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A Note on the Fermat Equation

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Fix one of the base variables x_1, x_2, x_3 in the equation $x_1^{x_4} + x_2^{x_4} = x_3^{x_4}$ subject to $x_1, x_2, x_3, x_4 \in \mathbb{Z}$, $x_1x_2x_3 \neq 0$, $x_4 \geqslant 3$, $\gcd(x_1, x_2, x_3) = 1$. Then the solutions are bounded. There are no solutions (x_1, x_2, x_3, x_4) with $P(x_i) < \varepsilon_0(\log\log|x_i|)^{1/3}$ for some $i \in \{1, 2, 3\}$.

1. INTRODUCTION

It was stated by Fermat, without proof, that the equation

$$x_1^{x_4} + x_2^{x_4} = x_3^{x_4} \tag{1}$$

has no solution in integers with $x_4 \geqslant 3$ and $x_1x_2x_3 \neq 0$. In order to investigate the possibility of finitely many solutions it is customary to restrict the variables to

$$x_1, x_2, x_3, x_4 \in \mathbb{Z},\quad x_1x_2x_3 \neq 0,\quad x_4 \geqslant 3,\quad \gcd(x_1, x_2, x_3) = 1. \tag{2}$$

One method of looking for integral points (x_1, x_2, x_3, x_4) in four-space satisfying (1) and (2) is to search by (three–dimensional hyper–) planes covering all integral points. In this respect we note the following well-known facts [4]: The plane $x_4 = k$ contains no solutions (that is, points satisfying (1) and (2)) if k enjoys one of various properties. It would follow immediately from the truth of a famous conjecture of Mordell on rational points on curves of genus $\geqslant 2$ that, for every k ($\geqslant 4$), the plane $x_4 = k$ contains only finitely many solutions. Stewart [5] (see also [3]) proved the following: for every $k \in \mathbb{Z}$ with $|k| > 2$ and for $i, j \in \{1, 2, 3\}$ with $i \neq j$, all solutions in the plane $x_i - x_j = k$ have their coordinates bounded (by an effectively computable number depending on k). The cases $|k| = 1, 2$ are still open. In this note we consider the planes $x_i = k$, where $k \in \mathbb{Z}$ and $i \in \{1, 2, 3\}$. Since there exist no solutions in $x_4 = k$ for $3 \leqslant k \leqslant 125.000$ [4]

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we also have, by Lemma 1, that $x_i = k$ $(i = 1, 2, 3)$ contains no solutions if $P(k) < 125.000$. Here $P(k)$ denotes the greatest prime divisor of k if $|k| > 1$ and $P(\pm 1) = 1$. We prove that for every $k \in \mathbb{Z}$ and $i \in \{1, 2, 3\}$ all solutions in $x_i = k$ have their coordinates bounded (by an e.c. number depending on k). Moreover, if $P(k) < \varepsilon_0(\log \log |k|)^{1/3}$, then $x_i = k$ contains no solutions $(i = 1, 2, 3)$.

2. Proofs

LEMMA 1 (Zsigmondy, Birkhoff and Vandiver). *Let $n, x, y \in \mathbb{Z}$ with $n \geqslant 2$ and $x > y > 0$. Then*

$$P(x^n - y^n) > n \qquad \text{except if} \quad (x, y, n) = (3, 1, 2), \tag{3}$$

$$P(x^n + y^n) > 2n \qquad \text{except if} \quad (x, y, n) = (2, 1, 3). \tag{4}$$

Proof. Simple consequences of their (elementary) results on primitive divisors of numbers of the form $a^n - b^n$. See [2]. ∎

LEMMA 2. *Let $n, x, y \in \mathbb{Z}$ with $n \geqslant 3$ and* $\gcd(x, y) = 1$. *Put $P := P(x^n \pm y^n)$. There exists an* e.c. *constant C such that*

$$\max\{|x|, |y|\} < \exp\exp(Cn^2 P). \tag{5}$$

Proof. Let $F \in \mathbb{Z}[X, Y]$ be homogeneous such that $F(X, 1)$ has at least three simple zeros. It is well known (see [1, p. 63, Theorem 1]) that there exists an e.c. number $c(F)$ depending only on F such that for every $x, y \in \mathbb{Z}$ with $\gcd(x, y) = 1$ one has $\max\{|x|, |y|\} < \exp\exp(c(F) \cdot P)$, where $P := P(F(x, y))$. In the special cases where $F = X^n \pm Y^n$, $n \geqslant 3$, one can prove that this inequality holds with $c(F) = C \cdot n^2$ for some e.c. constant C. See [6] for details. ∎

Remark. If instead of (5) one uses that $\max\{|x|, |y|\} < \exp\exp(C(n) \cdot P)$ for some unspecified function $C(n)$ (tending to infinity with n), then one obtains in the sequel bounds which tend to infinity in some unspecified manner.

THEOREM. *Suppose*

$$x_1^{x_4} + x_2^{x_4} = x_3^{x_4} \tag{1}$$

and

$$x_1, x_2, x_3, x_4 \in \mathbb{Z}, \quad x_1 x_2 x_3 \neq 0, \quad x_4 \geqslant 3, \quad \gcd(x_1, x_2, x_3) = 1. \tag{2}$$

There exists an effectively computable constant C such that

$$x_4 < P(x_j) \qquad \text{for} \quad j \in \{1, 2, 3\}, \tag{6}$$

$$|x_i^{x_4}| < \exp\exp(C(P(x_j))^3)) \qquad \text{for} \quad i, j \in \{1, 2, 3\}. \tag{7}$$

Proof. Let (i, j, k) be some permutation of $(1, 2, 3)$. We may assume that x_i, x_j, x_k are positive and distinct with $\pm x_j^{x_4} = x_i^{x_4} \pm x_k^{x_4}$. Since $x_4 > 3$ it follows from Lemma 1 that $P(x_j) = P(x_i^{x_4} \pm x_k^{x_4}) > x_4$, which gives (6). From Lemma 2 we obtain, using $x_4 < P(x_j)$, that $\max\{|x_i^{x_4}|, |x_k^{x_4}|\} < \exp\exp \{Cx_4^2 P(x_j) + \log x_4\} < \exp\exp\{(C+1)(P(x_j))^3\}$. Since $|x_j^{x_4}| \leqslant |x_i^{x_4}| + |x_k^{x_4}|$ this gives (7). ∎

COROLLARY. *Suppose*

$$x_1^{x_4} + x_2^{x_4} = x_3^{x_4}, \tag{1}$$

$$x_1, x_2, x_3, x_4 \in \mathbb{Z}, \;\; x_1 x_2 x_3 \neq 0, \;\; x_4 \geqslant 3, \; \gcd(x_1, x_2, x_3) = 1 \tag{2}$$

and moreover that

$$x_i = k$$

for some $i \in \{1, 2, 3\}$ *and* $k \in \mathbb{Z}$. *Then*

$$x_4 < P(k), |x_j^{x_4}| < \exp\exp(C(P(k))^3)$$
$$\text{for} \quad j = 1, 2, 3 \text{ and } P(k) > C^{-1/3} (\log\log |k|)^{1/3}.$$

Remarks. Our results can be extended to equations $a_1 x_1^{x_4} + a_2 x_2^{x_4} = a_3 x_3^{x_4}$, where a_1, a_2, a_3 are any given nonzero integers. See [6]. This does not seem to be the case for the results mentioned in the introduction about the planes $x_i - x_j = k$. We are unable to prove anything for solutions of (1) and (2) in an arbitrary plane $b_1 x_1 + b_2 x_2 + b_3 x_3 = k$ $(b_1, b_2, b_3, k$ given integers).

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