otimes H_n(t)$). For suppose $\lambda = U_1^{b_1} \cdots U_n^{b_n}$. We can choose i such that

$$\begin{aligned} b_i &\geqslant t'\left(1 + \sum_{j \neq i} b_j\right) \\ &\geqslant \left(t \sum_j a_j\right)\left(1 + \sum_{j \neq i} b_i\right) \\ &\geqslant t + t\left(\sum_{j \neq i} b_j\right) + t \sum_j a_j \\ &\geqslant t\left(1 + \sum_{j \neq i} (a_j + b_j)\right) \\ \Rightarrow a_i + b_i &\geqslant t\left(1 + \sum_{j \neq i} (a_j + b_j)\right), \end{aligned}$$

as required.

Now suppose that the equivalent conditions hold but that, we have a counterexample $R \hookrightarrow S$ to the direct summand conjecture, where R is regular, S is module-finite over R and R is a D-algebra. As in the proof of Theorem (6.1) we can assume (a decrease in n is harmless) that R is local with regular system of parameters $x_1, \ldots, x_n$ and the failure of the direct summand conjecture then means that for some integer $t \geqslant 0$ there exist $y_1, \ldots, y_n \in S$ such that

$$(x_1 \cdots x_n)^t = \sum_{i=1}^n y_i x_i^{t+1}. \qquad (\neq)$$

By localizing at a maximal ideal of S we obtain a local ring (which we still denote S) of dimension n with system of parameters $x_1,\ldots,x_n$ in which $(\neq)$ holds. We can now define a ring homomorphism from $D \otimes R_{n,t}$ to S which extends the algebra structure map $D \to S$ by mapping $X_i \mapsto x_i$, $Y_i \mapsto y_i$, $W \leqslant i \leqslant n$. Let $\mathfrak{m}$ be the maximal ideal of S: $(x) = (x_1,\ldots,x_n)S$ is $\mathfrak{m}$-primary. We know $H^n_{\mathfrak{m}}(S) \neq 0$, but on the other hand

$$H^n_{\mathfrak{m}}(S) \cong H^n_{(S)}(S) \cong H^n_{(X)}(S) \cong H^n_{(X)}(D \otimes R_{n,t}) \otimes S \cong 0 \otimes S \cong 0,$$

a contradiction. Q.E.D.

A number of comments need to be made here. We note that we can give an immediate proof of the direct summand conjecture in characteristic $p > 0$ from the fact that the groups $H_{n,t,c}$ are divisible, for then

$$(\mathbb{Z}/p\mathbb{Z}) \otimes H_{n,t,c} = 0$$

and we may apply Proposition (6.14).

Likewise, to establish the direct summand conjecture in mixed characteristic p, it would suffice to show that

$$\mathbb{Z}_{(p)} \otimes H_{n,t,c} = 0$$

for all n, t, c. We shall establish a finer result below.

The proof shows that for fixed n, t, the vanishing of $H^n_{(X)}(D \otimes R_{n,t})$ is equivalent to the vanishing of $D \otimes H_{n,t,c}$ for all c, and that these equivalent conditions imply that the equation

$$x_1^t \cdots x_n^t = \sum_{i=1}^{n} y_i x_i^{t+1}$$

cannot hold in a Noetherian D-algebra if $x_1,\ldots,x_n$ generate a proper ideal of height n.

We do not know the converse. The direct summand conjecture may be true even if the groups $H_{n,t,c}$ fail to vanish. It is worth remarking that we can recover the $R_{n,t}$-module structure on

$$H = H^n_{(X)}(R_{n,t}) = \bigoplus_{c \in \mathbb{Z}^n} H_{n,t,c}$$

easily when it is viewed as $\bigoplus_{c \in \mathbb{Z}^n} H_{n,t,c}$. Consider the image of a monomial $U_1^{a_1} \cdots U_n^{a_n}$ in $H_{n,t,c}$. The result of multiplying it by $x^{d_1} \cdots x_n^{d_n}$ is the image of that same monomial in $H_{n,t,c+d}$, where $d = (d_1,\ldots,d_n)$, while the result of multiplying it by $y_1^{b_1} \cdots y_n^{b_n}$ is the image of

$$U_1^{a_1+b_1} \cdots U_n^{a_n+b_n}$$

in $H_{n,t,c+f}$, where f is the multidegree of $y_1^{b_1} \cdots y_n^{b_n}$, i.e., $f = (f_1, \ldots, f_n)$ with $f_i = t(\sum_{j \neq i} b_j) - b_i$.

We next remark that because it suffices to prove the "monomial" form of the direct summand conjecture (as indicated in statement (8) of Theorem (6.1) and statement (8°) in the remarks preceding the proof) in mixed characteristic p when the sequence of elements in p, $x_2, \ldots, x_n$, the preceding analysis could have been carried through for $R'_{n,t} = R_{n,t}/(X_1 - p)$ instead, and then the vanishing of

$$_pH'_{n,t} = H^n_{(p,x_2,\ldots,x_n)}(R'_{n,t})$$

for all t would be sufficient to establish the direct summand conjecture for the case of mixed characteristic p in dimension $\leqslant n$. By the right exactness of $H^n_{(X)}$, we have

$$_pH'_{n,t} = H_{n,t}/(p - X_1)\, H_{n,t}.$$

Note, however, that $p - X_1$ is *not* a multiform; so that we lose the original multigrading in studying $_pH'_{n,t}$.

Let us define the "tilt" of a monomial $U_1^{a_1} \cdots U_n^{a_n}$ in $\mathbb{Z}[U_1, \ldots, U_n]$ to be

$$\max_j \left\{ a_i \Big/ \left(1 + \sum_{j \neq i} a_j\right) : 1 \leqslant i \leqslant n \right\}$$

and then extend the notion to polynomials by defining the "tilt" of a polynomial to be the smallest tilt of any monomial which occurs in it with nonzero coefficient. Thus, P has tilt $\geqslant t$ if and only if it is in $G_n(t)$. A monomial has tilt $\geqslant t$ if and only if it is $(t, c(t))$-unbalanced.

In this terminology we can rephrase part of the conclusion of Proposition (6.14) as follows: $H^n_{(X)}(D \otimes R_{n,t}) = 0$ for all t if and only if in $D[U_1, \ldots, U_n]$ 1 is congruent to polynomials of arbitrarily great tilt modulo $(\sum_j U_j - 1)$.

We conclude with a result which shows that it would be enough if the torsion part of $H_{n,t}$ (or $H'_{n,t}$) vanished.

(6.15) PROPOSITION. *Let $x_1, \ldots, x_n$ be elements of a commutative ring R, let $(x) = (x_1, \ldots, x_n)R$, let p be a prime integer, and suppose that $H^n_{(x)}(R)$ has no p-torsion and is p-divisible.*

Then there does not exist a homomorphism $\phi\colon R \to S$ such that S is a local ring of residual characteristic p and $\phi(x_1), \ldots, \phi(x_n)$ is a system of parameters.

Proof. Suppose such a ϕ exists. Then S is a V-algebra, where $V = \mathbb{Z}_{(p)}$, and we can replace R by $V \otimes R$ without loss of generality: we still have that $H^n_{(x)}(V \otimes R) = V \otimes H^n_{(x)}(R)$ has no p-torsion, is still p-divisible, and we still have a homomorphism $V \otimes R \to S$. Thus, we may assume that R is a V-algebra without loss of generality. But then since multiplication by p is an

automorphism of $H^n_{(x)}(R)$, $H^n_{(x)}(R)$ is $V[1/p]$-module, i.e., a $\mathbb{Q}$-vector space, where $\mathbb{Q}$ is the field of rational numbers, and it then follows that $H^n_{\mathfrak{m}}(S) = H^n_{(x)}(S) = H^n_{(x)}(R) \otimes_R S$ is a $\mathbb{Q}$-vector space as well, where $\mathfrak{m}$ is the maximal ideal of S. But if S has residual characteristic p, every element of $H^n(S)$ is killed by a power of p. But then $H^n_{\mathfrak{m}}(S) = 0$, a contradiction. Q.E.D.

(6.16) COROLLARY. *If $H_{n,t}$ has no p-torsion for all t, or if $H'_{n,t}$ has no p-torsion for all t, then the direct summand conjecture holds in mixed characteristic p for dimensions $\leqslant n$.*

REFERENCES

1. M. AUSLANDER, Modules over unramified regular local rings, *Illinois J. Math.* **5** (1961), 631–645.
2. M. AUSLANDER, Modules over unramified regular local rings, *in* "Proceedings, Intern. Congress of Math.," pp. 230–233, 1962.
3. H. BASS, On the ubiquity of Gorenstein rings, *Math. Z.* **82** (1963), 8–28.
4. W. BRUNS, The Eisenbud–Evans principal ideal theorem and determinantal ideals, *Proc. Amer. Math. Soc.* **83** (1981), 19–24.
5. D. EISENBUD AND E. G. EVANS, JR., A generalized principal ideal theorem, *Nagoya Math. J.* **62** (1976), 41–53.
6. E. G. EVANS, JR. AND P. GRIFFITH, The syzygy problem, *Ann. of Math.* (2) **114** (1981), 323–333.
7. J. A. EAGON AND M. HOCHSTER, R-sequences and indeterminates, *Quart. J. Math. Oxford Ser.* (2) **25** (1974), 61–71.
8. H.-B. FOXBY, Bounded complexes of flat modules, *J. Pure Appl. Algebra* **14** (1979), 149–172.
9. H.-B. FOXBY, A homological theory of complexes of modules, Københavens Univ. Mathematisk Inst. Preprint Series No. 19b, September, 1981.
10. P. GRIFFITH, A representation theorem for complete local rings, *J. Pure Appl. Algebra* **7** (1976), 303–315.
11. A. GROTHENDIECK, (notes by R. Hartshorne), "Local Cohomology," Lecture Notes in Math., Vol. 41, Springer–Verlag, Berlin/Heidelberg/New York, 1967.
12. M. HOCHSTER, Cohen–Macaulay modules, *in* "Proceedings, Kansas Commutative Algebra Conference," Lecture Notes in Math., Vol. 311, pp. 120–152, Springer–Verlag, Berlin/Heidelberg/New York, 1973.
13. M. HOCHSTER, Contracted ideals from integral extensions of regular rings, *Nagoya Math. J.* **51** (1973), 25–43.
14. M. HOCHSTER, Topics in the homological theory of modules over commutative rings, C.B.M.S. Regional Conference Series in Math., No. 24, Amer. Math. Soc., Providence, R.I., 1975.
15. M. HOCHSTER, Big Cohen–Macaulay modules and algebras and embeddability in rings of Witt vectors, *Queen's Papers Pure Appl. Math.* **42** (1975), 106–195.
16. M. HOCHSTER, The dimension of an intersection in an ambient hypersurface, in "Proceedings of the First Midwest Algebraic Geometry Seminar," Springer–Verlag Lecture Notes in Math., Vol. 862, pp. 93–106, Berlin/Heidelberg/New York, 1981.
17. M. HOCHSTER, Associated graded rings derived from integrally closed ideals and the local homological conjectures, *in* "Colloque d'Algèbre, Université de Rennes I," pp. 1–27, 28 à 31 Mai 1980.

18. M. HOCHSTER, The local homological conjectures, *in* "Commutative Algebra: Durham 1981," Cambridge Univ. Press, London Math. Soc. Lecture Note Series 72, 1982, 32–54.
19. M. HOCHSTER, AND J. MCLAUGHLIN, Quadratic extensions of regular local rings, *Illinois J. Math.* **27** (1983), 94–103.
20. M. HOCHSTER AND J. L. ROBERTS, Rings of invariants of reductive groups acting on regular rings are Cohen–Macaulay, *Advan. in Math.* **13** (1974), 115–175.
21. M. HOCHSTER AND J. L. ROBERTS, The purity of the Frobenius and local cohomology, *Advan. in Math.* **21** (1976), 117–172.
22. B. IVERSEN, Amplitude inequalities for complexes, *Ann. Sci. École Norm. Sup.* (4) **10** (1977), 547–558.
23. I. KAPLANSKY, Projective modules, *Ann. of Math.* (2) **68** (1958), 372–377.
24. J. H. KOH, The direct summand conjective and behavior of codimension in graded extensions, Thesis, Univ. of Michigan, Ann Arbor, 1983.
25. C. PESKINE AND L. SZPIRO, Dimension projective finie et cohomologie locale, *Inst. Hautes Études Sci. Publ. Math.* **42** (1973), 323–395.
26. C. PESKINE, Syzygies et multiplicités, *C. R. Acad. Sci. Paris Sér. A–B* **278** (1974), 1421–1424.
27. P. ROBERTS, Two applications of dualizing complexes over local rings, *Ann. Sci. École Norm. Sup.* (4) **9** (1976), 103–106.
28. P. ROBERTS, Cohen–Macaulay complexes and an analytic proof of the new intersection conjecture, *J. Algebra* **66** (1980), 220–225.