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THE CHROMATIC INDEX OF GRAPHS WITH LARGE MAXIMUM DEGREE

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By Vizing's Theorem, any graph G has chromatic index equal either to its maximum degree $\Delta(G)$ or $\Delta(G)+1$. A simple method is given for determining exactly the chromatic index of any graph with $2s+2$ vertices and maximum degree $2s$.

1. Introduction

We consider only undirected graphs without loops or multiple edges. The set of vertices of a graph is denoted by $V(G)$; the cardinality $p(G)$ of $V(G)$ is the *order* of G. Similarly the edge-set of G is denoted by $E(G)$, and its cardinality, $q(G)$, is the *size* of G. Given a graph G and a set $R \subset V(G)$, we let $\langle R\rangle$ denote the subgraph of G induced by the vertices of R, and let $G-R$ denote the subgraph of G induced by those vertices of G which are not contained in R.

A *coloring* of a graph is an assignment of colors to its edges so that no two adjacent edges are assigned the same color. The *chromatic index* of a graph G, denoted $\chi'(G)$, is the minimum number of colors used among all colorings of G. Vizing [11] has shown that for any graph G, $\chi'(G)$ is either its maximum degree $\Delta(G)$ or $\Delta(G)+1$. If $\chi'(G)=\Delta(G)$ then G is in *Class 1*; otherwise G is in *Class 2*. A vertex v in a colored graph is said to *miss* a color C (and similarly C misses v) if no edge which is assigned color C is incident with v. We follow [5] for most other terminology and notation.

Despite the tight bounds supplied by Vizing's Theorem, the problem of determining exactly the chromatic index of an arbitrary graph is an extremely difficult one; Holyer [6] has recently shown it to be NP-complete. Nevertheless, some good results have been obtained for graphs with additional properties.

Theorem A [8]. *If G is bipartite, then G is in Class* 1.

Theorem B [12]. *The complete graph K_p is in Class* 1 *if p is even and in Class* 2 *if p is odd.*

Theorem C [1]. *Let G have order* $2s+1$ *and maximum degree r. If the size of G is at least* $sr+1$, *then G is in Class* 2.

Theorem D [10]. *If G has order $2s$ and maximum degree $2s-1$, then G is in Class 1. If G has order $2s+1$ and maximum degree $2s$, then G is in Class 2 if and only if the size of G is at least $2s^2+1$.*

A non-increasing sequence $F=(f_1,\ldots,f_n)$ of non-negative integers is *feasible* for a graph G if there exists an n-coloring of G for which the cardinalities of the n color classes are precisely $f_1,\ldots,f_n$. The following result was obtained independently by McDiarmid [9] and De Werra [2].

Theorem E. *If the non-increasing sequence $F=(f_1,\ldots,f_n)$ is feasible for G, then so is any non-increasing sequence $F'=(f'_1,\ldots,f'_n)$ such that*

$$\sum_{i=1}^{n} f'_i=\sum_{i=1}^{n} f_i \quad \text{and} \quad \sum_{i=1}^{k} f'_i\leqslant\sum_{i=1}^{k} f_i \quad \textit{for } k=1,\ldots,n-1.$$

A coloring is *equitable* if the cardinalities of any two color classes differ by at most one. So if the graph G can be n-colored, then by Theorem E there is an equitable n-coloring of G.

For a non-increasing sequence $F=(f_1,\ldots,f_n)$ and non-negative integer f, let $f^*=\max\{i \mid f_i\geqslant f\}$. Thus f^* is the number of elements in F which are greater than or equal to f. We will use the following result, due to Folkman and Fulkerson.

Theorem F [3]. *Assume $f_1=f_2=\cdots=f_h\geqslant f_{h+1}=\cdots=f_n$, and let $F=(f_1,\ldots,f_n)$. Let B be a bipartite graph with size $\sum_{f=1}^{\infty} f^*$. Then F is feasible for B if and only if*

$$q(B-X)\geqslant\sum_{f=|X|+1}^{\infty} f^* \quad \textit{for all } X\subset V(B).$$

2. The chromatic index of graphs with order $2s+2$ and maximum degree $2s$

Theorem D seems to indicate that graphs with maximum degree 'close to' the order of the graph behave more predictably than arbitrary graphs. We extend those results with the following theorem.

Theorem 1. *Let G be a graph with order $2s+2$ and maximum degree $2s$. Then G is in Class 2 if and only if it contains a vertex v such that $G-v$ has size at least $2s^2+1$.*

Proof. If there is such a vertex v, then $\chi'(G-v)=2s+1$ by Theorem D. It then follows that $\chi'(G)=2s+1$, so that G is in Class 2.

The necessity 'half' of the proof is much more difficult. Suppose there is no such vertex v. Then $q(G)-\delta(G)\leqslant 2s^2$, where $\delta(G)$ denotes the minimum degree of G. We want to construct a $2s$-coloring of G.

Case 1. $q(G)-\delta(G)=2s^2$.

Let u be a vertex of minimum degree in G. By theorem D, $\chi'(G-u)=2s$. Consider any $2s$-coloring of $G-u$. Since every color class in the coloring can have cardinality at most s, each must have cardinality exactly s; that is, each color misses exactly one vertex of $G-u$. Now to each edge uw_i of G incident with u assign any color presently missing w_i (there must be at least one color missing w_i, for otherwise w_i has degree $2s+1$ in G). This yields a valid $2s$-coloring of G since if $w_i \neq w_j$ then no color misses both w_i and w_j.

Case 2. $q(G)-\delta(G)<2s^2$.

If there is an edge in the complement of G which can be added to G without increasing $\Delta(G)$, we add it. Continue this process until a graph H is obtained which has order $2s+2$ and maximum degree $2s$, contains G as a subgraph, and additionally has one of the following two properties.

(i) $q(H)-\delta(H)=2s^2$.

(ii) $q(H)-\delta(H)<2s^2$ and no edge of $\bar{H}$ can be added to H without increasing $\Delta(H)$.

If $q(H)-\delta(H)=2s^2$, then by Case 1 the graph H is $2s$-colorable, so that G is $2s$-colorable.

In order to handle the second possibility we introduce some new terminology. A graph H is *saturated* if no edge of $\bar{H}$ can be added to H without increasing its maximum degree. A saturated graph H with $2s+2$ vertices is *weakly saturated* if $q(H-v)<s\cdot\Delta(H)$ for any vertex v of H. Then by the previous discussion, to complete the proof it suffices to show that any weakly saturated graph with order $2s+2$ and maximum degree $2s$ is $2s$-colorable. It should be noted that such weakly saturated graphs do exist; the complements of the five smallest are given in Table 1.

Table 1. The complements of the five smallest weakly saturated graphs with order $2s+2$ and maximum degree $2s$

Order	Complement
12	$3K_{1,3}$
12	$4K_{1,2}$
14	$2K_{1,4}\cup K_{1,3}$
14	$2K_{1,3}\cup 2K_{1,2}$
14	$4K_{1,2}\cup K_{1,1}$

Since H is saturated and $\Delta(H)=p(H)-2$, $\bar{H}$ is the disjoint union of stars. Let $\bar{H}$ be $K_{1,t_1}\cup K_{1,t_2}\cup\cdots\cup K_{1,t_n}$ where $t_i\geq t_{i+1}$ for $1\leq i\leq n-1$. Note that $2s+2=n+\sum_{i=1}^{n} t_i$ and $\sum_{i\neq j} t_i>s$ for any j because H is weakly saturated. Let $z=\sum_{i=1}^{n} t_i$.

Subcase 1. *n is even.*

Let w_i denote the vertex with degree t_i in component i of $\bar{H}$ (if $t_i=1$ let w_i be either of the two vertices in component i). Let $W=\{w_1,\ldots,w_n\}$ and let $U=V(G-W)=\{u_1,u_2,\ldots,u_z\}$. The edges of H will be $2s$-colored in three stages.

Stage 1. *Coloring the edges of* $\langle U\rangle$.

Notice that $\langle U\rangle$ is simply the complete graph with z vertices. Since n is even, z must be also. It follows from Theorem B that $\langle U\rangle$ is $(z-1)$-colorable, that is, $\langle U\rangle$ has a factorization into $z-1$ disjoint 1-factors. Color $n-1$ of these 1-factors with the colors $C_1, C_2, \ldots, C_{n-1}$. Color $\frac{1}{2}(z-n)$ of the remaining 1-factors equitably with the colors $M_1,\ldots,M_{(z-2)/2}$; color the other $\frac{1}{2}(z-n)$ 1-factors equitably with the colors $N_1,\ldots,N_{z/2}$. Then letting $C=\{C_1,\ldots,C_{n-1}\}$, $M=\{M_1,\ldots,M_{(z-2)/2}\}$ and $N=\{N_1,\ldots,N_{z/2}\}$, it is easily verified that

(i) the colors of C each miss 0 vertices of $\langle U\rangle$;

(ii) $\frac{1}{2}(z-n)$ colors of M each miss $n-2$ vertices of U, while the remaining $\frac{1}{2}(n-2)$ colors of M each miss n vertices of $\langle U\rangle$;

(iii) the colors of N each miss n vertices of $\langle U\rangle$.

Stage 2. *Coloring the edges joining* $\langle U\rangle$ *and* $\langle W\rangle$.

We want to extend the coloring from Stage 1 to include all edges between $\langle U\rangle$ and $\langle W\rangle$. First we require the following observation.

Lemma 1. *Let H_1 and H_2 be graphs, each constructed by beginning with the complete graph K_z and n isolated vertices, and adding edges from the isolated vertices to the vertices of K_z in such a way that each vertex from K_z is now adjacent to all but one of the previously isolated vertices. If the degree sequences of the previously isolated vertices are identical (except possibly for ordering) in H_1 and H_2, then H_1 and H_2 are isomorphic.*

Proof of Lemma 1. The graphs $\bar{H}_1$ and $\bar{H}_2$ are both K_n with z vertices of degree 1 attached. Since the degree sequences of $\bar{H}_1$ and $\bar{H}_2$ are identical, there is an obvious isomorphism between these two graphs. Therefore, H_1 and H_2 are isomorphic.

Returning now to the proof of Theorem 1, let U_i be the set of colors missing vertex u_i after Stage 1 for $i=1,\ldots,z$. Then each U_i has cardinality $n-1$. Construct the bipartite graph B with vertex set

$$V(B)=\{U_1,\ldots,U_z,M_1,\ldots,M_{(z-2)/2},N_1,\ldots,N_{z/2}\}$$

and edge set

$$E(B)=\{U_iM_j \mid M_j\in U_i\}\cup\{U_iN_j \mid N_j\in U_i\}.$$

By Lemma 1, extending the coloring from Stage 1 to include all edges between U and W is equivalent to finding an n-coloring of B with colors $D_n, D_{n-1},\ldots,D_1$ and cardinality sequence $(z-t_n,\ldots,z-t_1)$; coloring edge U_iM_j with color D_k indicates that an edge between u_i and w_k in H should be assigned color M_j, and similarly coloring the edge U_iN_j with color D_k in B corresponds to coloring an edge between u_i and w_k with the color N_j in the graph H.

In order to determine the existence of such a coloring of B, it suffices to show that the cardinality sequence

$$F=(z-1,\dots,z-1,\tfrac{1}{2}(z+n-2),\tfrac{1}{2}(z+n-2))$$

of length n is feasible for B. Then the desired sequence will be feasible by Theorem E, since $z-t_i \geqslant \frac{1}{2}(z+n-2)$ for any i, because otherwise the complement of $H-w_i$ has less than $\frac{1}{2}(z+n-2)=\frac{1}{2}(2s)=s$ edges, a contradiction. It is straightforward to verify that the graph B and the sequence F together satisfy the conditions of Theorem F: a subset X of $V(B)$ with order at most $\frac{1}{2}(z+n-2)$ can be incident with at most $\frac{1}{2}n(z+n-2)$ edges of B since $\Delta(B)=n$, and if X has cardinality r between $1+\frac{1}{2}(z+n-2)$ and $z-1$ it covers at most $\frac{1}{2}n(z+n-2)+\frac{1}{2}(n-2)(2r-z-n+2)$ edges of B because each U_i is adjacent to $\frac{1}{2}n$ of the N_i. Consequently, F is a feasible sequence for B and it follows by the preceding argument that the edges between $\langle U\rangle$ and $\langle W\rangle$ can be colored using only colors in M and N.

Stage 3. *Coloring* $\langle W\rangle$.

In Stage 2, no colors of C were assigned to any edges incident with vertices of $\langle W\rangle$. Thus, by Theorem B, we can complete the coloring of H by coloring $\langle W\rangle$ with the colors $C_1,\dots,C_{n-1}$. A $2s$-coloring of H is thereby obtained, and the proof of the theorem is complete for the subcase when n is even.

Subcase 2. *n is odd.*

If n is odd, define U and W as in the proof of Subcase 1. Choose any vertex u in $\langle U\rangle$ which is not adjacent to w_1. Remove u from U and place it in W, renaming it w_{n+1}. A proof similar to, although somewhat more complicated than, that of Subcase 1 can now be used to obtain the theorem.

3. Final remarks

A connected graph G is *critical* if $\chi'(G)=\Delta(G)+1$ and $\chi'(G-e)<\chi'(G)$ for each edge e of G. Several authors [1, 7] conjectured that all critical graphs have an odd number of vertices. Goldberg [4] recently disproved this Critical Graph Conjecture. However, because of Theorem D, we can restate Theorem 1 in the following equivalent form.

Theorem 1$'$. *There are no critical graphs with order* $2s+2$ *and maximum degree* $2s$.

It is natural to ask if the predictability of the chromatic index of graphs with high maximum degree that is demonstrated in Theorems D and 1 can be extended any further. We close with the following conjectures. Conjecture 1 is a special case of a conjecture due to Vizing [13].

Conjecture 1. A graph G with order $2s+1$ and maximum degree $2s-1$ is in Class 2 if and only if it has size at least $2s^2-s+1$.

Conjecture 2. A graph G with order $2s+2$ and maximum degree $2s-1$ is in Class 2 if and only if $q(G)-\delta(G)\geqslant 2s^2-s+1$.

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