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Cohen–Macaulay Rees Algebras and Their Specialization

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INTRODUCTION

The *Rees algebra* of an ideal J in a commutative ring R is by definition the graded algebra.

$$\mathscr{R}(J, S) := S \oplus J \oplus J^2 \oplus \cdots = S[Jt] \subset S[t],$$

where t is an indeterminate.

In this paper we are concerned with proving that certain Rees algebras are Cohen–Macaulay, and with answering, under strong hypotheses, the question: if $R = S/N$ is a factor-ring of S and $I = JR$, when is $\mathscr{R}(I, R)$ a specialization of $\mathscr{R}(J, S)$; that is, when is the natural epimorphism

$$R \otimes \mathscr{R}(J, S) \to \mathscr{R}(I, R)$$

an isomorphism?

These questions have interest partly because if S and $\mathscr{R}(J, S)$ are Cohen–Macaulay, then so is $\mathrm{gr}_J\, S := S/J \oplus J/J^2 \ldots$ [16] and under these hypotheses if N is perfect and $R \otimes \mathscr{R}(J, S) = \mathscr{R}(I, R)$, then R and $\mathscr{R}(I, R)$ are Cohen–Macaulay too. Thus, $\mathrm{gr}_I R$ is Cohen–Macaulay, and its torsion freeness and normality, for exmple, can be characterized in terms of analytic spreads, as in [16]; see Section 3 for details.

We deal with the specialization question in Section 1. It is answered for the case where, as above, S and $\mathscr{R}(J, S)$ are Cohen–Macaulay and N is

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perfect (Theorem 1.1) in terms of the analytic spreads of J localized at certain primes.

We then turn to a consideration of examples. In Section 2 we prove that $\mathscr{R}(J, S)$ is Cohen–Macaulay in some "generic" cases, where J is the ideal generated by the $r \times r$ minors of an $r \times s$ generic matrix, by the $2n \times 2n$ pfaffians of a $(2n+1) \times (2n+1)$ alternating matrix, and where J is a certain 4 generator ideal of height 3 which has some interesting specializations. The first two of these three cases are handled as applications of a more general result to the effect that if S is an algebra with straightening law in the sense of [8] (or see [10]; the definition is recalled below in Section 2) then, for certain ideals J, $\mathscr{R}(J, S)$ will also satisfy a straightening law.

Some special cases of the above results have been discovered previously. Notably, the case of the $r \times r$ minors of an $r \times (r+1)$ matrix has been treated by several authors [12, 14], and the specialization question was studied in this case in [15] and later generalized in [1, 22].

In Section 3 we give a number of examples where the specialization process of Section 1 can be applied to the generic examples of Section 2. Some of these examples have treated elsewhere by ad hoc methods; references will be found in Section 3.

One easy but pleasant subsidiary result we obtain (Proposition 1.3) is that if S is a ring and J an ideal, then the set of primes P for which the analytic spread of J_p is $\geqslant$ a given number is closed in Spec S—even without the hypothesis that the residue class fields of S are infinite.

We remark that there has been other recent work in the direction of showing that rings of the form $\mathrm{gr}_I R$ are Cohen–Macaulay, or domains; see for example [8, 17]. Vasconcelos and Simis [21, 22] have recently related this question to the Cohen–Macaulayness of the Koszul homology of I. Herzog has obtained new results in the classical case where R is local and I is the maximal ideal [13]. Brodman has studied the local cohomology of $\mathscr{R}(I, R)$ for certain primary ideals I in non-Cohen–Macaulay rings R [3].

Throughout this paper, all rings will be assumed commutative, unitary, noetherian, and universally catenary. This last condition, which is satisfied, of course, by any ring remotely resembling an affine ring, has the following immediate consequence, which we will use several times. If (S, m) is a local ring (in our sense) and A is a finitely generated S-algebra, $Q \subseteq M$ are two prime ideals of A such that $M \cap S = m$ and $Q \cap S = P$, then

$$\dim(A/Q)_M = \dim S/P + \dim S_P \otimes (A/Q)_M.$$

For the rest, we use standard terminology as found, for example, in the book of Matsumura [18].

1. Specialization of Cohen–Macaulay Rees Algebras

Throughout this section we let S be a Cohen–Macauley ring and let $R = S/N$, where N is a perfect ideal of S, that is, the projective dimension of R as an S-module is equal to the height of N—so that in particular R is also Cohen–Macauley. For example, N might be generated by a regular sequence in S, as in the examples of Section 3.

Suppose that J is an ideal of S of which we know that $\mathscr{R}(J, S)$ is Cohen–Macauley. Writing $I = JR$, we wish to know when $\mathscr{R}(I, R) \cong R \otimes \mathscr{R}(J, S)$ and when $\mathscr{R}(I, R)$ is Cohen–Macaulay.

Recall that if P is a prime of S, then the *analytic spread* of J_P in S_P, written $l(J_P)$, is defined [19] by

$$l(J_P) = \dim S_P/PS_P \otimes \mathscr{R}(J, S).$$

It is proved by Northcott and Rees that if $J \subset P$, then

$$\mathrm{ht}(J) \leqslant l(J_P) \leqslant \mathrm{ht}\, P.$$

Our main result is:

THEOREM 1.1. *Let S be a Cohen–Macaulay ring, N a perfect ideal of S, and J any ideal of S. Suppose that $\mathscr{R}(J, S)$ is Cohen–Macaulay. Set $R = S/N$ and $I = JR$. The following conditions are equivalent*:

(a) *$\mathscr{R}(I, R) \cong R \otimes \mathscr{R}(J, S)$ by the natural map.*

(b) *For every prime $P \subset S$ which is the contraction to S of a minimal prime of $\mathrm{gr}_J S/N \,\mathrm{gr}_J S$, we have $l(J_P) \leqslant \mathrm{ht}\, P/N$.*

If these conditions are satisfied, then $\mathscr{R}(I, R)$ is Cohen–Macaulay.

Remarks. (1) We will see in the proof that condition (a) implies condition (b) for *any* prime P of S containing $J + N$.

(2) It seems reasonable to hope that the conclusion $R \otimes \mathscr{R}(J, S) \cong \mathscr{R}(I, R)$ would remain valid even if the Cohen–Macaulay hypotheses are dropped, provided one assumes something like depth $P/N \geqslant l(J_P)$ for primes $P \supset J + N$.

(3) It is easy to show from the theorem that, under the given hypotheses, the primes P of S which are contractions of minimal primes in $\mathrm{gr}_J S/N \,\mathrm{gr}_J S$ are precisely those for which $l(J_P) = \mathrm{ht}\, P/N$; it would be nice to have a general result describing these primes.

(4) We will see in the proof that the conditions of the theorem imply

$\operatorname{ht} I = \operatorname{ht} J$, so $\operatorname{ht}(J+N) = \operatorname{ht} J + \operatorname{ht} N$. But much more follows; in fact $\ker(R \otimes \mathscr{R}(J,S) \to \mathscr{R}(I,R))$ is (without any hypotheses) isomorphic to

$$\sum_{k \geqslant 0} J^k \cap N/J^kN = \sum_{k \geqslant 0} \operatorname{Tor}_1(R/J^k, R/N).$$

Thus, at least modulo Auslander's conjecture on the rigidity of Tor, and in the case of regular rings, the conclusion of the theorem is equivalent to $\operatorname{grade}(N + J^k/J^k) = \operatorname{grade} N$ for every k.

Proof of Theorem 1.1. (a) $\Rightarrow$ (b):

$$\begin{aligned} l(J_P) &= \dim(S_P/PS_P) \otimes \mathscr{R}(J,S) \\ &= \dim(R_{PR}/PR_{PR}) \otimes \mathscr{R}(J,S) \\ &= \dim(R_{PR}/PR_{PR}) \otimes \mathscr{R}(I,R) \\ &= l(I_{PR}) \\ &\leqslant \operatorname{ht} P/N, \end{aligned}$$

the last inequality following from the inequality of Northcott and Rees mentioned above.

(a) $\Rightarrow \mathscr{R}(I,R)$ is Cohen–Macaulay: We may without loss of generality assume that S is local, say with maximal ideal M. Write $\mathscr{S} = \mathscr{R}(J,S)$. Since N is perfect, $\mathscr{S}/N\mathscr{S} = \mathscr{R}(I,R)$ will be Cohen–Macaulay as soon as grade $N\mathscr{S} \geqslant \operatorname{grade} N$. Because $\mathscr{S}$ is supposed Cohen–Macaulay, it is enough to prove $\operatorname{ht} N\mathscr{S} \geqslant \operatorname{ht} N$.

Set $\mathscr{M} = M \oplus J \oplus J^2 \oplus \cdots \subset \mathscr{S}$; it is a maximal ideal which, since $\mathscr{S}$ is graded and $N\mathscr{S}$ is homogeneous, contains all the minimal primes of $N\mathscr{S}$. Thus, it is enough to show that $\operatorname{ht}(N\mathscr{S}_{\mathscr{M}}) \geqslant \operatorname{ht} N$. Of course, since $\mathscr{S}_{\mathscr{M}}$ is local and Cohen–Macaulay, it is equidimensional, so $\operatorname{ht} N\mathscr{S}_{\mathscr{M}} = \dim \mathscr{S}_{\mathscr{M}} - \dim \mathscr{S}_{\mathscr{M}}/N\mathscr{S}\mathscr{M} = \dim \mathscr{S}_{\mathscr{M}} - \dim \mathscr{R}(I,R)_{\mathscr{M}}$. Thus, the following well-known lemma will complete the proof of this implication:

LEMMA 1.2. *Let (T, N) be a local ring, and let $K \subset N$ be any ideal. If $\mathscr{N} = N \oplus K \oplus \cdots \subset R(K,T)$, then*

$\dim \mathscr{R}(K,T)_{\mathscr{N}} = \dim \mathscr{R}(K,T) = \max(\dim T, \{1 + (\dim T/Q \mid Q$ *is a prime ideal of* K, *and* $Q \not\supseteq K\})$.

In particular, $\dim \mathscr{R}(K,T)_{\mathscr{N}} \leqslant \dim T + 1$ *with equality if* $\operatorname{ht} K \geqslant 1$.

Proof. We leave this as an easy exercise for the reader. Also see [20, Remark 3.7].

(b) $\Rightarrow$ (a): We may without loss of generality assume that S is local, with maximal ideal M, say; we set

$$\begin{aligned} \mathcal{S} &:= \mathcal{R}(J, S), \\ \mathcal{M} &:= M \oplus J \oplus J^2 \oplus \cdots, \\ \bar{\mathcal{S}} &:= \mathcal{S}_{\mathcal{M}}/N\mathcal{S}_{\mathcal{M}}. \end{aligned}$$

Since $\mathcal{S}$ is graded and $N\mathcal{S}$ is homogeneous, all the associated primes of $N\mathcal{S}$ are contained in $\mathcal{M}$, and it is enough to show that the epimorphism

$$\bar{\mathcal{S}} = R \otimes \mathcal{S}_{\mathcal{M}} \to \mathcal{R}(I, R)_{\mathcal{M}}$$

is an isomorphism. Let K be its kernel.

If we invert any element of J, and the corresponding element of I, then $\mathcal{S}$ becomes the polynomial ring $S[t]$, while $\mathcal{R}(I, R)$ becomes $R[t]$ (with the natural identification) and one sees that the natural map $\bar{\mathcal{S}} \to \mathcal{R}(I, R)_{\mathcal{M}}$ becomes an isomorphism. It follows that, for some integer k, $I^k K = 0$, and, thus,

$$\dim \bar{\mathcal{S}} = \max(\dim \bar{\mathcal{S}}/I\bar{\mathcal{S}}, \dim \mathcal{R}(I, R)_{\mathcal{M}}).$$

We first show that $\dim \bar{\mathcal{S}}/I\bar{\mathcal{S}} \leqslant \dim R$: Note that $\bar{\mathcal{S}}/I\bar{\mathcal{S}} = (\operatorname{gr}_J S/N \operatorname{gr}_J S)_{\mathcal{M}}$. Thus, there is a prime Q of $(\operatorname{gr}_J S)_{\mathcal{M}}$, minimal over $N \operatorname{gr}_J S$, such that $\dim \bar{\mathcal{S}}/I\bar{\mathcal{S}} = \dim(\operatorname{gr}_J S/QS)_{\mathcal{M}}$. Writing Q again for the preimage of Q in $\mathcal{S}$, and P for the intersection with S, we have

$$\begin{aligned} \dim \bar{\mathcal{S}}/I\bar{\mathcal{S}} &\leqslant \dim \mathcal{S}/Q \\ &= \dim S/P + \dim(S_P \otimes \mathcal{S}/Q) \\ &\leqslant \dim S/P + \dim(S_P \otimes \mathcal{S}/P) \\ &= \dim S/P + l(J_P). \end{aligned}$$

By our hypothesis $l(J_P) \leqslant \operatorname{ht} P/N$. This gives $\dim \bar{\mathcal{S}}/I\bar{\mathcal{S}} \leqslant \dim R$.

We now wish to compute $\dim \mathcal{R}(I, R)$, using Lemma 1.2; to do this we need $\operatorname{ht} I \geqslant 1$. If P/N is a prime of R containing I, then by hypothesis $\operatorname{ht} P/N \geqslant l(J_P) \geqslant \operatorname{ht} J$, so $\operatorname{ht} I \geqslant \operatorname{ht} J$, which is by hypothesis $\geqslant 1$. Thus, by Lemma 1.2 we have

$$\begin{aligned} \dim \mathcal{R}(I, R) &= \dim \mathcal{R}(I, R) = 1 + \dim R, \\ \dim \mathcal{S}_{\mathcal{M}} &= \dim \mathcal{R}(J, S) = 1 + \dim S. \end{aligned}$$

These calculations imply $\dim \overline{\mathscr{S}} = 1 + \dim R$, so using the fact that $\mathscr{S}_{\mathscr{N}}$ is Cohen–Macaulay and equidimensional,

$$\begin{aligned}\operatorname{grade} N\mathscr{S}_{\mathscr{N}} = \operatorname{ht}(N\mathscr{S}_{\mathscr{N}}) &= \dim \mathscr{S}_{\mathscr{N}} - \dim \overline{\mathscr{S}}\\ &= \dim S - \dim R.\end{aligned}$$

Since N was perfect, this shows that $N\mathscr{S}_{\mathscr{N}}$ is perfect, and thus $\overline{\mathscr{S}}$ is Cohen–Macaulay. Further, since $\dim \overline{\mathscr{S}}/I\overline{\mathscr{S}} = \dim R < \dim \overline{\mathscr{S}}$ we see that $\operatorname{ht} I\overline{\mathscr{S}} \geqslant 1$ and, thus, I contains a nonzerodivisor. Since $I^k K = 0$ this implies $K = 0$, and we are done with the proof.

The conditions of Theorem 1.1 are not necessary for the Cohen–Macaulayness of $\mathscr{R}(I, R)$, as the following simple example shows:

EXAMPLE. Let $S = k[u_{ij}]$, $1 \leqslant i \leqslant 2$, $1 \leqslant j \leqslant 3$, where for convenience k is a field. We regard the u_{ij} as the entries of a 2×2 matrix, and let J be the ideal generated by the 2×2 minors of this matrix. It is known that the symmetric algebra $\operatorname{Sym}(J)$ is isomorphic to the Rees algebra of J by the canonical map $\operatorname{Sym}(J) \to \mathscr{R}(J, S)$. Now let $R = k[u_{11}, u_{12}]$, and consider the map S onto R carrying the matrix (u_{ij}) to

$$\begin{pmatrix} u_{11} & u_{12} & 0 \\ 0 & u_{11} & u_{12} \end{pmatrix}.$$

The kernel N of this map is generated by the regular sequence u_{13}, u_{21}, $u_{11} - u_{22}$, $u_{12} - u_{23}$, and is in particular perfect. We have

$$R \otimes \mathscr{R}(J, S) = R \otimes \operatorname{Sym}(J) = \operatorname{Sym}(J/NJ).$$

Since the generators of N form a regular sequence $\operatorname{mod} J$, we have $(N \cap J)/NJ = \operatorname{Tor}_1(S/J, S/N) = 0$, and so $\operatorname{Sym}(J/NJ) \cong \operatorname{Sym}(I)$.

On the other hand, I may be generated by the elements

$$u_{11}^2, \qquad u_{11}u_{12}, \qquad u_{12}^2,$$

and these satisfy the quadratic relation $(u_{11}^2)(u_{12}^2) = (u_{11}u_{12})^2$, so $\operatorname{sym}_2 I \not\cong I^2$. Thus, $R \otimes \mathscr{R}(J, S) \not\cong \mathscr{R}(I, R)$. However, it is easy to verify that they are both Cohen–Macaulay; one can easily check that $\mathscr{R}(I, R)$ is isomorphic to the quotient of $R[T_1, T_2, T_3]$ by the ideal of 2×2 minors (a perfect ideal) of the matrix

$$\begin{pmatrix} u_{11} & T_1 & T_2 \\ u_{12} & T_2 & T_3 \end{pmatrix},$$

under the map $T_1 \mapsto u_{11}^2 t$, $T_2 \mapsto u_{11}u_{12}t$, $T_3 \mapsto u_{12}^2 t$ in $R[It]$, while

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$R \otimes \mathscr{R}(J, S) = \mathrm{Sym}(I)$ is isomorphic to $R[T_1, T_2, T_3]$ modulo the two minors of that matrix which contain the first column (a regular sequence).

We will apply Theorem 1.1 in the sequel with a reformulation of part (b). Before formulating this, we need to prove the closedness of the locus on which a given ideal has analytic spread $\geqslant$ a given number.

Let K be an ideal in a Noetherian ring T, and let $k \geqslant \mathrm{ht}(K)$ be an integer.

DEFINITION. $L_k(K) \subset T$ is the intersection of all prime ideals P of T such that $l(K_P) \geqslant k$.

PROPOSITION 1.3. *Let Q be a prime ideal of T, and let K be any ideal of T. If k is an integer $\geqslant \mathrm{ht}\, K$, then $Q \supset L_k(K)$ iff $l(K_Q) \geqslant k$.*

Equivalently, if $k \geqslant \mathrm{ht}\, K$, then $\{Q \in \mathrm{Spec}\, T \mid l(K_Q) \geqslant k\}$ is a closed subset of spec T.

For the proof we will use some facts about the analytic spread, proved by Northcott and Rees [19]. Recall that an ideal K is integral over an ideal $K' \subset K$ iff, for some $n \geqslant 1$, $K'K^n = K^{n+1}$. If P is a prime ideal, we write $\mu(K_P)$ for the minimal number of generators of K_P in T_P.

PROPOSITION 1.4. *Let (T, M) be a local ring, $K \subset T$ an ideal.*

(1) *If $K' \subset K$ and K is integral over K', then $l(K) = l(K')$.*

(2) *$l(K) \leqslant \mu(K)$; if T/M is infinite, then there exists an ideal $K' \subset K$ with K integral over K'; and $l(K) = \mu(K')$.*

(3) *If P is a prime ideal of T, then $l(K_P) \leqslant l(K)$.*

Proof of Proposition 1.3. We prove that the complementary set $U = \{Q \mid l(K_Q) < k\}$ is open. By the criteria in [18, (22.13)] it is enough to show that $P \in U$, $Q \subset P$ implies $Q \in U$, which is immediate from Proposition 1.4(3), and that if P is in U, then a nonempty set subset of $V(P) = \{Q \mid Q \supseteq P\}$ is in U.

If P is maximal, this second condition is trivial. If P is not maximal, then the residue class field of T_p is infinite, so there is an ideal $K' \subset K$ and an integer n such $K'_P K^n_P = K^{n+1}_P$ and $\mu(K'_p) < k$. We may suppose that K' itself is generated by $<k$ elements. Let H be the ideal $(K'K^n : K^{n+1})$ in T; clearly, $H \nsubseteq P$, and $\{Q \supseteq P \mid Q \nsupseteq H\}$ is an open subset of Spec T/Q whose elements satisfy $l(K_Q) < k$, as required.

We can now reformulate Theorem 1.3:

COROLLARY 1.5. *Let $S \supset N, J$ and $R = S/N \supset I = JR$ be as in Theorem* 1.3, *with $\mathscr{R}(J, S)$ Cohen–Macaulay. The following conditions are equivalent to* (a) *and* (b) *of Theorem* 1.1:

(c) *For all primes P of S containing $J+N$ we have $l(J_P) \leqslant \operatorname{ht} P/N$.*

(d) *For each $k \geqslant \operatorname{ht} J$ we have $\operatorname{ht}(L_k(J)R) \geqslant k$.*

(e) *There exists a sequence of ideals*

$$J = J_0 \subset J_1 \subset \cdots \subset J_e \subset J_{e+1} = S$$

such that

$$\operatorname{ht}(J_k R) \geqslant \sup\{l(J_Q) \mid Q \not\supseteq J_k\}.$$

Proof. The equivalence of (c), (d), and (e) is a straightforward manipulation, which we leave to the reader. Condition (c) is formally stronger than condition (b) of the theorem and is implied by condition (a), as was shown in the proof of (a) $\Rightarrow$ (b) above.

As a special case, we deduce:

COROLLARY 1.6. *Let $S \supset N, J$ and $R = S/N \supset I = JR$ be as in Theorem 1.1, with $\mathscr{R}(J, S)$ Cohen–Macaulay. If for all primes $P \supset J+N$ of S we have*

$$\mu(J_p) \leqslant \operatorname{ht} P/N,$$

then $\mathscr{R}(I, R) \cong R \otimes \mathscr{R}(J, S)$ and $\mathscr{R}(I, R)$ is Cohen–Macaulay.

Proof. Since $\mu(J_P) \geqslant l(J_P)$, this is clear from Corollary 1.5.

Of course, Corollary 1.6 can be reformulated, in the style of Corollary 1.5(e), in terms of the Fitting ideals of J.

We conclude this section with a consequence of Theorem 1.1 for the specialization of associated graded algebras. Recall that in [13] it is shown that if K is an ideal of the Cohen–Macaulay ring T and if $\mathscr{R}(K, T)$ is Cohen–Macaulay, then $\operatorname{gr}_K(T)$ is Cohen–Macaulay. Also, Hochster has shown [14] that if T is regular and $\operatorname{gr}_K T$ is a Cohen–Macaulay domain, then $\operatorname{gr}_K T$ is Gorenstein.

COROLLARY 1.7. *Let $S \supset N, J$ and $R = S/N \supset I = JR$ be as in Theorem 1.1. If the equivalent conditions* (a) *and* (b) *are satisfied, then $R \otimes \operatorname{gr}_J S \cong \operatorname{gr}_I R$, and $\operatorname{gr}_I R$ is Cohen–macaulay. Moreover, if S and R are regular and $\operatorname{gr}_J S$ is a domain, then $\operatorname{gr}_I R$ is Gorenstein.*

Proof. Note that $\operatorname{gr}_J S = S/J \otimes \mathscr{R}(J, S)$, and similarly for $\operatorname{gr}_I R$. Thus, $\mathscr{R}(I, R) \cong R \otimes \mathscr{R}(J, S)$ implies $\operatorname{gr}_I R = R \otimes \operatorname{gr}_J S$ (and in particular, the ideal of leading forms of elements of N in $\operatorname{gr}_J S$ is generated by $N + J/J \subset S/J$). The fact that $\mathscr{R}(I, R)$ is Cohen–Macaulay implies the same for $\operatorname{gr}_I R$ by the result of [16] already mentioned.

Now suppose R and S are regular. Since $\operatorname{gr}_I R$ is graded, it is enough to show that it is Gorenstein after localizing it at a maximal ideal of the form $P/I \oplus I/I^2 \oplus \cdots$; in particular, we may without loss of generality suppose that R and S are local. This implies that N is generated by a regular sequence. Thus, from the fact that $\mathrm{gr}_J S$ is Gorenstein [14], and $\mathrm{gr}_I R = S/N \otimes \mathrm{gr}_J S$, it follows that $\mathrm{gr}_I R$ is Gorenstein.

2. Some "Generic" Cohen–Macaulay Rees Algebras

In this section we prove that Rees algebras which arise in several "generic" examples are Cohen–Macaulay. The most important of these examples are determinantal varieties and we use the theory of Hodge algebras, or algebras with straightening law, of DeConcini, Eisenbud, and Procesi [8] to prove the Rees algebras of such varieties are Cohen–Macaulay. We first review their definition and salient features.

Let A be a commutative ring and let H be a finite partially ordered set (poset) with a map $H \to A$ sending, say, α to $\bar{\alpha}$. We say a monomial $m = \bar{\alpha}_1 \cdots \bar{\alpha}_k$ is *standard* if $\alpha_1 \leqslant \cdots \leqslant \alpha_k$.

Definition [8, 10]. Let R be a commutative ring, A a commutative R-algebra and H a finite poset as above. The algebra A is said to be a graded Hodge algebra on H over R (or simply an ASL) if the elements $\bar{\alpha}$ for α in H are homogeneous of degree greater than 0 such that the standard monomials form a free basis for A over R, and such that if α, β are noncomparable elements of H, then $\bar{\alpha}\bar{\beta}$ is a (necessarily unique) sum of standard monomials

$$\bar{\alpha}\bar{\beta} = \sum r_j \bar{\gamma}_{j1} \cdots \bar{\gamma}_{jn_j}$$

with $\gamma_{j1} \leqslant \alpha$ and $\gamma_{j1} \leqslant \beta$. We will refer to this relation as the *straightening* of $\bar{\alpha}\bar{\beta}$.

Definition. If H is a poset, a subset $I \subset H$ is said to be an *ideal* if whenever α is in I and $\beta \leqslant \alpha$, then β is in I. If A is an ASL on the poset H, and $I \subset H$, we let $\bar{I}$ denote the ideal generated by all elements $\bar{\alpha}$ such that α is in I.

If H is a poset and α, β are in H, then β is said to *cover* α if $\alpha < \beta$ and there is no γ in H such that $\alpha < \gamma < \beta$.

Definition [8]. If H is a finite poset, H is said to be *wonderful* (or locally semimodular in the more standard terminology) if, after adjoining least and greatest elements, H has the following property: if β_1 and β_2 are less than or equal to γ and are covers of the same element α, then there is a δ in H which is a common cover of β_1 and β_2 such that $\delta \leqslant \gamma$.

PROPOSITION 2.1 [8]. *Suppose A is an algebra with straightening law on H over R, and suppose*

(1) *R is a Cohen–Macaulay Noetherian ring, and*

(2) *H is wonderful.*

Then A is Cohen–Macaulay.

Let H be a finite poset and suppose I is an ideal of H. Let $H \ltimes I$ be the disjoint union of H and I, where we distinguish the additional copy of I in $H \ltimes I$ by an asterisk. We place an order on $H \ltimes I$ as follows: the subset $H \subset H \ltimes I$ has the ordering it already has, while the subset $I^* \subset H \ltimes I$ has the order it inherits from H. If α is in H and β is in I, then $\beta^* < \alpha$ if and only if $\beta \leqslant \alpha$. Otherwise β^* and α are noncomparable. We will say that $I \subset H$ is *self-covering* if whenever β_1, β_2 in I both cover an element α, any common cover of β_1 and β_2 is in I.

LEMMA 2.2. *Let H be a finite wonderful poset and suppose I is a self-covering ideal of H. Then $H \ltimes I$ is wonderful.*

Proof. Suppose we have adjoined greatest and least elements to $H \ltimes I$ and β_1, β_2 are in $H \ltimes I$, are $\leqslant \gamma$, and are covers of a common element α. We may take the greatest element to lie in H. We distinguish three cases.

Case 1. Suppose β_1, β_2 are in $H \subset H \ltimes I$. Then γ must also lie in H. Since H is wonderful, if in addition $\alpha \in H$, there is a common cover δ of β_1 and β_2 such that $\delta \leqslant \gamma$. Clearly, this element still has this property in $H \ltimes I$. If $\alpha = \pi^*$ is in I^*, then there is a unique element of H which covers α, namely, π. Consequently, this case does not arise.

Case 2. Suppose β_1, β_2 are in I^*, say $\beta_1 = \gamma_1^*$ and $\beta_2 = \gamma_2^*$. Since $\alpha \leqslant \beta_1, \beta_2$, $\alpha = \pi^*$ where $\pi \in I$. Define an element τ in H as follows: if $\gamma \in H$, let $\tau = \gamma$, if not then $\gamma = \tau^* \in I^*$ and let τ be the corresponding element. In either case, γ_1 and $\gamma_2 \leqslant \tau$. By assumption there is a common cover δ in H of γ_1 and γ_2 such that $\delta \leqslant \tau$. Since I is self-covering, τ is in I, and so $\gamma = \tau^*$, δ is in I, and γ^* is a common cover of β_1 and β_2.

Case 3. Suppose $\beta_1 = \gamma_1^* \in I^*$ and $\beta_2 \in H$. Since $\gamma \geqslant \beta_2$, $\gamma \in H$. As $\alpha \leqslant \beta_1$, $\alpha = \pi^*$ where $\pi \in I$. As β_2 is in H and is a cover of α, it easily follows that $\beta_2 = \pi$. Since γ_1^* is a cover of π^*, γ_1 is a cover of π. We may conclude that γ_1 is a common cover of β_1 and β_2 and is $\leqslant \gamma$.

If H is a poset and $\alpha \in H$, we set $I_\alpha = \{\beta \in H \mid \beta < \alpha\}$.

THEOREM 2.3. *Let A be an algebra with straightening law on H over R. Suppose I is a self-covering ideal of H and*

(1) *R is a Cohen–Macaulay Noetherian ring,*

(2) *H is wonderful, and*

(3) *if α, β are noncomparable elements of I, then $\bar{\alpha}\bar{\beta}$ is in $\bar{I}_\alpha \bar{I}$.*

Then $\mathscr{R}(I, A)$ is an algebra with straightening law on $H \ltimes I$ over R and is Cohen–Macaulay.

Proof. We identify $\mathscr{R}(I, A)$ with the subring $A[It]$ of the polynomial ring $A[t]$. If $\alpha^* \in I^* \subseteq H \ltimes I$, we let $\overline{\alpha^*} = \bar{\alpha}t$. It is trivial to verify that $A[t]$ is an ASL where we adjoin t as an element greater than any element of H. It easily follows that the standard monomials of $A[It]$ form a free R-basis of $A[It]$. To show $A[It]$ is an ASL we must show that the expression for a product $\bar{\alpha}\bar{\beta}$ with α, β noncomparable in $H \ltimes I$ has the correct form.

Case 1. Assume α, β are in $H \subseteq H \ltimes I$. It is clear the straightening of $\bar{\alpha}\bar{\beta}$ in A will suffice.

Case 2. Assume α and β are in I^*. Set $\alpha = \delta^*$ and $\beta = \gamma^*$. Then δ and γ are in I and are noncomparable. By assumption (3), the straightening,

$$\bar{\delta}\bar{\gamma} = \sum_k r_k \bar{\gamma}_{k1} \cdots \bar{\gamma}_{kn_k},$$

has the property that $\gamma_{k1} \leqslant \delta, \gamma$ and $\gamma_{k2} \in I$. Then

$$\overline{\delta^*}\,\overline{\gamma^*} = (\bar{\delta}t)(\bar{\gamma}t) = \sum_k r_k(\bar{\gamma}_{k1}t)(\bar{\gamma}_{k2}t)\,\bar{\gamma}_{k3} \cdots \bar{\gamma}_{kn_k}$$

is the desired straightening.

Case 3. Assume α is in H and $\beta = \delta^*$ is in I^*. There is a straightening,

$$\bar{\alpha}\bar{\delta} = \sum_k r_k \bar{\gamma}_{k1} \cdots \bar{\gamma}_{kn_k},$$

of $\bar{\alpha}\bar{\delta}$ in A. Since $\gamma_{k1} \leqslant \delta$, γ_{k1} is in I. Then

$$\bar{\alpha}\bar{\beta} = \bar{\alpha}(\bar{\delta}t) = \sum_k r_k(\bar{\gamma}_{k1}t)\,\bar{\gamma}_{k2} \cdots \bar{\gamma}_{kn_k}$$

is the straightening of $\bar{\alpha}\bar{\beta}$ in $A[It]$.

We have shown $A[It]$ is an ASL on $H \ltimes I$ over R. By Lemma 2.2, $H \ltimes I$ is wonderful. Furthermore $A[It]$ is graded through the grading on A by giving t degree 1. Proposition 2.1 shows $A[It] = \mathscr{R}(I, A)$ is Cohen–Macaulay.

Before passing to the main cases of interest we mention a simple special case:

PROPOSITION 2.4. *Let R be a ring, and let $S = R[X_1, \ldots, X_n]/L$ be a polynomial ring modulo on ideal generated by square-free monomials. Let*

$J \subset S$ be an ideal generated by any set of pairwise relatively prime square-free monomials in $X_1,\dots, X_n$. Then $\mathscr{R}(J, S)$ is an algebra with straightening law on the poset $H \ltimes I$ where H is the poset of nonzero square-free monomials in S, partially ordered by $m \leqslant n$ if and only if $n \mid m$, and I is the ideal of H consisting of those monomials which are in J.

We leave the proof of this result as an easy exercise for the reader. It would be interesting to know, supposing R and S are Cohen–Macaulay, when $\mathscr{R}(J, S)$ is Cohen–Macaulay, or, equivalently [8], when $H \ltimes I$ is a Cohen–Macaulay poset. The case of a wonderful H and a self-covering I is not so interesting because H will, in this setup, be wonderful if and only if any two maximal square-free nonzero monomials of S differ in at most one factor—a condition rarely attained.

We will apply Theorem 2.3 to two specific examples which were treated in [8, 10], the maximal minors of a generic matrix, and the Pfaffians of a generic alternating matrix. We describe these in detail.

Let $r \leqslant s$ and let $X = (x_{ij})$ be a $r \times s$ matrix whose entries are algebraically independent over a commutative ring R. Set $A = R[x_{ij}]$.

Any k by k minor of X is determined by choosing k rows, $1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant r$, and k columns, $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant s$. We denote this minor by the expression $(j_1,\dots,j_k \mid i_1,\dots, i_k)$ and order these as follows:

$$(j_1,\dots,j_k \mid i_1,\dots, i_k) \leqslant (j'_1,\dots,j'_m \mid i'_1,\dots, i'_m)$$

if and only if $k \leqslant m$ and $j_1 \leqslant j'_1,\dots,j_m \leqslant j'_m$, $i_1 \leqslant i'_1,\dots, i_m \leqslant i'_m$. Let H be the poset of such expressions.

PROPOSITION 2.5. *A is an algebra with straightening law on H over R.*

Now let $I_r \subset H$ be the ideal of maximal minors of X and let $I_r(X) = \bar{I}_r$ be the ideal in A they generate.

PROPOSITION 2.6. *If R is Cohen–Macaulay, then the Rees algebra $\mathscr{R}(I_r(X), A)$ is Cohen–Macaulay.*

Proof. We apply Theorem 2.3. Conditions (1)–(3) are easily verified. In addition, it is clear that I_r is a self-covering ideal of H. If $\bar{\alpha}$ and $\bar{\beta}$ are two noncomparable maximal minors, the straightening of $\bar{\alpha}\bar{\beta}$ is given by a standard Plücker relation; in particular we may write $\bar{\alpha}\bar{\beta}$ as a quadratic form in other maximal minors in such a way that each monomial has one minor less than $\bar{\alpha}$ and $\bar{\beta}$. Thus, condition (3) of Theorem 2.3 is also satisfied. We may conclude $\mathscr{R}(I_r(X), A)$ is Cohen–Macaulay.

We note that the theory of algebras with straightening law developed in [8] now allows us to conclude $\mathrm{gr}_{I_r(X)}(A)$ is also Cohen–Macaulay without using [16]; in fact $\mathrm{gr}_{I_r(X)}(A)$ is also an algebra with straightening law.

Next we consider the generic Pfaffians. Let

$$X = \begin{pmatrix} 0 & x_{12} & \cdots & x_{12n} \\ -x_{12} & 0 & & \vdots \\ \vdots & & & x_{2n-12n} \\ -x_{12n} & \cdots & -x_{2n-12n} & 0 \end{pmatrix}$$

be a generic $(2n+1) \times (2n+1)$ skew-symmetric matrix with zeroes down the diagonal. Let R be a commutative ring and set $A = R[X]$, by which we mean R with the entries of X adjoined.

If we choose $2k$ integers, $1 \leqslant i_1 < \cdots < i_{2k} \leqslant 2n+1$, then the minor of x determined by the $i_1, ..., i_{2k}$ columns and rows is a square of a polynomial function of the entries of this matrix, called the Pfaffian. We will denote this polynomial by $[i_1, ..., i_{2k}]$ and place an order on them by

$$[i_1, ..., i_{2k}] \leqslant [j_1, ..., j_{2m}]$$

if and only if $k \geqslant m$ and $i_s \leqslant j_s$ for $s = 1, ..., 2m$. We denote the poset of these expressions by P.

PROPOSITION 2.7 [7]. *A is an algebra with straightening law on P over R.*

Let $Pf_{2n} \subset P$ be the subset of Pfaffians of order $2n$, and place $I = \mathrm{Pf}_{2n}(X)$, the ideal generated by these Pfaffians.

PROPOSITION 2.8. *If R is a Cohen–Macaulay Noetherian ring, then $\mathscr{R}(I, A)$ is Cohen–Macaulay.*

Proof. We apply Theorem 2.3. Conditions (1)–(3) are easily verified and moreover it is clear that Pf_{2n} is self-covering. If $\bar{\alpha}$ and $\bar{\beta}$ are two noncomparable Pfaffians of maximal order, the straightening of $\bar{\alpha}\bar{\beta}$ also consists only of maximal Pfaffians [7]. This implies condition (4) of Theorem 2.3 and implies Proposition 2.8.

Again we note that $\mathrm{gr}_I(A)$ is again on algebra with straightening law which is Cohen–Macaulay [8].

We next consider a class of imperfect ideals I with four generators all with resolutions of the form

$$0 \to S \to S^4 \to S^4 \to I \to 0.$$

Our class is a subclass of divisors on scrolls. (See [11].)

Let R be a regular local ring and let

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \qquad Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & -y_{11} \end{pmatrix}$$

be matrices whose entries are algebraically independent over R (except for the one repeated entry of Y.) Set $S = R[X, Y]$, and let I be generated by the two by two minors of the matrix

$$Z = (X, YX).$$

This matrix is a 2×4 matrix and there are thus six 2×2 minors. However, due to the form of Y, it is clear that two of these are linear combinations of the others: denoting the 2×2 minor of Z determined by columns i, j $(i < j)$ by Δ_{ij}, we have $\Delta_{34} = (\det Y)\, \Delta_{12}$ and $\Delta_{23} = \Delta_{14}$.

A resolution for I can be obtained as follows. The matrix x determines an exact sequence:

$$0 \to \Lambda^2 S^2 \xrightarrow{(x_{11}, x_{12}, x_{21}, x_{12})} S^2 \otimes S^2 \xrightarrow{f} S_2 S^2$$
$$e_i \otimes e_j \mapsto \sum_k x_{ik} e_k e_j,$$

where $\{e_1, e_2\}$ is a chosen basis of S^2.

We may define a map of this complex to the complex $0 \to S \xrightarrow{\Delta_{12}} S$ by using the matrix YX and the minors of Z. Map $S_2 S^2 \to S$ by sending e_1^2 to Δ_{13}, $e_1 e_2$ to Δ_{23} and e_2^2 to Δ_{24}. This gives a map of complexes,

$$\begin{array}{ccccccc} & & & & S & \xrightarrow{\Delta_{12}} & S \\ & & & \begin{pmatrix} p_{11} \\ p_{12} \\ p_{21} \\ p_{22} \end{pmatrix} & \Big\uparrow & & \Big\uparrow \begin{pmatrix} \Delta_{13} \\ \Delta_{23} \\ \Delta_{24} \end{pmatrix}, \\ 0 \to & \Lambda^2 S^2 & \xrightarrow{(x_{11}, x_{12}, x_{21}, x_{22})} & & S^2 \otimes S^2 & \xrightarrow{f} & S_2 S^2 \end{array}$$

where

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = YX.$$

The mapping cylinder of this map is a resolution of the ideal $I = (\Delta_{12}, \Delta_{13}, \Delta_{23}, \Delta_{24})$. This ideal is a height two ideal of projective dimension 2.

The four linear relations on the generators of I may be derived by choosing any three columns of Z and using the usual linear relation on the 2×2 determinants of this 2×3 matrix. Specifically they are

$$\begin{aligned}
&x_{11}\Delta_{23} - x_{12}\Delta_{13} + p_{11}\Delta_{12} = 0,\\
&x_{21}\Delta_{23} - x_{22}\Delta_{13} + p_{21}\Delta_{12} = 0,\\
&x_{11}\Delta_{24} - x_{12}\Delta_{23} + p_{12}\Delta_{12} = 0,\\
&x_{21}\Delta_{24} - x_{22}\Delta_{23} + p_{22}\Delta_{12} = 0.
\end{aligned} \tag{1}$$

In addition, there is a quadratic relation coming from the usual Plücker relation of the 2×2 minors of a 2×4 matrix. This is

$$(\det Y)\,\Delta_{12}^2 - \Delta_{23}^2 + \Delta_{13}\Delta_{24} = 0. \tag{2}$$

We consider the Rees algebra $\mathscr{R}(I, S)$. It is a homomorphic image of $S[T_1, T_2, T_3, T_4]$ by the ideal Q generated by all homogeneous polynomials F in $S[T_1,..., T_4]$ such that $F(\Delta_{12}, \Delta_{13}, \Delta_{23}, \Delta_{24}) = 0$. Consequently, the five equations

$$\begin{aligned}
L_1 &= x_{11}T_3 - x_{12}T_2 + p_{11}T_1 = 0,\\
L_2 &= x_{21}T_3 - x_{22}T_2 + p_{21}T_1 = 0,\\
L_3 &= x_{11}T_4 - x_{12}T_3 + p_{21}T_1 = 0,\\
L_4 &= x_{21}T_4 - x_{22}T_3 + p_{22}T_1 = 0,\\
L_5 &= (\det y)\,T_1^2 - T_3^2 + T_2T_4 = 0
\end{aligned} \tag{3}$$

are in Q. We claim these generate Q. Let T be the ideal in $S[T_1,..., T_4] = T$ they generate. By inspection, one sees that $L_1,..., L_5$ are the Pfaffians of the matrix

$$A = \begin{pmatrix}
0 & x_{11} & -x_{21} & x_{22} & x_{12}\\
-x_{11} & 0 & T_2 & y_{11}T_1 - T_3 & -y_{12}T_1\\
x_{21} & -T_2 & 0 & -y_{21}T_1 & -T_3 - y_{11}T_1\\
-x_{22} & T_3 - y_{11}T_1 & y_{21}T_1 & 0 & T_4\\
-x_{12} & y_{12}T_1 & T_3 + y_{11}T_1 & -T_4 & 0
\end{pmatrix}.$$

PROPOSITION 2.9. *If T, J, S and I are as above, then $T/J \simeq \mathscr{R}(I, S)$ and $\mathscr{R}(I, S)$ is Gorenstein.*

Proof. We first wish to show that $\operatorname{ht} J \geqslant 3$; given the representation of J

as the ideal of 5×5 Pfaffians of A, the main result of [4] then implies that ht $J = 3$ and T/J is Gorenstein.

Set $\bar{T} = T/(T_1)$, and write "–" for reduction modulo T_1. The elements $\bar{L}_1, \ldots, \bar{L}_5$ are all of the six 2×2 minors of the matrix

$$M = \begin{pmatrix} \bar{x}_{11} & \bar{x}_{12} \\ \bar{x}_{12} & \bar{x}_{22} \\ \bar{T}_2 & \bar{T}_3 \\ \bar{T}_3 & \bar{T}_4 \end{pmatrix}$$

except for

$$\Delta = \begin{vmatrix} \bar{x}_{11} & \bar{x}_{12} \\ \bar{x}_{12} & \bar{x}_{22} \end{vmatrix}.$$

Now it is easy to see that the 2×2 minors of M generate an ideal $I_2(M)$ of height 3, as do $\bar{T}_2, \bar{T}_3, \bar{T}_4$. On the other hand, from the form of the matrix M, $(\bar{T}_1, \bar{T}_2, \bar{T}_3)\Delta \subseteq (\bar{L}_1, \ldots, \bar{L}_5)$, so that $(\bar{T}_1, \bar{T}_2, \bar{T}_3) I_2(M) \subseteq (\bar{L}_1, \ldots, \bar{L}_5)$ whence this latter ideal is of height $\geqslant 3$. This implies $\text{ht}(L_1, \ldots, L_5) \geqslant 3$ in T as stated.

We next wish to show the map $T/J \xrightarrow{f} \mathscr{R}(I, R)$ is an isomorphism. Since the ideal $(x_{11}, x_{12}, x_{21}, x_{22})$ has height 4 in T, it contains a nonzero divisor on T/J. Thus, it suffices to show that f is an isomorphism after localizing T at an arbitrary prime ideal not containing $(x_{11}, x_{12}, x_{21}, x_{22})$. Now after doing this localization, the skew symmetric matrix A can be reduced to the form

$$\left(\begin{array}{cc|c} 0 & 1 & \multirow{2}{*}{O} \\ -1 & 0 & \\ \hline \multicolumn{2}{c|}{O} & B \end{array}\right),$$

where B is a 3×3 skew symmetric matrix and, thus, $(L_1, \ldots, L_5)$ will, after this localization, become a complete intersection, and the map $\text{Sym}\, I \xrightarrow{g} \mathscr{R}(I, R)$ becomes an isomorphism. Since g factors through an epimorphism $\text{Sym}\, I \to T/J$, we are done.

Finally we note some of the specializations of the ideal I defined above.

EXAMPLE 2.10. Let p be the ideal in $k[x_{11}, x_{21}, x_{12}, x_{22}]$ defining the projection of the twisted quartic, $k[t^4, t^3w, tw^3, w^4]$. Then p is defined by the 2×2 minors of the matrix

$$\begin{pmatrix} x_{11} & x_{12} & x_{21}^2 & x_{22}x_{21} \\ x_{21} & x_{22} & x_{11}x_{12} & x_{12}^2 \end{pmatrix}.$$

If we set

$$Y = \begin{pmatrix} 0 & x_{21} \\ x_{12} & 0 \end{pmatrix},$$

this is of the form above.

EXAMPLE 2.11 (see Hochster [14]). Let p be the ideal in $k[x_{11}, x_{21}, x_{22}, x_{12}]$ defining the surface $k[u^2, u^3, uv, v]$. This ideal is defined by the 2×2 minors of the matrix

$$\begin{pmatrix} x_{11} & x_{12} & x_{21} & x_{22} \\ x_{21} & x_{22} & x_{11}^2 & x_{12}x_{11} \end{pmatrix}.$$

If specialize

$$Y = \begin{pmatrix} 0 & 1 \\ x_{11} & 0 \end{pmatrix},$$

this is a homomorphic image of $k[X, Y]$ in which I goes to p.

EXAMPLE 2.12. Let I be ideal $(x, y) \cap (z, x)$ in $k[x, y, z, w]$. Supose $\frac{1}{2}$ is in k. Then $I = (xz, xw, yz, yw)$ is also defined by the 2×2 minors of

$$\begin{pmatrix} x & y & x & y \\ z & w & -z & -w \end{pmatrix}.$$

If we specialize

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

to $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ and Y to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, we obtain this ideal.

3. SOME SPECIALIZATIONS

In this section we apply the results of the previous two sections to show that certain Rees algebras are Cohen–Macaulay. First, however, we discuss what other information this gives. We recall two propositions found in Huneke [16].

PROPOSITION 3.1. *Suppose R is Cohen–Macaulay (respectively Gorenstein) and I is an ideal of R. If $\mathscr{R}(I, R)$ is Cohen–Macaulay (respectively Gorenstein), then* $\mathrm{gr}_I(R)$ *is Cohen–Macaulay (respectively Gorenstein).*

PROPOSITION 3.2. *Let R be a Noetherian ring and I an ideal of R. Suppose* $\mathrm{gr}_I(R)$ *is Cohen–Macaulay.*

(1) *If R/I is a domain, then* $\mathrm{gr}_I(R)$ *is a domain if and only if*

$$l(I_P) \leqslant \max\{\mathrm{ht}\, P - 1, \mathrm{height}(I)\}$$

for all prime ideals P of R.

(2) *If R/I is integrally closed, then* $\mathrm{gr}_I(R)$ *is integrally closed if and only if*

$$l(I_P) \leqslant \max\{\mathrm{ht}\, P - 2, \mathrm{height}(I)\}$$

for all prime ideals P of R.

PROPOSITION 3.3. *Let (R, m) be a local ring and let I be an ideal of R of height at least one. Suppose* $\mathrm{gr}_I(R)$ *is Cohen–Macaulay. Then,*

$$l(I) = \dim(R) - \inf \mathrm{depth}(R/I^n), \tag{1}$$

and if $\mathrm{depth}(R/I^k) = \inf \mathrm{depth}(R/I^n)$, *then* $\mathrm{depth}(R/I^{k+1}) = \mathrm{depth}(R/I^k)$.

We remark this sharpens, for the Cohen–Macaulay case, the inequality of Burch [5]. (See also [3].)

Proof. Set $k = \inf \mathrm{depth}(R/I^n)$. We may choose elements $x_1, \ldots, x_k$ in R which form an R-sequence on the modules R/I^n for all $n \geqslant 1$. (See [2, 5].) It follows that $x_1^*, \ldots, x_k^* \in R/I$ are a $\mathrm{gr}_I(R)$-sequence and

$$\mathrm{gr}_I(R)/(x_1^*, \ldots, x_k^*) \simeq \mathrm{gr}_{\bar{I}}(\bar{R}),$$

where "–" denotes reduction by $(x_1, \ldots, x_k)$. Since the x_i may be chosen so that $\dim(R) = \dim(\bar{R}) + k$, it follows that $l(\bar{I}) = l(I)$ and, thus, it is enough to verify Proposition 4.3 under the assumption $\inf \mathrm{depth}(R/I^n) = 0$.

Let $J = m\, \mathrm{gr}_I(R)$. By definition, $l(I) = \dim(\mathrm{gr}_I(R)/J)$. To show $\dim(R) = l(I)$ it is enough to show $\mathrm{ht}(I) = 0$ as $\mathrm{gr}_I(R)$ is Cohen–Macaulay. If $\mathrm{ht}(J) > 0$, we may choose an element x in m, x not in I, such that $x^* \in R/I$ is a nonzero divisor on $\mathrm{gr}_I(R)$. It would follow that x is a nonzero divisor on R/I^n for all $n \geqslant 1$, contradicting $\inf \mathrm{depth}\, R/I^n = 0$. This shows $l(I) = \dim(R)$.

Suppose $\mathrm{depth}(R/I^k) = \inf \mathrm{depth}(R/I^n) = 0$. Since $\mathrm{gr}_I(R)$ is Cohen–Macaulay and $\mathrm{ht}\, I^* \geqslant 1$, we may choose an x in I, not in I^2, such that x^* is a non-zero-divisor on $\mathrm{gr}_I(R)$. Then x induces an embedding

$$0 \to R/I^k \xrightarrow{x} R(I^{k+1}),$$

and consequently $\operatorname{Ass}(R/I^k) \subseteq \operatorname{Ass}(R/I^{k+1})$. This shows $\operatorname{depth}(R/I^{k+1}) = 0$ as required.

THEOREM 3.4. *Let R be a Cohen–Macaulay domain and I an ideal of R of finite projective dimension. Suppose either*

(a) *height $(I) = 2$ and R/I is Cohen–Macaulay, or*

(b) *height $(I) = 3$ and R/I and R are Gorenstein.*

Then the following statements hold.

(1) *If for all prime ideals $q \supseteq I$, $\mu(I_q) \leqslant \operatorname{ht} q$, then $\mathscr{R}(I, R)$ and $\operatorname{gr}_I(I)$ are Cohen–Macaulay. If, in addition, R is a regular ring, then $\operatorname{gr}_I(R)$ is Gorenstein.*

(2) *If I is a prime ideal and for all prime ideals $q \supseteq I$,*

$$\mu(I_q) \leqslant \max\{\dim R_q - 1, \operatorname{height}(I)\},$$

then $\operatorname{gr}_I(R)$ is a Cohen–Macaulay domain.

(3) *If R/I is integrally closed and for all prime ideals $q \supseteq I$,*

$$\mu(I_q) \leqslant \max\{\operatorname{ht} q - 2, \operatorname{height}(I)\},$$

then $\operatorname{gr}_I(R)$ is an integrally closed Cohen–Macaulay domain.

Proof. Without loss of generality we may suppose that R is local. We treat cases (a) and (b) separately. Assume we are in case (a). The structure theorem of Hilbert and Burch shows I is generated by the $n \times n$ minors of an $n \times (n+1)$ matrix $A = (a_{ij})$ with coefficients in R. Let $X = (x_{ij})$ be a generic $n \times (n+1)$ matrix over R and set $S = R[x_{ij}]$. Let $N = (x_{ij} - a_{ij})$. Then $S/N \simeq R$. If we denote the map from S to R by f and let $J = I_n(X)$ be the ideal generated by the maximal minors of S, then $(J+N)/N = I$.

By Proposition 2.5, $\mathscr{R}(J, S)$ is a Cohen–Macaulay ring. Now Corollary 1.6 immediately implies $\mathscr{R}(I, R)$ is Cohen–Macaulay, and now the rest of the statements follow from Corollary 1.7, Proposition 3.2, and the fact that $l(I_q) \leqslant \mu(I_q)$.

Now assume (b). By the structure theorem of Buchsbaum and Eisenbud [4], we may realize I as the highest-order Pfaffians of a skew–symmetric matrix A with zeroes down the diagonal. Let X be the generic skew-symmetric matrix with zeroes down the diagonal and set $S = R[X]$. We may define a map $f: S \to R$ by sending x_{ij} to a_{ij}, where these are the respective entries of X and A in the ith row and jth column. If we let $J = \operatorname{Pf}_{2n}(X)$ where X is a $(2n+1) \times (2n+1)$ matrix, then $f(J) = I$. By Proposition 2.8, $\mathscr{R}(J, S)$ is Cohen–Macaulay.

Statements (1), (2), and (3) now follow immediately from Corollaries 1.5, 1.6, and 1.7.

THEOREM 3.5. *Let R be a Cohen–Macaulay Noetherian domain and let $A=(a_{ij})$ be an $r \times s$ matrix ($r \leqslant s$) with coefficients in R. Let $I = I_r(A)$.*

(1) *If* $\operatorname{height}(I_t(A)) \geqslant (r-t+1)(s-r)+1$ *for all* $1 \leqslant t \leqslant r$, *then* $\mathscr{R}(I, \mathscr{R})$ *and* $\operatorname{gr}_I(R)$ *are Cohen–Macaulay. If further R is a regular ring,* $\operatorname{gr}_I(R)$ *is Gorenstein.*

(2) *If* $\operatorname{height}(I) \geqslant s-r+1$ *and moreover*

$$\operatorname{height}(I_t(A)) \geqslant (r-t+1)(s-r)+2 \quad (1 \leqslant t \leqslant r-1)$$

and I is prime, then $\operatorname{gr}_I(R)$ *is a domain.*

(3) *If* $\operatorname{height}(I) \geqslant s-r+1$, *$R/I$ is an integrally closed domain, and*

$$\operatorname{height}(I_t(A)) \geqslant (r-t+1)(s-r)+3 \qquad (1 \leqslant t \leqslant r-1),$$

then $\operatorname{gr}_I(R)$ *is integrally closed.*

Proof. Let $X=(x_{ij})$ be a generic $r \times s$ matrix over R and set $S = R[x_{ij}]$. Define a map $f\colon S \to R$ by sending x_{ij} to a_{ij}. Then $N = \ker(f) = (x_{ij} - a_{ij})$ is a perfect ideal and we are in a position to apply Theorem 1.1. Let $J = I_r(X)$ so that $(J+N)/N = I$. By Proposition 2.6, $\mathscr{R}(J, S)$ is Cohen–Macaulay. We will apply Corollary 1.5. Consider the chain of ideals,

$$J \subset I_{r-1}(X) \subset \cdots \subset I_1(X),$$

and set $v_k = \sup\{l(\tilde{I}_Q) \mid Q \in \operatorname{Spec}(S) \text{ and } Q \not\supseteq I_{k-1}(X)\}$. To show $R(I, R)$ is Cohen–Macaulay it is enough to prove

$$\operatorname{height}(f(I_k(X)) \geqslant v_k.$$

However, $f(I_k(X)) = I_k(A)$ and by assumption $\operatorname{height}(I_k(A)) \geqslant (r-k+1)(s-r)+1$. Thus, it is enough to verify $(r-k+1)(s-r)+1 \geqslant v_k$.

Suppose $Q \not\supseteq I_{k-1}(X)$. Then in S_Q, by elementary row and column operations we may change X to the matrix

$$\begin{array}{cc} & \overbrace{\hphantom{1 \quad 0}}^{k-1} \\ k-1\left\{\vphantom{\begin{matrix}1\\ \ddots \\ 0\end{matrix}}\right. & \left(\begin{matrix} 1 & 0 & \\ & \ddots & 0 \\ 0 & 1 & \\ & 0 & X' \end{matrix}\right). \end{array}$$

X' is an $(r-k+1) \times (s-k+1)$ matrix and $I_{r-k+1}(X') = (I_r(X))_Q = I_Q$.

Let Y be a generic $(r-k+1)\times(s-k+1)$ matrix. Then $l(I_Q)\leqslant l(I_{r-k+1}(Y))$. We claim $l(I_{r-k+1}(Y))=(r-k+1)(s-r)+1$. This is well-known [6] and can be seen as follows: Let $T=k[y_{ij}]$, let Δ_α be the maximal minors of $Y=(y_{ij})$, and let $m=(y_{ij})$. Then

$$T/m\oplus I_{r-k+1}(Y)/mI_{r-k+1}(Y)\oplus\cdots\simeq k[\Delta_\alpha],$$

which is the homogeneous coordinate ring for the Grassmanian $G(r-k+1, s-k+1)$ which has dimension $(r-k+1)(s-r)+1$.

We will show (2); (3) follows similarly. Since $\mathrm{gr}_I(R)$ is Cohen–Macaulay, to show it is a domain it is enough to show

$$l(I_P)\leqslant\max\{\dim(R_P)-1, \text{height } I\}.$$

This is a local question so we may assume (R,P) is local and we need to show

$$l(I)\leqslant\max\{\dim(R)-1, \text{height } I\}.$$

Suppose $P\supseteq I_t(A)$ but $P\not\supseteq I_{t-1}(A)$. Then from the calculations above,

$$l(I_P)\leqslant(r-t+1)(s-r)+1.$$

However, since $P\supseteq I_t(A)$, $\dim R\geqslant\text{height}(I_t(A))\geqslant(r-t+1)(s-r)+2$. This establishes our claim and an application of Proposition 3.2 proves (2).

Now let R be a Cohen–Macaulay domain, $S=R[x_{11},x_{12},x_{21},x_{22}, y_{11},y_{12},y_{21}]$, and let $\tilde{I}=$ ideal generated by the 2×2 minors of $Z=(X, YX)$, as in Section 2. Let $f\colon S\to R$ be any R-homomorphism and put $I=f(\tilde{I})$. Put $J=f((x_{11},x_{12},x_{21},x_{22}))$.

THEOREM 3.6. *If* $\text{height}(I)\geqslant 2$, $\text{height}(J)\geqslant 3$, *then* $\mathscr{R}(I,R)$ *is a Cohen–Macaulay ring. If* $\text{height}(J)\geqslant 4$, *and I is prime, then* $\mathrm{gr}_I(R)$ *is a domain. If, in addition, R is regular,* $\mathrm{gr}_I(R)$ *is Gorenstein.*

Proof. By Proposition 2.9, $\mathscr{R}(\tilde{I},S)$ is Cohen–Macaulay. We apply Corollary 1.6 to the chain of ideals,

$$\tilde{I}\subseteq(x_{11},x_{12},x_{21},x_{22}).$$

It is enough to show that

$$\text{height}(I)\geqslant\sup\{l(\tilde{I}_Q)\mid Q\not\supseteq\tilde{J}\}$$

and

$$\text{height}(J)\geqslant\sup\{l(\tilde{I}_Q)\mid Q\in\operatorname{Spec}(S)\}.$$

The first inequality follows since if $Q \not\supseteq \tilde{J}$, then $\tilde{I}_Q$ is generated by two elements and so $l(\tilde{I}_Q) = 2$. The second inequality holds because the calculations of Section 3 show Δ_{23} is integral over the other three generators of $\tilde{I}$, so $l(\tilde{I}_Q) \leqslant 3$ for every Q.

COROLLARY 3.7. *Let* $p \subseteq k[x_{11}, x_{12}, x_{21}, x_{22}] = R$ *be a prime and consider three cases*:

(a) $R/p \simeq k[u^2, u^3, uv, v]$,

(b) $R/p \simeq k[u^4, u^3 v, uv^3, v^4]$,

(c) $p = (x_{11}, x_{12}) \cap (x_{21}, x_{22})$ *and* $\frac{1}{2}$*is in* k.

Then in all cases $\mathrm{gr}_p(R)$ *is a Gorenstein ring, and in cases* (a) *and* (b) *it is also a domain.*

Proof. This follows immediately from Theorem 3.6 and Proposition 3.1.

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