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The Existence of Periodic Travelling Waves for Singularly Perturbed Predator–Prey Equations via the Conley Index

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1. INTRODUCTION

A.

This paper is concerned with the existence of periodic travelling wave solutions of reaction–diffusion equations, which arise as models of predator–prey interactions in mathematical ecology. The relevant equations are somewhat similar to the Fitz-Hugh–Nagumo equations, and our results apply to this latter system as well.

In studying the Fitz-Hugh–Nagumo equations, Conley ([3]), initiated a new topological approach to this problem. His method depends on the construction of an isolating neighborhood (cf. Section 2A) about a "singular" periodic solution. Associated with such a neighborhood is a topological invariant, the *Conley Index*, which, when nontrivial, implies the existence of a smooth solution near the singular one.

The new aspects of the present discussion are in the method used to construct the isolating neighborhood (in $\mathbb{R}^4$), and in the computation of the index. With regard to the isolating neighborhood, our construction is global in that a parameter taken to be small in Conley's equations, is now allowed to assume large values. In order to compute the index, we "continue" our equations to those studied by Conley. Actually, we go one step further and continue Conley's equations to the Van der Pol equations crossed with a pair of linear equations admitting a repelling critical point. The index for this latter system is easily computed and turns out to be non-trivial. It follows from the invariance of the index under continuation [2], that the desired

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periodic solution exists. (It is worth noting that the global continuation of periodic orbits is a subject of independent interest, c.f. [1].)

We assume our equations to be of a rather general form, and our hypotheses are made only to ensure that the "singular" solution exists for sufficiently small wave speeds; see hypotheses (H_1)–(H_4) in Section 2D, below, and the discussion therein.

Related problems have been studied by a number of investigators. In particular, Fife [5], and Mimura *et al.* [6], have applied classical perturbation theory to obtain existence theorems for boundary-value problems. Their equations also contain another parameter which they require to be small; we show that our hypothesis (H_4), which we obtain from phase plane considerations, is actually a weaker condition than theirs.

In the remainder of this section, we shall describe the problem and the main result. In Section 2 we discuss the existence of the singular solution. It is obtained from the singularly perturbed equations, which turn out to be a mixed algebraic–differential system. The algebraic equation is of the form $uf(u, v) = 0$, and the condition $f(u, v) = 0$ is equivalent to the equation $u = P(v)$ in the relevant region. Thus u "jumps" from the surface $u = 0$ to the surface $u = P(v)$. Associated to each of these surfaces is a phase plane for the (v, v') equations. We obtain the singular solution by suitably "glueing" these two phase planes together. Our hypotheses are such as to imply that the discontinuity in u is approximable by a transition layer for the unperturbed equations. The actual existence of the singular solution is via a continuity method together with estimates on certain geometric quantities in the (glued!) phase plane.

In Section 3 we construct a family N_ε of shrinking isolating neighborhoods containing the singular solution. Each N_ε is homeomorphic to the product of a circle with a three-cell and is the union of four tubular regions in $\mathbb{R}^4$. This construction is highly intuitive, but the actual verification that it indeed is an isolating neighborhood, is a little involved, and requires the checking of several cases. This is done in Section 4, where we also show that the orbits which stay interior to N_ε must traverse the tubes in a precise order, and cannot "turn around."

In Section 5 we continue the equations to simpler ones so as to be able to compute the index. This is done in three stages. In the first stage we introduce an "artificial viscosity" parameter μ into the equations, and under suitable changes of variables, we show that the equations are continuable to the (simpler) system obtained by setting $\mu = 0$. We next deform the nonlinear functions to those which appear in the Van der Pol equations. In the last stage, we introduce a second "artificial viscosity" parameter ν, and we again continue the equations to the case $\nu = 0$. These final equations are of the desired form; namely, the Van der Pol equations crossed with "a repelling critical point." All of these changes require modifications of the earlier constructed isolating neighborhoods, and we prove that at each stage in the continuation, the relevant isolated invariant sets have the same index.

In Section 6 we compute the index of the isolated invariant set. It turns out to be $\Sigma^2 \vee \Sigma^3$, the wedge product of a pointed two-sphere with a pointed three-sphere. This means that our original isolating neighborhood N_ε contains a complete orbit in its interior. The fact that this orbit must cycle the union of tubes implies that a Poincaré map is well-defined in some neighborhood of the orbit. The proof is concluded by computing the degree of the Poincaré map, using the homotopy invariance of the degree.

B. *The Equations*

The equations have the form

$$\begin{aligned} \varepsilon u_t &= \varepsilon^2 u_{xx} + uf(u, v), \\ v_t &= v_{xx} + vg(u, v); \end{aligned} \tag{1}$$

here u and v are the population densities of a prey and predator species, respectively, and f and g are their growth rates. We take ε to be a small positive parameter. It will be assumed that

$$\partial f/\partial v < 0 \qquad \text{and} \qquad \partial g/\partial u > 0, \tag{H_0}$$

and for definiteness we assume that the zero sets of f and g are as indicated in Fig. 0. Our model is an example of the modified Rosenzweig–MacArthur equations (see [4]).

C. *Travelling Waves*

We seek solutions of (1) which depend on the single variable $\xi = x - \theta t$, where θ is a constant, called the wave-speed. Substituting in (1) leads to a system of first-order ordinary differential equations in $\mathbb{R}^4$; namely,

$$\begin{aligned} \dot{u} &= w, & \dot{v} &= z, \\ \varepsilon^2 \dot{w} &= \varepsilon\theta w - uf(u, v), & \dot{z} &= -\theta z - vg(u, v). \end{aligned} \tag{2_ε}$$

Here the "dot" represents $d/d\xi$. We shall use the notation $U = (u, w)$ and $V = (v, z)$.

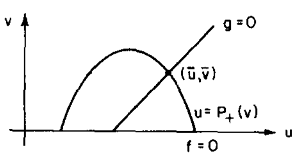

FIGURE 0

It will be convenient to change to a slow time scale by introducing the variable $\eta = \varepsilon\xi$. With this change of variables, $(2)_\varepsilon$ becomes

$$\begin{aligned} \tilde{u}' &= \tilde{w}, & \tilde{v}' &= \tilde{z}, \\ \tilde{w}' &= -\theta\tilde{w} - \tilde{u}f(\tilde{u}, \tilde{v}), & \tilde{z}' &= -\varepsilon\theta\tilde{z} - \varepsilon^2\tilde{v}g(\tilde{v}, \tilde{u}), \end{aligned} \tag*{$(3)_\varepsilon$}$$

where the "prime" is $d/d\eta$. We shall use the notation $\tilde{U} = (\tilde{u}, \tilde{w})$, $\tilde{V} = (\tilde{v}, \tilde{z})$. Note that $\tilde{u} = u$, $\tilde{w} = \varepsilon w$, $\tilde{v} = v$ and $\tilde{z} = \varepsilon z$. If R is a region in U (resp. V) space, $\tilde{R}$ will denote the corresponding region in $\tilde{U}$ (resp. $\tilde{V}$) space, and vice versa. Similarly, N and $\tilde{N}$ will be used to denote corresponding regions in (U, V) and $(\tilde{U}, \tilde{V})$ space, respectively.

D. *Hypotheses*

Equations $(2)_0$ admit solutions in which u jumps from $u = 0$ to $u = P_+(v)$, the right-hand branch of the "parabola" in Fig. 1. There are, of course, many such solutions; however, the only ones which represent "realistic" behavior are those which can be recovered as limits of solutions of $(2)_\varepsilon$ as ε tends to zero. In order to locate such solutions, we make the following reasonable hypotheses.

There exists a unique value $\sigma_0 > 0$ such that $\qquad (\mathrm{H}_1)$

$$\int_0^{P_+(\sigma_0)} uf(u, \sigma_0)\, du = 0.$$

The v-component of the rest point P in Fig. 0 is larger than σ_0. $\qquad (\mathrm{H}_2)$

For each θ, $-1 \leqslant \theta < 0$, there exist unique values $\sigma_i = \sigma_i(\theta)$, $i = 1, 2$, such that $0 < \sigma_1 < \sigma_0 < \sigma_2$, and the equations $\qquad (\mathrm{H}_3)$

$$\tilde{u}' = \tilde{w}, \qquad \tilde{w}' = -\theta\tilde{w} - \tilde{u}f(\tilde{u}, \sigma_i), \tag*{$(4)_i$}$$

admit bounded solutions which, for $(4)_1$ connect $(0, 0)$ at $\eta = -\infty$ to $(P_+(\sigma_1), 0)$ at $\eta = +\infty$, and which for $(4)_2$ connect $(P_+(\sigma_2), 0)$ at $\eta = -\infty$ to $(0, 0)$ at $\eta = +\infty$.

A few words are in order concerning these hypotheses. Condition (H_1) is

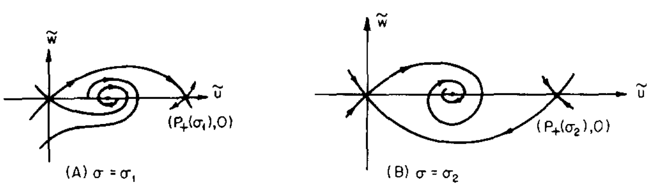

FIGURE 1

really a condition on the parameterised family of equations $\tilde{u}' = \tilde{w}$, $\tilde{w}' = \tilde{u}f(\tilde{u}, \sigma)$; it states that there is a unique $\sigma = \sigma_0$ such that there is a heteroclinic orbit connecting the two critical points $(0, 0)$ and $(P_+(\sigma_0), 0)$, i.e., σ_0 determines a unique "equal-area" function. Condition (H_2) ensures us that the projection $\bar{v}$ of the critical point $P(\bar{u}, \bar{v})$ of $(2)_\varepsilon$, on the v-axis, lies in the region $v > \sigma_0$. Finally (H_3) implies that the equations $\tilde{u}' = \tilde{w}$, $\tilde{w}' = -\theta\tilde{w} - \tilde{u}f(\tilde{u}, \sigma_i)$ have phase planes as depicted in Fig. 1.

These hypotheses ensure that a discontinuity in u for Eqs. $(2)_0$ can be approximated by a transition layer in u for Eqs. $(2)_\varepsilon$, for small ε; intuitively, u must ride across from $P_+(v)$ to $u = 0$, and back, via the connecting orbits described in (H_3); cf. Fig. 1, and also Fig. 2, where the "singular" orbit is depicted. For example, if $f(u, v) = -v - (u-1)(u-3)$, and $g(u, v) = -v + m(u - \gamma)$, $m > 0$, $1 < \gamma < 3$, then we find $P_+(v) = 2 - \sqrt{1 - v}$, $\sigma_0 = 5/9$, and one easily checks by elementary phase plane methods, that hypotheses (H_1)–(H_3) are valid, for a suitable range of m and γ.

We require one additional hypothesis, whose relevance cannot be seen until the next section. We must first introduce some notation.

Let

$$g_-(v) = vg(0, v), \qquad g_+(v) = vg(P_+(v), v), \qquad h_\sigma(u) = uf(u, \sigma),$$

and note that $g_+(\sigma_0) < 0 < g_+(\sigma_0)$. Next, let

$$G_\pm(v) = \int_0^v g_\pm(s)\, ds, \qquad \Sigma^{-1} = -\int_0^{P_+(\sigma_0)} sf_v(s, \sigma_0)\, ds,$$
$$H_\sigma(u) = \int_0^u h_\sigma(s)\, ds. \tag{5}$$

Observe that $\Sigma > 0$, $G_-(\sigma_0) < 0$ and that $H_{\sigma_0}(u) \leqslant 0$, $0 \leqslant u \leqslant P_+(\sigma_0)$. Next, let

$$\rho_v = \int_0^{\sigma_0} \sqrt{-2G_-(s)}\, ds, \qquad \rho_u = \int_0^{P_+(\sigma_0)} \sqrt{-2H_{\sigma_0}(s)}\, ds. \tag{6}$$

Now consider the equations $\dot{v} = z$, $\dot{z} = -\theta z - vg_-(v)$. We consider the solution γ which passes through $(0, 0)$, and we let (σ_1, e) be the point on γ

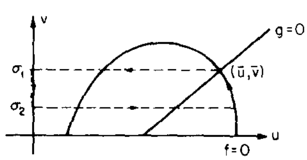

FIGURE 2

which passes through the line $v = \sigma_1$. Observe too that on γ we have, at $v = \sigma_0$, $z^2 + 2G_\pm(\sigma_0) = 0$. Our last condition is to require

$$\rho_v - 2e\Sigma \Delta \rho_u > 0, \tag{H$_4$}$$

where Δ is evaluated along γ, e is as in Figure 5, and

$$\Delta = |dz/dv|_{v=\sigma_0} = \left|\frac{-\theta z - g_\pm(\sigma_0)}{z}\right| = \left|-\theta - \frac{g_\pm(\sigma_0)}{\sqrt{-2G_\pm(\sigma_0)}}\right|.$$

We note that Δ is nothing more than the largest slope of γ on the interval $0 \leqslant v \leqslant \sigma_1$. This condition may appear difficult to understand, but we shall see in Section 2C that it is really quite natural, and it is obtained from elementary phase-plane considerations.

Finally, we call attention to the fact that if we modify Eqs. (1) by replacing the term vg by λvg, then (H$_4$) will hold for sufficiently small λ. To see this, note that the "new" quantities are

$$\tilde{\rho}_v = \sqrt{\lambda}\,\rho_v, \qquad \tilde{e} = c\sqrt{\lambda} + O(\theta), \qquad c > 0, \qquad \tilde{\Delta} = O(|\theta| + \sqrt{\lambda}).$$

Thus, for these quantities, (H$_4$) becomes

$$\sqrt{\lambda}\,[\rho_v - 2(c + O(\theta))\,\Sigma O(|\theta| + \sqrt{\lambda})\,\rho_u],$$

which is positive, if λ and θ are small.

Thus, for the modified equations, (H$_4$) is valid if λ and θ are small. This modfication of the equations by the addition of the parameter λ is the actual form of the equations considered in [5, 6]. It is required there that λ be sufficiently small. The methods of [5, 6] are analytical, and their condition is used in order to apply the implicit function theorem.We shall see below in Section 2C that our condition is obtained from phase plane considerations.

E. *The Main Theorem*

Under hypotheses (H$_0$)–(H$_4$), the following theorem will be proved.

THEOREM A. *Suppose that $0 < -\theta \ll 1$. Then there exists a (singular) periodic solution of $(2)_0$, in which the u-component satisfies alternatively $u = 0$ or $u = P_+(v)$, and admits discontinuities according to the following rule: u jumps from 0 to $P_+(\sigma_1)$ whenever $v = \sigma_1$ and $v' > 0$, and u jumps from $P_+(\sigma_2)$ to 0, whenever $v = \sigma_2$ and $v' < 0$. The (v, v') components of this solution are continuous.*

B. *For sufficiently small $\varepsilon > 0$, there exists a periodic solution of $(2)_\varepsilon$, "near" the singular solution described in* A. *More precisely, there are periodic solutions of $(2)_\varepsilon$ which converge to the singular solution of $(2)_0$ as*

$\varepsilon \to 0$. *This convergence is uniform on compact ξ-intervals which are disjoint from those ξ values in which the singular solution is discontinuous.*

2. Solutions with Discontinuities

A. *The Singular Flow*

The u-component of $(2)_0$ must satisfy the algebraic equation $uf = 0$, whenever the v component satisfies

$$\int_{-\infty}^{\infty} (v\ddot{\phi} - \theta\dot{\phi}v + vg(u, v)\phi)\, d\xi = 0,$$

for every test function $\phi(\xi)$. It follows easily from this, that $(v, \dot{v})$ is continuous, even though u may not be.

Now consider the equations

$$\dot{v} = z, \qquad \dot{z} = -\theta z - g_\pm(v), \tag{7$_\pm$}$$

where $g_\pm(v)$ are defined in Section 1D. The phase planes for $(7)_\pm$ are depicted in Fig. 3. Suppose now that u admits discontinuities according to the rule described in part A of the theorem stated in Section 1E. We may then "glue" the above two phase planes together as indicated in Fig. 4. (We have assumed for definiteness that the orbit OR meets an orbit in the $(+)$ plane when $v = \sigma_1$ which lies below that orbit in the $(+)$ plane which tends to the rest point on the positive v-axis. This assumption can be reversed without significantly altering the proof.)

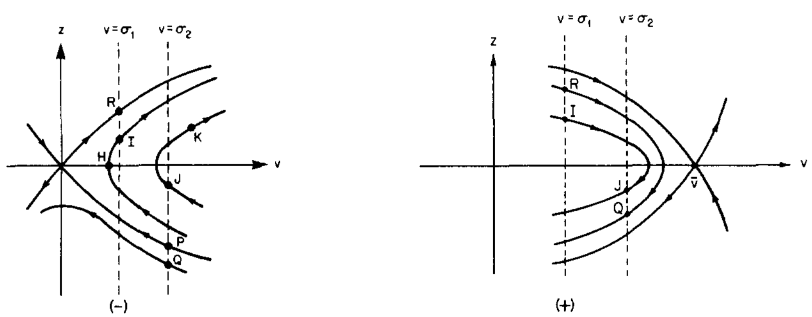

Figure 3

B. *Construction of a Periodic Solution*

Assume for a moment that the orbit segments *PORQ* and *GHIJ* have the form indicated in Fig. 4. Then the orbit segment *PORQ* exits $v > 0$, and never again reaches $v = \sigma_1$, with $v' > 0$. Also, the orbit segment *GHIJ* never reaches $v = \sigma_1$ after it reaches *J*, and at subsequent values of ξ, the orbit enters, and remains in the positive quadrant. It is now easily seen from the continuity of the (v, v') flow that these two orbits are separated by a periodic solution.

It is clear by changing *G* (or *J*) in an appropriate manner that an orbit such as *GHIJ* must always exist. However, it is not always the case that "*Q*" lies below "*P*" on the orbit *PORQ*; clearly this property is essential to the above argument. It is here where we shall use (H_4).

C. *Behavior of PORQ for Small θ*

We shall compare the solutions of $(7)_\pm$ with those of the equations

$$\ddot{v} + g_\pm(v) = 0. \tag{8}$$

The former equations are gradient-like, while the latter are Hamiltonian. The relevant solutions are depicted in Fig. 5, where the dotted curves are solutions of (8), while the heavy curves are solutions of $(7)_\pm$ with $\theta < 0$.

If $E_\pm(v, \dot{v}) = \dot{v}^2/2 + G_\pm(v)$, where $G'_\pm = g_\pm$, then $\dot{E}_\pm = 0$ along solutions of (8) and $\dot{E}_\pm = -\theta\dot{v}^2 > 0$ along solutions of $(7)_\pm$ with $\theta < 0$. This implies that the solid curves cross the dotted ones transversally, in the manner indicated in Fig. 5.

The *z*-coordinate of the various points along the lines $v = \sigma_1$ or $v = \sigma_2$ are indicated by the letters *a–e* and A, B, C respectively. Our goal will be to show that $A > C$. This will be the case if $e > b$, since then, the orbit through S_4, S_5, S_6 will start above the orbit (with $\theta = 0$) through S_1, S_2, S_3, and by the remark in the previous paragraph the former orbit can never cross the latter.

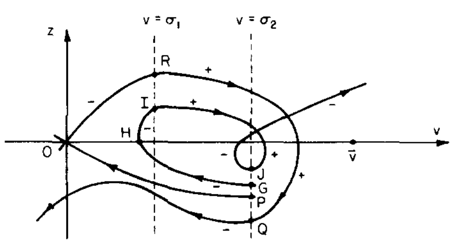

FIGURE 4

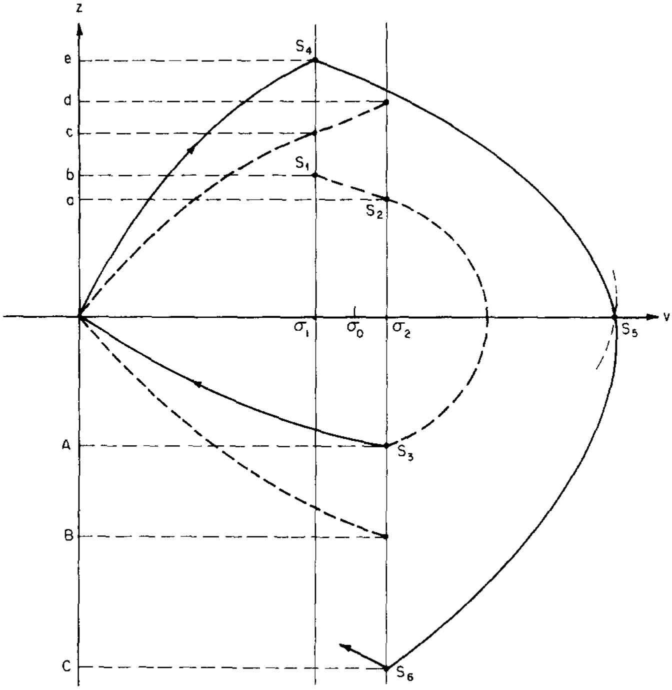

FIGURE 5

We begin by estimating $e-c$ and $A-B$. Now $E_-(\sigma_1, c) = e^2/2 + G_-(\sigma_1) = e^2/2 - c^2/2$, so

$$(e^2 - c^2)/2 = E_-(\sigma_1, c) - E_-(0,0) = -\theta \int_{-\infty}^{\xi} \dot{v}(\xi)^2 \, d\xi,$$

where $v(\xi_1) = \sigma_1$. Since v is montone for $-\infty < \xi < \xi_1$, we can invert $v = v(\xi)$, as $\xi = \xi(v)$, with $\dot{\xi} = 1/\dot{v}$. Thus

$$(e^2 - c^2)/2 = -\theta \int_0^{\sigma_1} \dot{v}(v) \, dv.$$

Now if θ is small, $\dot{v}(v) = \sqrt{-2G_-(v)} + O(\theta)$. Also, since $e > c$, $(e+c)/2 \leqslant e$. Thus, with ρ_v as in (6), it follows that

$$e - c \geqslant |\theta| \, e^{-1} \rho_v + O(\theta^2). \tag{9}$$

Clearly a similar estimate holds for $A - B$, with an even larger ρ_v.

We next obtain an estimate for $\sigma_0 - \sigma_1$. To this end, we consider the equation

$$-\theta \tilde{u}' = \tilde{u}'' + h_\sigma(\tilde{u}), \tag{10}$$

where, we recall, $h_\sigma(\tilde{u}) = \tilde{u} f(\tilde{u}, \sigma)$. Now let $\Lambda(\tilde{u}, \tilde{u}') = (\tilde{u}')^2/2 + H_\sigma(\tilde{u})$ (see (5)). Then if $(\tilde{u}_1, \tilde{u}_1')$ is the solution of (10) with $\sigma = \sigma_1$, which connects 0 to $P_+(\sigma_1)$ (cf. (H_3)), we have that

$$H_{\sigma_1}(P_+(\sigma_1)) = \Lambda(P_+(\sigma_1), 0) - \Lambda(0, 0) = -\theta \int_{-\infty}^{\infty} [\tilde{u}_1'(\eta)]^2 \, d\eta.$$

Let $\Gamma(\sigma) = H_\sigma(P_+(\sigma))$. Then by definition of σ_0 (see (H_1), $\Gamma(\sigma_0) = 0$. Moreover,

$$\Gamma(\sigma_1) = \Gamma(\sigma_1) - \Gamma(\sigma_0) = \Gamma'(\sigma_*)(\sigma_1 - \sigma_0),$$

for some σ_*, $\sigma_1 < \sigma_* < \sigma_0$. Since $f(P_+(\sigma), \sigma) = 0$, and $f_v < 0$, we have $\Gamma'(\sigma) = \int_0^{P_+(\sigma)} s f_v(s, \sigma) \, ds < 0$. Thus,

$$|\sigma_1 - \sigma_0| \leqslant \Sigma \, |\theta| \int_{-\infty}^{\infty} |\tilde{u}_1'(\eta)|^2 \, d\eta, \qquad \Sigma = |\Gamma'(\sigma_0)^{-1}|.$$

Now we invert $\tilde{u} = \tilde{u}_1(\eta)$, to get $\eta = \eta_1(\tilde{u})$. If $(\tilde{u}_0(\eta), \tilde{u}_0'(\eta))$ is the orbit which connects 0 to $P_+(\sigma_0)$ when $\theta = 0$ (see (H_3)), we can also write $\tilde{u} = \tilde{u}_0(\eta)$ as $\eta = \eta_0(\tilde{u})$. Hence, there exists a $K > 0$ such that

$$|\tilde{u}_1'(\tilde{u}) - \tilde{u}_0'(\tilde{u})| < K \, |\theta|, \qquad 0 \leqslant \tilde{u} \leqslant P_+(\sigma_1). \tag{11}$$

Thus

$$\begin{aligned} \int_{-\infty}^{\infty} [\tilde{u}_1'(\eta)]^2 \, d\eta &= \int_0^{P_+(\sigma_1)} \tilde{u}_1'(u) \, du = \int_0^{P_+(\sigma_1)} \tilde{u}_0'(u) \, du + O(\theta) \\ &= \int_0^{P_+(\sigma_0)} \tilde{u}_0'(u) \, du + O(\theta), \end{aligned}$$

where the last inequality follows from (11). Since $\tilde{u}_0'(\tilde{u}) = \sqrt{-2H_{\sigma_0}(\tilde{u})}$, we have that

$$|\sigma_1 - \sigma_2| \leqslant \Sigma \, |\theta| \, \rho_u + O(\theta^2),$$

where ρ_u is as in (H_4). Similarly, we can show $|\sigma_2 - \sigma_0| \leqslant \Sigma \, |\theta| \, \rho_u + O(\theta^2)$, and thus we obtain

$$|\sigma_1 - \sigma_2| \leqslant 2\Sigma \, |\theta| \, \rho_u + O(\theta^2). \tag{12}$$

We can now show that $c > a$. Since solutions of (7) with $\theta = 0$ are symmetric about $v = 0$, we have $A - B = d - a$. This, combined with (9) and (12) yields

$$d - c \leqslant \Delta\,|\sigma_1 - \sigma_2| \leqslant 2\Delta\,|\theta|\,\rho_u + O(\theta^2),$$

and

$$d - a = A - B \geqslant e^{-1}\rho_v\,|\theta| + O(\theta^2).$$

From these we obtain

$$c - a \geqslant |\theta|\,(\rho_v e^{-1} - 2\Delta\Sigma\rho_u) + O(\theta^2).$$

It follows from (H_4), that $c - a > 0$ for small θ.

We shall now show that $e > b$ to complete the proof. First, $b \leqslant a + \Delta\,|\sigma_1 - \sigma_2|$, and from (9),

$$e - a = (e - c) + (c - a) \geqslant e - c \geqslant |\theta|\,e^{-1}\rho_v + O(\theta^2).$$

Using (12), we get

$$a - b \geqslant -\Delta\,|\sigma_1 - \sigma_2| \geqslant -2\Delta\Sigma\,|\theta|\,\rho_u + O(\theta^2).$$

Combining these last two inequalities gives

$$e - b \geqslant |\theta|\,(e^{-1}\rho_v - 2\Delta\Sigma\rho_u) + O(\theta^2).$$

Thus (H_4) implies that $e - b > 0$ for small θ.

3. The Perturbed Flow

We shall construct a neighborhood N_ε, in the phase space of $(2)_\varepsilon$, "near" the discontinuous solution of $(2)_0$, which we have just obtained. For each $\varepsilon > 0$, N_ε will be compact; however, the w coordinate will grow without bound as ε tends to zero, since the u-component of the limiting solution has jump discontinuities.

N_ε will consist of the union of four tubular regions in $\mathbb{R}^4$; these will be denoted by T_i, $1 \leqslant i \leqslant 4$. In regions T_1 and T_2, the flow will closely approximate the flows of $(7)_-$ and $(7)_+$, respectively. In regions T_3 and T_4, v remains near σ_1 and σ_2, respectively, and u undergoes a rapid transition. N_ε will be homeomorphic to $S^1 \times D^3$.

A. *The Conley Index*

Suppose there is given a flow on a compact space X. Let N be a compact neighborhood in X, and let $S(N)$ denote all those points on solution curves which stay in N for all time. If $S(N) \subset \operatorname{int}(N)$, then N is called an *isolating neighborhood*, and $S(N)$ is an *isolated invariant set.*

A homotopy invariant can be associated with $S(N)$ as follows (see [2] for details): There exist compact subsets N_1 and N_2 of N such that

(i) N_i is positively invariant relative to N, $i = 1, 2$.

(ii) $S(N) \subset N_1 \backslash N_2$.

(iii) Any orbit in N, which leaves N in positive time meets N_2.

Then (N_1, N_2) is called an *index pair* and the homotopy type of N_1/N_2 is called the *Conley Index* of $S(N)$, and is denoted by $h(S(N))$. $h(S(N))$ is independent of the pair (N_1, N_2), and it is invariant under deformations of both the flow, and N, provided that $S(N)$ remains interior to N throughout the deformation. If $S(N)$ is empty, then $N_1 = N = N_2$ determines an index pair, so that the index is the homotopy type of a (pointed) point which we denote by $\bar{0}$. Thus, if $h(S(N)) \neq \bar{0}$ then $S(N) \neq \varnothing$.

In Section 5 we shall show that $S_\varepsilon \equiv S(N_\varepsilon)$ has index $\Sigma^2 \vee \Sigma^3$, the topological sum of a pointed two-sphere and a pointed three-sphere. Thus $S_\varepsilon \neq \varnothing$; this will imply the existence of the desired periodic solution.

B. *Construction of* T_1

When $\varepsilon = 0$ and $u = 0$, the (u, z) plane is depicted in Fig. 3(−). We shall use this to construct T_1. Let $\mathscr{B}$ be the rectangle in (u, v) spaced depicted in Fig. 6a. The point v_1 should be close enough to 0 so that the orbit of the singular flow (Fig. 3) through $(v_1, 0)$ remains near the curve ORQ and therefore exits $v \geqslant 0$ near L. v_2 should be so large that the orbit segment RQ is contained in $\mathscr{B}$.

Let A_0 be the "diamond" indicated in Fig. 6b. It can easily be shown that for small $|\theta|$, and any $\varepsilon > 0$, such an A_0 can be constructed; that is, the flow of the (u, w) components of $(2)_\varepsilon$ through any point with $(v, z) \in \mathscr{B}$ and

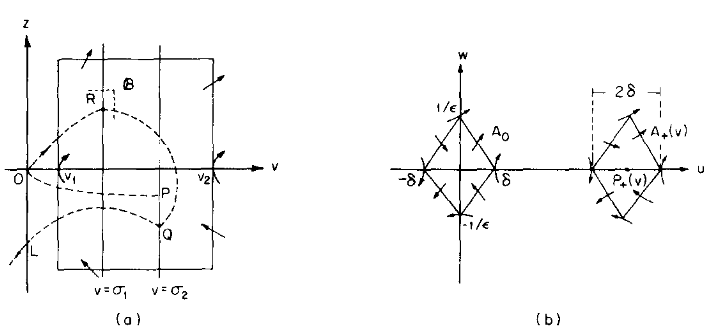

FIGURE 6

$(u, w) \in \partial A_0$, are indicated. Indeed, A_0 is mapped into a similar diamond $\tilde{A}_0$ in the $(\tilde{u}, \tilde{w})$ plane, with vertices $(0, \pm 1)$ on the $\tilde{w}$-axis. It is easily checked that $(0, 0)$ is a hyperbolic rest point of the $(\tilde{u}, \tilde{w})$ flow of $(3)_0$, and that if $|\theta|$ is small, the flow is always transverse to the edges of the diamond. Hence this property holds for $(3)_\varepsilon$ for small $\varepsilon > 0$, and the (u, w) components of $(2)_\varepsilon$ cross ∂A_0 as indicated.

The flow of $(2)_0$ with $u = 0$ through any face of $\partial \mathscr{B}$ (with the exception of $v = v_1$) is either transvese to $\partial \mathscr{B}$ or is externally tangent to it. If $\delta > 0$ in Fig. 6b is small enough, a similar remark can be applied to the (v, z) components of $(2)_\varepsilon$, provided $(u, w) \in A_0$.

Now we set $T_1 = A_0 \times \mathscr{B}$. The only points on ∂T_1 which may have internal tangencies are those with $v = v_1$. This will be discussed in Section 4B.

C. *Construction of* T_2

We now assume that when $\varepsilon = 0$, the u coordinate is restricted to lie on $P_+(v)$. The (v, z) flow of $(2)_0$ is as in Fig. 3(+). If $u = P_+(v)$ were a vertical line beneath the rest point P in Fig. 2, then the construction of $T_2 = A_+ \times \mathscr{B}$ (see Fig. 6) would be analogous to that of T_1, since in this case $u = P_+(v)$ is constant. If $u = P_+(v)$ is non-constant (as in Fig. 1), T_2 is modified as follows. For each $\bar{v}$, $v_1 \leqslant \bar{v} \leqslant v_2$, let $\mathscr{B}(\bar{v}) = \{(v, z) \in \mathscr{B} : v = \bar{v}\}$, and let $A_+(\bar{v})$ be the diamond in Fig. 6b centered at $P_+(\bar{v})$. We set

$$T_2 = \bigcup_{v_1 \leqslant \bar{v} \leqslant v_2} A_+(\bar{v}) \times \mathscr{B}(\bar{v});$$

T_2 is homeomorphic to a four-cell, but it is no longer the product of 2 two-cells. Note that $T_1 \cap T_2 = \varnothing$.

The discussion of the flow $(2)_\varepsilon$ through points on ∂T_2 will be discussed in Section 4.

D. *Construction of* T_3 *and* T_4

When the (u, w) components of $(2)_\varepsilon$ are in transition from $u = 0$ to $u = P_+(v)$, it is reasonable to expect that the $(\tilde{u}, \tilde{w})$ components of $(3)_\varepsilon$ remain near the connecting solution of $(4)_1$. Let $\tilde{\mathscr{R}}$ be the compact neighborhood of this orbit as depicted in Fig. 7a, when the top boundary is an orbit of the equations $\tilde{u}' = \tilde{w}$, $\tilde{w}' = -\tilde{u}f(\tilde{u}, \sigma_0)$. Let $\mathscr{R}$ be the corresponding region in (u, w) space. We assume that A_0 is interior to $\mathscr{R}$. If $\tilde{v}$ is constant, then solutions of $(3)_0$ either cross $\partial \tilde{\mathscr{R}}$ transversally, or they are externally tangent to $\partial \tilde{\mathscr{R}}$. Since $\tilde{z} = \varepsilon z$ is of order ε, it follows that solutions of $(3)_\varepsilon$ whose $(\tilde{u}, \tilde{w})$ components meet $\partial \tilde{\mathscr{R}}$ also exit $\tilde{\mathscr{R}}$ in at least one time direction.

We let $T_3 = \mathscr{R} \times D$, where D is the region depicted in Fig. 7b. We note

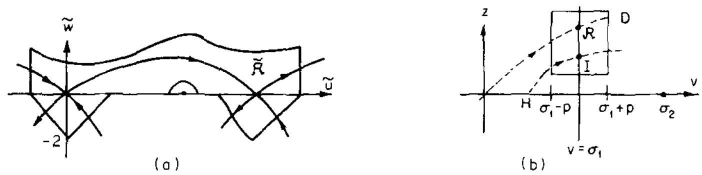

FIGURE 7

that $D \subseteq \mathscr{R}$ and the top (resp. bottom) of D lies above the orbit OR (resp. below the orbit HI); these being orbits of $(7)_-$; see Fig. 4. Note too that $z > 0$ in D, and that v lies within ρ of σ_1, where ρ is chosen smaller than $\sigma_0 - \sigma_1/2$.

In a similar manner, we set $T_4 = \mathscr{S} \times E$, where $\tilde{\mathscr{S}}$ and E are depicted in Fig. 8. Note that $z < 0$ in E. We also required v to stay within ρ of σ_2. From this we see that $T_3 \cap T_4 = \varnothing$.

4. An Isolating Neighborhood

We let $N_\varepsilon = \bigcup_{1 \leqslant i \leqslant 4} T_i$, and set $S_\varepsilon = S(N_\varepsilon)$. We shall show that $S_\varepsilon \cap \partial N_\varepsilon = \varnothing$ for sufficiently small $\varepsilon > 0$; this will imply that N_ε is an isolating neighborhood for $(2)_\varepsilon$.

Let $p = (u_0, w_0) \times (v_0, z_0) \in \partial N_\varepsilon$; there are several cases to be checked; namely,

(i) $p \in \partial T_1 \cup \partial T_2$,

(ii) $p \in \partial T_3 \cup \partial T_4$,

(iii) p lies in at least two of the T_i's. (Since $T_1 \cap T_2 = \varnothing = T_3 \cap T_4$, we only need check the points in $T_i \cap T_j$, $i = 1, 2$; $j = 3, 4$.)

A. *Two Lemmas*

We will now show that the (v, z) components of any solution in S_ε closely approximate the singular flow in Fig. 4. This will imply that internal

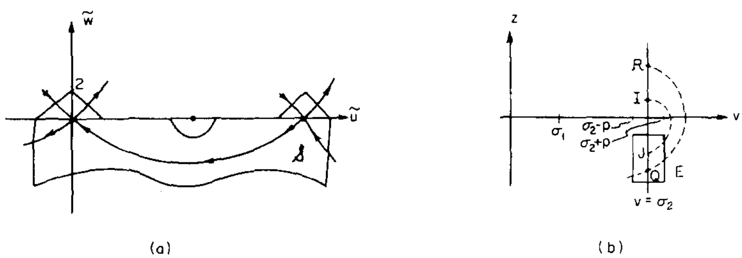

FIGURE 8

tangencies in N_ε, such as the one depicted in Fig. 6a through $(v_1, 0)$, do not lie in S_ε, since points in the singular flow exit $\mathscr{R}$ in finite time.

LEMMA 1. *If $\varepsilon > 0$ is sufficiently small, then solution curves in S_ε traverse N_ε through the cycle of tubes in the order T_1, T_3, T_2, T_4, T_1, in the positive direction, without "turning around."*

Proof. First, it is easy to see that a solution in S_ε cannot remain in any one of the T_i's for all time. Thus, if such a solution stayed in T_1 or T_2 this would force u to be near 0 or $P_+(v)$, $v_1 \leqslant v \leqslant v_2$, respectively, for all time, and so the (v, z) components would then approximate the singular solution flows in $(7)_-$ or $(7)_+$. But such solutions eventually exit N_ε in the positive (resp. negative) time direction if they are in T_1 (resp. T_2). On the other hand, if such a solution remained in T_3 or T_4 for all time, v would have to stay near σ_1 or σ_2. Hence the corresponding solution in the slow coordinates $(3)_\varepsilon$ would remain near solutions of $(3)_0$, as long as the $(\tilde{u}, \tilde{w})$ components stayed bounded away from the two rest points in the $(\tilde{u}, \tilde{w})$ plane. Since any such solution is carried out of the region $\tilde{\mathscr{R}}$ (in Fig. 7a) in finite time, the corresponding orbit in S_ε would leave N_ε. This is impossible. Thus, solutions in S_ε ultimately enter T_1 and T_2.

Suppose that a solution curve starting in T_3 enters $T_1 \backslash T_3$. We claim that this must occur in backwards time. To see this, first note that the solution curve must meet $T_1 \cap \partial T_3$. Since $T_1 = A_0 \times \mathscr{R}$, $T_3 = \mathscr{R} \times D$, and $A_0 \subset \mathscr{R}$, this must occur at a point (U, V) in $(A_0, \partial D)$. For such points (U, V), $\dot{v} = z$ and $\dot{z} = -\theta z - vg(U)$ are both positive. Thus, if the orbit entered T_1 and exited T_3 in forward time, V would lie in the "top" or "right" side of D (Fig. 7b). From the above equations for $\dot{v}$ and $\dot{z}$, is clear that the components of V continue to increase in forward time, as long as u is constrained to lie in a small neighborhood of zero. Since V will now lie increasingly above and to the right of D, the solution can never get back to T_3 (the orbit must leave T_1 in forward time, and can only do this via entering T_3). Thus the orbit goes from T_1 to T_3 in positive time. The order of the flow in the remaining cases, from T_3 to T_2 to T_4 to T_1, is proved in a similar manner. The proof is complete.

Now suppose that $S_\varepsilon = \varnothing$ for all sufficiently small $\varepsilon > 0$. Then of course, all these N_ε are isolating neighborhoods. Thus, assume $S_{\varepsilon_n} \neq \varnothing$ for a sequence ε_n tending to zero, and let $\gamma_n(\xi) = (U_n(\xi), V_n(\xi))$ be a solution curve in S_{ε_n}. From Lemma 1, we know that $\gamma_n(\xi)$ passes through all four of the tubes T_i, and so it must meet a point in T_1 with $z_n = 0$. We re-parametrize γ_n so that $z_n(0) = 0$.

Since $|\dot{V}_n(\xi)|$ is uniformly bounded, $\{V_n(\xi)\}$ has a convergent subsequence, which converges to a limit $V_0(\xi)$, uniformly on compact ξ-intervals.

LEMMA 2. *$V_0(\xi)$ consists of solution segments of* $(7)_-$ *and* $(7)_+$. *The transition from* $(7)_-$ *to* $(7)_+$ *occurs when* $v_0(\xi)=\sigma_1$ *and* $z_0(\xi)<0$.

Proof. Since $V_n(0)\in \mathscr{B}\backslash D$, we have that $U_n(0)\in A_0$, so that $\dot z_n(0)>0$, and $z_n(\xi)>0$, at least until $v_n=\sigma_1$ (since $z_n>0$ when $V_n\in D$). Thus, $v_0(\xi)$ is also strictly increasing until $v_0=\sigma_1$. Let $\xi_1>0$ be the smallest $\xi>0$ for which $v_0(\xi)=\sigma_1$.

We claim that $V_0(\xi)$ is a solution of $(7)_-$ for $0\leqslant\xi<\xi_1$. This will be immediate if we show that $\lim u_n(\xi)=0$ as $n\to\infty$, uniformly on compact subsets of $[0,\xi_1)$.

First, let $\bar\xi\in(0,\xi_1)$ be such that $V_0(\bar\xi)\notin D$. Then $V_n(\xi)\notin D$ for n sufficiently large, and for $0\leqslant\xi\leqslant\bar\xi$. We shall first show that $u_n(\xi)\to 0$, pointwise if $\xi\in[0,\bar\xi]$. If this were false, then there would exist $\xi^*\in[0,\bar\xi]$, and a subsequence of $U_n(\xi^*)$ (again denoted by $U_n(\xi^*)$), such that $\lim u_n(\xi^*)=\alpha\neq 0$. Note that $0<|\alpha|<\delta$ (cf. Fig. 6b), since $U_n(\xi^*)\in A_0$ for all n. Let $\eta=(\xi-\xi^*)\varepsilon$, and let $(\tilde U_n(\eta),\tilde V_n(\eta))$ be the solution of $(3)_{\varepsilon_n}$ corresponding to $\gamma_n(\xi)$. Since $|\tilde U_n'(\eta)|$ and $|\tilde V_n'(\eta)|$ are both uniformly bounded, some subsequence, again denoted by $(\tilde U_n,\tilde V_n)$, converges uniformly to a limit $(\tilde U_0(\eta),\tilde V_0(\eta))$, on compact η-intervals. It follows that $\tilde V_0=(\tilde v_0,0)$, where $\tilde v_0$ is a constant, $\tilde v_0<\sigma_1$, and that $(\tilde U_0(\eta),\tilde V_0(\eta))$ is a solution of $(3)_0$. Since $\tilde u_0(0)=u_0(\xi^*)=\alpha$, and $\tilde v_0<\sigma_1$, $|\tilde u(\eta)|$ must assume values greater than $-\delta$ at some finite time, in at least one time direction. This can be seen from Fig. 9. Since solutions of $(3)_{\varepsilon_n}$ remain near solutions of $(3)_0$ on finite η-intervals, it follows that the $\tilde U_n$ components exit the projection of $\tilde N_\varepsilon$ in the U-plane, in finite time, and so the solution curves cannot lie in S_{ε_n}. This contradiction proves the pointwise convergence of $u_n(\xi)$ to zero, $0\leqslant\xi\leqslant\bar\xi$.

We now show that the convergence is uniform. From what we have just shown, given any $\mu>0$, there is an N such that $n>N$ implies that $|u_n(\xi)|<\mu$ if $\xi=0$ or $\xi=\bar\xi$. Note that since $U_n(\xi)\in A_0$, $0\leqslant\xi\leqslant\bar\xi$, $|u_n(\xi)|\leqslant\delta$. If the convergence were non-uniform, then for each μ, there would exist $n>N$ and $\xi\in(0,\bar\xi)$ such that $|u_n(\xi)|>\mu$. Thus, at some $\xi^*\in(0,\bar\xi)$, u_n would assume either a positive local maximum, or a negative local minimum. In the former

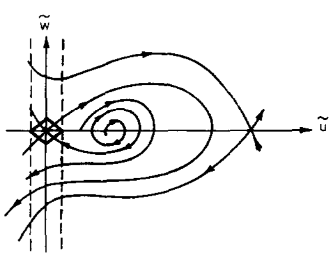

FIGURE 9

case, at this point $u_n'' = v_n' = -\varepsilon_n^{-2} u_n f(U_n) > 0$, and in the latter case, $u_n'' < 0$. Both of these are impossible so that the convergence is uniform.

Now suppose that $\bar{\xi} \in (0, \xi_1)$, and that $V_n(\bar{\xi}) \in D$. We know that $v_0(\bar{\xi}) < \sigma_1$ and $v_n(\xi) < \sigma_1$ for large n. If $U_n(\bar{\xi}) \in A_0$ for all large n, then as before, $u_n \to 0$ uniformly on $[0, \bar{\xi}]$. But it is also a priori possible that $U_n(\bar{\xi}) \in \mathscr{R} \backslash A_0$, for n large. We shall show that this is actually impossible to complete the proof.

By Lemma 1, any solution with $U_n(\xi) \in \mathscr{R} \backslash A_0$ and $V_n(\xi) \in D$ must enter T_1 in backwards time, and so there exists $\xi_n < \bar{\xi}$ such that $U_n(\xi_n) \in \partial A_0$. Let $\eta = (\xi - \xi_n)\varepsilon$; if $(\tilde{U}_n(\eta), \tilde{V}_n(\eta))$ is the associated solution of $(3)_{\varepsilon_n}$, then $\tilde{U}_n(0) \in \partial \tilde{A}_0$ for all n. Again since $|\tilde{U}_n'|$ and $|\tilde{V}_n'|$ are uniformly bounded, a subsequence converges to a solution $(\tilde{U}_0(\eta), \tilde{V}_0(\eta))$ of $(3)_0$, uniformly on compact η-intervals. Thus, $\tilde{U}_0(0) \in \partial \tilde{A}_0$, and $\tilde{v}_0 < \sigma_1$. Again from Fig. 9, it is evident that this solution exits the projection of $\tilde{N}_{\varepsilon_n}$ in $\tilde{U}$-space, and thus such a solution of $(3)_{\varepsilon_n}$ cannot be in S_n. Thus $u_n \to 0$ uniformly on compact subsets of $[0, \xi_1)$.

From Lemma 1, we see that solutions in S_{ε_n} eventually enter T_2; in order to get from T_2 to T_3, $z_n(\xi)$ must cross zero while the solution curve is in T_2; let $\xi_2 > \xi_1$, be the smallest such value of ξ. An argument analogous to the above shows that $V_0(\xi)$ is a solution of $(7)_+$ for $\xi \in (\xi_1, \xi_2]$. Finally, it is clear from the symmetric character of N_ε, that this procedure can be continued for all ξ. The proof of the lemma is complete.

B. *Points on the Boundary of* T_1

Suppose that $p \in \partial T_1$; we shall show that $p \notin S_\varepsilon$ for all sufficiently small ε. In Section 3A, it was shown that every solution through such p leaves T_1 in at least one time direction, with the exception of points which lie on the face $v = v_1$ of $\mathscr{B}$. Suppose then, that there are solutions in S_ε passing through such points $p \in \partial T_1$ for a sequence of ε tending to zero. If $p_\varepsilon = (U_\varepsilon, V_\varepsilon)$, then from Fig. 6b, it is evident that $V_\varepsilon = (v_1, 0)$. From Lemma 2, we see that a subsequence of $V_\varepsilon(\xi)$ converges to a limit $V_0(\xi)$, the components of which are solution segments of $(7)_-$ and $(7)_+$, with transitional behavior as in Fig. 4. Now $V_0(\xi)$ passes through $(v_1, 0)$, and this point is close to the orbit *PORQ* in Fig. 6a. Since *PORQ* exits the projection of N_ε in V-space in finite time, the same is true of the orbit through $(v_1, 0)$. Thus $V_\varepsilon(\xi)$ exits this region of V-space in finite time for all sufficiently small ε; this solution cannot lie in S_ε.

C. *Points on the Boundary of* T_2

If $p = (U_0, V_0)$, and U_0 lies in the interior of $A_+(v_0)$, then U lies interior to $A_+(v_0)$ for all (U, V) near enough to (U_0, V_0). Thus, if V_0 is interior to $\mathscr{B}$, p is an interior point. If $V_0 \in \partial \mathscr{B}$, then the orbit eventually leaves N_ε; the proof is similar to what we have done in Section 3B above.

Suppose then that $U_0 \in \partial A_+(v_0)$; in the slow variables, $\tilde{U}_0 \in \partial \tilde{A}_+(\tilde{v}_0)$. The problem here is that $A_+(\tilde{v})$ changes as $\tilde{v}$ changes. We resolve this by noting that $|\tilde{v}'(\eta)| \leqslant K\varepsilon$ for some $K > 0$, so that $A_+(\tilde{v}(\eta))$ changes at a rate $O(\varepsilon)$. Moreover, $\tilde{U}_0 \in \partial A_+(\tilde{v}_0)$, and therefore $\tilde{U}_0$ changes at a rate $O(1)$, for small ε, since $\tilde{U}_0$ lies at a distance at least δ from the critical point $(P_+(\tilde{v}_0), 0)$ of $(3)_0$. Since the U-components of solutions of $(3)_0$ are externally tangent or transverse to $\partial A_+(\tilde{v}_0)$, it follows that the U-components of such solutions of $(3)_\varepsilon$ leave $A_+(v(\eta))$ in at least one time direction.

D. *Points on ∂T_3 or ∂T_4*

We consider only the case $p = (U_0, V_0) \in \partial T_3$; the other case is treated in a similar manner. If $U_0 \in \partial \mathscr{R}$, it has been noted in Section 3C that the solution leaves N_ε.

Now suppose that V_0 lies in the "top" or "bottom" of ∂D. The argument above in Section 3B shows that the V components of the solution through p closely approximate orbit segments of $(7)_\pm$ as in Fig. 4. Since the top (resp. bottom) of ∂D is above $PORQ$ (resp. below $GHIJ$), it follows that the solution exits N_ε, and thus $p \notin S_\varepsilon$.

If, on the other hand, V_0 lies in the "left" or "right" edge of ∂D, then since $z_0 > 0$, we must have that $U_0 \in A_0$ or $U_0 \in A_+(v_0)$ (otherwise V immediately leaves D in one time direction while U remains exterior to A_0 or $A_+(v_0)$). Such solutions therefore exit N_ε. If U_0 is interior to A_0 or $A_+(v_0)$, then p is an interior point, while if U_0 lies on the boundary of one of these sets, then $p \in \partial T_1 \cap \partial T_3$ or $p \in \partial T_2 \cap \partial T_3$; this case is treated in Section 3E below.

E. *Points in $\partial T_i \cap \partial T_j$*

We shall only check the points in $\partial T_1 \cap \partial T_3$; the other cases are similar.

First recall that D is interior to $\mathscr{B}$ and A_0 is interior to $\mathscr{R}$. It follows, then, that

$$\partial T_1 \cap \partial T_3 = (\partial A_0 \times \mathscr{B}) \cap (\mathscr{R} \times \partial D) = \partial A_0 \times \partial D.$$

Let $(U_0, V_0) \in \partial A_0 \times \partial D$; if V_0 lies in the "top" or "bottom" of ∂D, the solution exits N_ε (c.f. Section 3D above). Suppose then that V_0 lies in a vertical edge of D, say, $v = \sigma_1 - \rho$ (the other case is similar). As a first approximation of the flow of $(3)_\varepsilon$ through $(\tilde{U}_0, \tilde{V}_0)$, we set $\varepsilon = 0$, so that $\tilde{V}_0 = (\sigma_1 - \rho, 0)$. The flow of $(3)_0$ in U-space is then as in Fig. 9, and so U leaves the projection of $\tilde{N}_\varepsilon$ in U-space in finite time. Now since the flow $(3)_\varepsilon$ is an $O(\varepsilon)$ perturbation from the flow $(3)_0$, it follows that such solutions of (3) also exit $\tilde{N}_\varepsilon$ in finite time, for sufficiently small ε.

We have thus proved that $S_\varepsilon \cap \partial N_\varepsilon = \varnothing$, for sufficiently small ε. Thus N_ε is an isolating neighborhood for the flow $(2)_\varepsilon$. It follows too that

$\tilde{S}_\varepsilon \cap \partial\tilde{N}_\varepsilon = \varnothing$, where $\tilde{S}_\varepsilon = S(\tilde{N}_\varepsilon)$. Since $(2)_\varepsilon$ and $(3)_\varepsilon$ are the same flows modulo a change in time scale, it is clear that S_ε and $\tilde{S}_\varepsilon$ have the same index. In the sequel, it will be more convenient to work with $\tilde{S}_\varepsilon$.

5. Continuation to the Van der Pol Equations

As we have explained earlier, we must show $h(S_\varepsilon) \neq \bar{0}$ in order to conclude that $S_\varepsilon \neq \varnothing$. It is difficult to compute this index directly. Instead, we exploit the homotopy invariance of the index, by suitably deforming the equations. This will be done by introducing some new parameters. As we deform the equations, the sets N_ε will also have to be modified. At the end of the homotopy, we will be left with the Van der Pol equations crossed with a "two-dimensional repelling critical point."

A. *A Preliminary Lemma*

We modify Eqs. (1) by replacing the v_{xx} term by $\mu^2 v_{xx}$, where $0 < \mu \leqslant 1$. The $\dot{z}$ and $\tilde{z}'$ equations in $(2)_\varepsilon$ and $(3)_\varepsilon$ now become

$$\begin{aligned} \mu^2 \dot{z} &= -\theta z - vg(u, v), \\ \mu^2 \tilde{z}' &= -\theta\varepsilon\tilde{z} - \varepsilon^2 \tilde{v} g(\tilde{v}, \tilde{u}). \end{aligned} \tag{13}$$

Concerning these equations, we have the following lemma.

LEMMA 3. *If* (H_4) *holds with* $\mu = 1$, *then it also holds for all* μ, $0 < \mu < 1$, *for sufficiently small* $|\theta|$ $(\theta < 0)$.

Proof. Let $z_\mu(v)$ be the orbit segment OS_4 of (6) in Fig. 5, where the parameter μ appears as in (12). If $z_{\mu_1}(v)$ and $z_{\mu_2}(v)$ are two such curves with $\mu_2 > \mu_1$ which pass through the same point (v, z), then from (13), $\mu_1^2 z'_{\mu_1}(v) = \mu_2^2 z'_{\mu_2}(v)$ (here "prime" means d/dv), so that $z'_{\mu_1}(v) \geqslant z'_{\mu_2}(v)$. Since both curves pass through the origin, it follows that $z_{\mu_1}(v) > z_{\mu_2}(v)$ for $v > 0$.

The parameters ρ_v, and Δ of Section 1D, and (Fig. 5) will now depend on μ; these expression now become

$$\rho_v(\mu) = \int_0^{\sigma_1} z_\mu(v)\, dv,$$

$$e_\mu = z_\mu(\sigma_1),$$

$$\Delta_\mu = \sqrt{-2G_-(\sigma_0)}/\mu,$$

for $0 \leqslant \mu \leqslant 1$.

Note that when $\mu = 1$, $\rho_v = \int_0^{\sigma_1} \sqrt{-2G_-(s)}\, ds$, and this latter quantity is within $O(\theta)$ of $\int_0^{\sigma_1} z_1(s)\,ds$. Thus, when $\mu = 1$, we can use either of the two expressions for $\rho_v(1)$. Now we have

$$\mu^2 \frac{z^2}{2} + G_-(x) = -\theta\rho_v(\mu), \tag{14}$$

and if e_μ, c_μ, etc., are the quantities in Fig. 5 with $\mu < 1$ (analogous to those with $\mu = 1$), we have

$$\mu^2(e_\mu^2 - c_\mu^2)/2 = -\theta\rho_v(\mu).$$

Thus

$$e_\mu - c_\mu \geqslant -\theta\rho_v(\mu)/\mu^2 e_\mu,$$

and similarly

$$A_\mu - B_\mu = d_\mu - e_\mu \geqslant -\theta\rho_v(\mu)/\mu^2 e_\mu.$$

Now note that

$$\Delta(\mu) = |dz/dv|_{v=\sigma_1} = \Delta/\mu.$$

Thus $d_\mu - c_\mu \leqslant 2\Sigma\Delta\rho_\mu |\theta|/\mu$, and

$$c_\mu - a_\mu \geqslant \frac{|\theta|}{\mu^2 e_\mu} [\rho_v(\mu) - 2\Sigma\Delta(\mu^2 e_\mu)\,\rho_u/\mu] + O(\theta^2). \tag{15}$$

Now for fixed μ, if the quantity in backets in (15) is positive, then $c_\mu - a_\mu > 0$ for small θ. In a similar manner, $e_\mu > b_\mu$, and this is the desired condition. However, care must be taken to ensure that if a θ works for $\mu = 1$, the *same* θ works for $\mu < 1$. We proceed first to show that the quantity in backets in (15) is positive.

Thus, from (14), we have (when $z = e_\mu$, $v = \sigma_1$) $\mu^2 e_\mu = \mu[2(-\theta\rho_v(\mu) - G(\sigma_1))]^{1/2}$, so the quantity in brackets in (15) is

$$Q(\rho_v(\mu)) \equiv \rho_v(\mu) - 2\Sigma\rho_u\Delta\sqrt{2(-\theta\rho_v(\mu) - G_-(\sigma_1))}.$$

As a function of ρ, $Q(\rho)$ has the form depicted in Fig. 10. Thus, since $\rho_v(\mu)$ increases as μ decreases, it follows that $Q(\rho_v(\mu)) > Q(\rho_v(1))$ for all $\mu < 1$.

Finally, we check that a θ which works in (15) for $\mu = 1$ also works for all $\mu < 1$. The quantity in brackets in (15) is of the form

$$\frac{|\theta|}{\mu^2 e_\mu} [\rho_v(\mu) - c_1\sqrt{2(-\theta\rho_v(\mu) + c_2)}] + O(\theta^2),$$

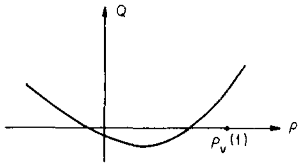

FIGURE 10

where c_1 and c_2 are positive constants. But since $\mu^2 e_\mu = \mu\sqrt{2(-\theta\rho_v(\mu) + c_3)}$, with c_3 a positive constant, we want that

$$\frac{|\theta|}{\mu\sqrt{-2\theta\rho_v(\mu) + c_3}}\, Q(\rho_v(\mu)) + O(\theta^2) > 0.$$

Since the graph of the function $Q(\rho)(-\theta\rho + c_3)^{-1/2} = \tilde{Q}(\rho)$ is as depicted in Fig. 11, where $\tilde{Q}(\rho)$ is bounded away from zero for $\rho \geqslant \rho_v(1)$, we see that if the quantity in brackets in (15) is positive for some θ when $\mu = 1$, it is also positive for that θ when $\mu < 1$.

B. *The Artificial Viscosity Equations*

It follows from this lemma, that an isolating neighborhood $\tilde{N}_\varepsilon(\mu)$ for $(3)_\varepsilon$ (modified via (13)), can be constructed for $0 < \mu_0 \leqslant \mu \leqslant 1$, and all sufficiently small ε *(depending on* μ_0), for some $\mu_0 > 0$. The construction proceeds exactly as in the case $\mu = 1$.

Next, we introduce a new variable

$$\pi(\eta) = -\varepsilon\theta\tilde{z}(\eta) - \varepsilon^2 G, \qquad \text{where} \quad G(\tilde{u}, \tilde{v}) = \tilde{v}g(\tilde{u}, \tilde{v}).$$

Then $\mu^2\tilde{z}' = \pi$ and

$$\begin{aligned}\pi' &= -\varepsilon\theta\tilde{z}' - \varepsilon^2\, dG \cdot (\tilde{w}, \tilde{z})\\ &= -\varepsilon\theta\pi/\mu^2 - \varepsilon^2\, dG \cdot (\tilde{w}, \pi/\varepsilon\theta - \varepsilon G/\theta).\end{aligned}$$

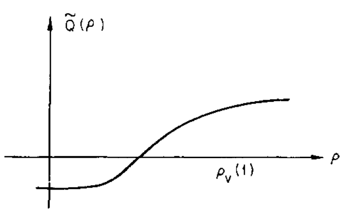

FIGURE 11

Finally, let

$$p(\eta) = \pi(\mu^2\eta)/\varepsilon^2; \tag{16}$$

then

$$\begin{aligned} p' &= \varepsilon^{-2}\mu^2[-\varepsilon\theta p/\mu^2 - \varepsilon G_v p/\theta + O(\varepsilon^2)] \\ &= \varepsilon^{-1}[-\theta - \mu^2 G_v/\theta]\, p + \mu^2 O(1) \\ &\equiv \varepsilon^{-1} c p + \mu^2 O(1). \end{aligned} \tag{17}$$

Now $-\theta > 0$, and θ is small. However, the smallness of $|\theta|$ (used in Section 2 to obtain the behavior of the singular flow) was taken relative to the quantity $Q(\rho_v(\mu))$. In Lemma 3, we saw that this quantity increases as μ decreases, and thus, a value of $\theta < 0$ which was admissible for $\mu = 1$ is also admissible for every $\mu \in (0, 1]$.

Since G_v is uniformly bounded, there exists $\mu_0 > 0$ so small that the coefficient c in (17) is positive and uniformly bounded away from zero. Thus for large $|p|$, the equation for p' represents (essentially) a repelling critical point. Under the change of variables $\tilde{z} \to p$ in $(3)_\varepsilon$, the $\tilde{V}$ equations become

$$\begin{aligned} \tilde{v}' &= -\varepsilon\theta^{-1} G(\tilde{U}) - \theta\varepsilon p(\mu^{-2}\eta), \\ p' &= \varepsilon^{-1} c p + \mu^2 O(1). \end{aligned} \tag{18}$$

For each (U, v), let $I(U, v) = \{z\colon (U, v, z) \in N_\varepsilon(\mu_0)\}$. If U lies in A_0 or $A_+(\mathrm{v})$, then I is an interval of the form $|z| \leqslant K$ for some (large) $K > 0$. If U is outside of these regions, then I is an interval of the form $k \leqslant z \leqslant K$ or $-K \leqslant z \leqslant -k$, for some $k > 0$. We must now extend the regions D and E in Figs. 7b and 8b so that I is always an interval of the form $|z| \leqslant K$. When this is the case, then we can make the change of variables $\tilde{z} \to p$ (see (16)) and the projection on the p-axis would always contain an interval of the form $|p| \leqslant K_1$, for some $K_1 > 0$. It would then be possible to let μ tend to zero *independently* of ε, since when $\mu = 0$, the bounded solutions of $(18)_\varepsilon$ lie on $p = 0$. Therefore these points must lie in our isolating neighborhood; see the discussion following the proof the next lemma.

LEMMA 4. *Let*

$$\begin{aligned} D^* &= \{V\colon |v_1 - \sigma_1| \leqslant \rho, |z| \leqslant K\}, \\ E^* &= \{V\colon |v_1 - \sigma_2| \leqslant \rho, |z| \leqslant K\}. \end{aligned}$$

and let $N_\varepsilon^* = N_\varepsilon(\mu_0) \cup (\mathscr{R} \times D^*) \cup (\mathscr{S} \times E^*)$. *Then* $S(N_\varepsilon^*) = S(N_\varepsilon(\mu_0))$; *the latter set being taken with respect to the equations* $(2)_\varepsilon$.

Proof. Suppose that an orbit of $(2)_\varepsilon$ passes through a point $(U_0, V_0) \in (\mathscr{R} \times D^*)\backslash N_\varepsilon$, when $\xi = 0$. It follows that U_0 is exterior to A_0 and $A_+(v)$. If this orbit remains in N_ε^*, U must enter $A_+(v(\bar{\xi}))$ in positive time $\bar{\xi} = O(\varepsilon)$. Since $z = v'$ is bounded, $v(\bar{\xi})$ is within $O(\varepsilon)$ of $v(0)$, and hence, $v(\bar{\xi}) < \sigma_2 - \rho$ for small ε. Since the "base" of D was assumed to be near the v-axis, it follows that every solution of $(7)_+$ starting in (or near) $D^*\backslash D$ remains outside of E^* at all future times. The behavior of the perturbed equations will therefore be similar; in particular the solution, having entered T_2 can never reach T_4, and $V(\xi)$ will closely approximate the flow of $(7)_+$ for all $\xi \geqslant \bar{\xi}$. It follows that $V(\xi)$ eventually exits the region $\mathscr{R}$ in Fig. 6a, and thus the orbit cannot lie in $S(N_\varepsilon^*)$; (Fig. 12).

From the lemma, it follows that $N_\varepsilon(\mu_0)$ can be continuously deformed to N_ε^* without changing the maximal invariant set; the index therefore remains unchanged throughout the homotopy. Note too that a similar remark applies, of course, to the associated neighborhood $\tilde{N}_\varepsilon^*$.

Observe that $\tilde{N}_\varepsilon^*$ is the product of a set $\tilde{M}$ in $(\tilde{U}, \tilde{v})$ space, with an interval $J_k = \{\tilde{z} : |\tilde{z}| \leqslant \varepsilon K\}$, and $\tilde{M}$ is independent of ε. Since $p = -\theta\varepsilon^{-1}\tilde{z} - G$, it follows that for large K, $K_1 = -\theta K - \max|G| > 0$, where the maximum is taken over the projection of $\tilde{M}$ on the $(\tilde{u}, \tilde{v})$ plane. Thus, under the change of variables $\tilde{z} \to p$, $\tilde{N}_\varepsilon^*$ is mapped into a region $\tilde{M} \times J_{K_1}$, where $J_{K_1} = \{p : |p| \leqslant K_1\}$. (The actual projection on the p-axis may be larger than J_{K_1}, but this will not be important.)

The equations are now

$$\begin{aligned} \tilde{u}' &= w, & \tilde{v}' &= -\theta^{-1}\varepsilon G(\tilde{u}, \tilde{v}) + O(\varepsilon p), \\ \tilde{w}' &= -\theta\tilde{w} - \tilde{u}f(\tilde{u}, \tilde{v}), & p' &= \varepsilon^{-1}(-\theta + O(\mu^2))p + O(\mu^2). \end{aligned} \tag{19$_\varepsilon$}$$

Since the critical points in $\tilde{M} \times J_{K_1}$ are repellors, we see that any solution of (19) in $\tilde{M} \times J_{K_1}$ along which $|p| = K_1$ leaves $\tilde{M} \times J_{K_1}$ in positive time; a similar remark applies to solutions of $(3)_\varepsilon$ in N_ε^* along which $|\tilde{z}| = K$. It follows that the index of $S(\tilde{M} \times J_{K_1})$ with respect to $(19)_\varepsilon$, equals the index of $S(N_\varepsilon^*)$ with respect to $(3)_\varepsilon$.

We now claim that $\tilde{M} \times J_{K_1}$ is an isolating neighborhood for (19), for each $\mu \in [0, \mu_0]$, and for all small $\varepsilon > 0$. The $(\tilde{U}, \tilde{v})$ flow of (19) can be closely

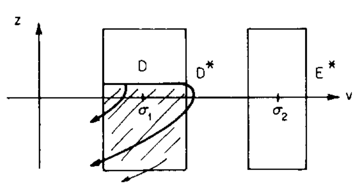

FIGURE 12

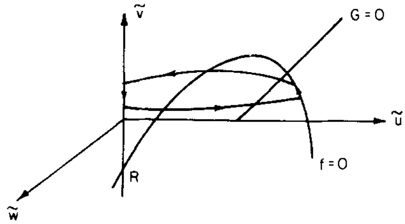

FIGURE 13

approximated by setting $p = \varepsilon = 0$. In this case, there exists a singular "periodic" solution which consists of two curves of critical points along $\tilde{u} = \tilde{w} = 0$, and along $\tilde{w} = 0$, $f = 0$, together with the two connecting orbits of (H_3) when $v = \sigma_1$ or σ_2 (see Fig. 13). It is easily seen that $\tilde{M}$ is a compact neighborhood of this singular solution. Moreover, it follows, with minor modifications from Conley's construction ([3]), that $\tilde{M} \times J_{k_1}$ is an isolating neighborhood for $(19)_\varepsilon$ when $\mu = 0$, for all sufficiently small $\varepsilon > 0$. Since isolating neighborhoods persist under perturbation, it follows that $M \times J_{K_1}$ is an isolating neighborhood for $(19)_\varepsilon$ for all small ε and for all $\mu \in [0, \mu_1]$, for some $\mu_1 > 0$. Now replace μ_0 by $\min(\mu_0, \mu_1)$ in the above arguments. We conclude that the index of $S(\tilde{N}_\varepsilon^*)$, with respect to $(3)_\varepsilon$, equals that of $S(\tilde{M} \times J_{K_1})$ with respect to $(19)_\varepsilon$, when $\mu = 0$. The equations have now been continued to

$$\begin{aligned} \tilde{u}' &= \tilde{w}, & \tilde{v}' &= -\theta^{-1}\varepsilon \tilde{v} g(\tilde{u}, \tilde{v}), \\ \tilde{w}' &= -\theta \tilde{w} - \tilde{u} f(\tilde{u}, \tilde{v}), & p' &= -\theta \varepsilon^{-1} p. \end{aligned} \tag{20$_\varepsilon$}$$

C. *Deformation of f and g*

Next we deform $\tilde{u} f(\tilde{u}, \tilde{v})$ and $\tilde{v} g(\tilde{u}, \tilde{v}) = G$ as indicated in Fig. 14. In particular, $\tilde{u} = 0$ and $f = 0$ are pulled apart (when they meet at R in Fig. 13) into two components. The upper component is deformed into a cubic; care must be taken to ensure that the equal area point, $\tilde{v} = \sigma_0$ remains positive and unique. At the same time, $g = 0$ is deformed into a vertical line and pushed to the left, so that it meets $f = 0$ along the branch with positive slope, as indicated in the third picture in Fig. 14. Also, the horizontal line $\tilde{v} = 0$ (in

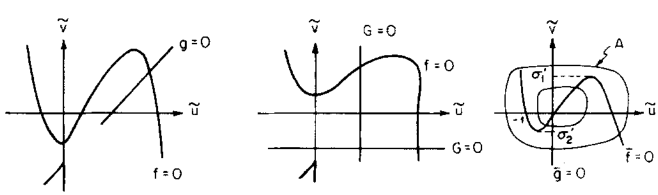

FIGURE 14

$G=0$) is pushed down into the region $\tilde{v}<0$. Finally, the nonlinear functions are translated so that the middle critical point occurs at the origin. Since the other portions of the curves $f=0$ and $g=0$ are far away from the region in which the desired solution is to lie, they can be ignored. We denote the final nonlinear functions by $\bar{f}$ and $\bar{g}$; they can be written as

$$\bar{f}=\tilde{u}-\tilde{u}^3-v, \qquad \bar{g}=\tilde{u};$$

that is, they are the functions appearing in the Van der Pol equation.

As the functions f and g change, we must modify the region $\tilde{M}$. Using Conley's construction ([3]), it is easily seen that a suitable $\tilde{M}$ can be found at each step. This follows since the crucial properties of f and g needed here are that (i) $f=0$ is qualitatively like a cubic; (ii) $g<0$ along the left branch of $f=0$, and $g>0$ along the right branch of $f=0$; and (iii) hypotheses analogous to (H_1)–(H_3) hold at each stage of the deformation. We omit these details.

D. *The Second Artificial Viscosity Equations*

The last step in the homotopy is the introduction of another parameter $\nu\in(0,1]$, as follows. The equations we look at are

$$\begin{aligned} \tilde{u}' &= \tilde{w}, & \tilde{v}' &= -\theta^{-1}\varepsilon\tilde{u}, \\ \nu\tilde{w}' &= -\theta\tilde{w}-\bar{f}(\tilde{u},\tilde{v}), & p' &= -\varepsilon^{-1}\theta p. \end{aligned} \tag{21$_\varepsilon$}$$

We also introduce a new variable

$$r(\eta)=\nu\tilde{w}'(\nu\eta). \tag{22}$$

As before, we find that $r'=(-\theta+O(\nu))r+O(\nu)$, and in these new variables, $(21)_\varepsilon$ becomes

$$\begin{aligned} \tilde{u}' &= -\theta^{-1}f(\tilde{u},\tilde{v})-\theta^{-1}r, & \tilde{v}' &= -\theta^{-1}\varepsilon\tilde{u}, \\ r' &= (-\theta+O(\nu))r+O(\nu), & p' &= -\varepsilon^{-1}\theta p. \end{aligned} \tag{23$_\varepsilon$}$$

The isolating neighborhood for the old equations will have to be modified as ν varies, to a neighborhood $\tilde{M}_\nu\times J_{K_1}$. In particular, as $\nu\to 0$, the value $\tilde{v}=\sigma_1$ (resp. σ_2), where $\tilde{u}$ goes into transition moves upward (resp. downward) towards σ_1' (resp. σ_2'); see Fig. 14. Let ν_0 be so small that $-\theta+O(-\nu_0)$, the coefficient of r in the r' equation in $(23)_\varepsilon$, is positive. The region M_{ν_0} will consist of two tubes in $(\tilde{u},\tilde{w},\tilde{v})$ space about the left and right branches of $\bar{f}(\tilde{u},\tilde{v})=0$, $\tilde{w}=0$, together with two neighborhoods in the $\tilde{U}$ plane of the connecting orbits of the singular equations (as in Figs. 7a, 8a), crossed with small neighborhoods of $\tilde{v}=\sigma_1$ and $\tilde{v}=\sigma_2$. The neighborhood $M_{\nu_0}\times J_{K_1}$ will be isolating for $(23)_\varepsilon$ for sufficiently small ε, depending on ν_0.

As ν tends to zero, the left and middle critical points move together in Fig. 7a, and the middle and right critical points move together in Fig. 8a, since σ_i approaches σ_i', $i=1,2$. As a result, ε will have to decrease to zero along with ν, in order that $\tilde{M}_\nu \times J_{K_1}$ remain an isolating neighborhood. In order to avoid this problem, we fix $\nu=\nu_0$ as above, and we enlarge $\tilde{M}_{\nu_0}$ in a suitable manner, which we shall soon describe. In the variables $(23)_\varepsilon$, the enlarged neighborhood will remain isolating for all $\nu \in [0, \nu_0)$, *independently* of ε. In particular, $\tilde{M}_{\nu_0}$ will be enlarged so that under the change of variables $\tilde{w} \to r$, the projection of the image of $\tilde{M}_{\nu_0}$ on the r-axis always contains an interval of the form $J_{K_2} = \{|r| < K_2\}$, for some $K_2 > 0$. This is *not* the case with $\tilde{M}_{\nu_0}$; e.g., at the middle critical point in Fig. 7a, $\tilde{w} = \tilde{f}(\tilde{u}, \sigma_1) = 0$, so that $r=0$; however $(\tilde{u}, 0, \sigma_1) \notin \tilde{M}_{\nu_0}$.

LEMMA 5. *Let the regions* $\tilde{\mathscr{R}}$ *and* $\tilde{\mathscr{S}}$ *(where* $\mathscr{R}$ *and* $\mathscr{S}$ *are in Fig.* 7a *and* 8a, *respectively) be replaced by the rectangle*

$$\tilde{T} = \{(\tilde{u}, \tilde{w}) : |\tilde{u}| < C, |\tilde{w}| \leqslant K\},$$

where C and K are large and positive. Let $\tilde{L}$ *consist of those points in* $\tilde{M}_{\nu_0}$, *together with the points in* $\tilde{T} \times \{|\tilde{v} - \sigma_i| < \rho\}$, $i=1,2$. *Then the maximal invariant set of* $(21)_\varepsilon$ *in* $\tilde{L} \times J_{K_1}$ *coincides with that of* $(21)_\varepsilon$ *in* $\tilde{M}_{\nu_0} \times J_{K_1}$, *when* $\nu = \nu_0$, *and for sufficiently small* $\varepsilon > 0$, *say* $0 < \varepsilon \leqslant \varepsilon_0$.

Proof. The $(\tilde{u}, \tilde{w})$ flow of $(21)_0$ when $\tilde{v} = \sigma_1$ is as in Fig. 15. Let $\tilde{D} \subseteq \{\tilde{u} < 0\}$ be a compact neighborhood of the middle critical point, disjoint from the other two critical points and from their connecting orbit; see Fig. 15. Since the middle critical point is a repellor, any orbit of $(21)_0$ for which $(\tilde{u}, \tilde{w}) \in \partial\tilde{D}$ exits $\tilde{D}$ in positive time, and enters $\tilde{D}$ in negative time. If $\varepsilon > 0$ is small, the $\tilde{U}$ components of $(21)_\varepsilon$ must also have the same behavior, if $\tilde{U}$ lies on $\partial\tilde{D}$. Thus, if a complete solution curve of $(21)_\varepsilon$ passed through a point at which $\tilde{U} \in \tilde{D}$ (say at $\eta = 0$), it follows that $\tilde{U}(\eta) \in \tilde{D}$ for all $\eta < 0$. Since $\tilde{D}$ lies in $\tilde{u} < 0$, $\tilde{v}' = \varepsilon\tilde{u}$ is negative and bounded away from zero for all $\eta < 0$. Hence $\tilde{v}(\eta)$ must decrease and eventually $\tilde{v}$ exits the interval $|\tilde{v} - \sigma_1| \leqslant \rho$. The solution therefore exits $\tilde{L} \times J_{K_1}$ in negative time.

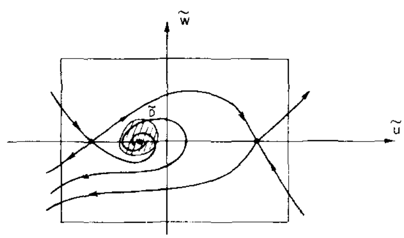

FIGURE 15

Now suppose that the $\tilde{U}$-component passed through a point in $\tilde{L}\backslash(\tilde{\mathscr{R}} \cup \tilde{D})$. When $\varepsilon = 0$, the $\tilde{U}$ component either exits $\tilde{L}$ in finite time, or enters $\tilde{D}$ in backward time. Since $\tilde{L}\backslash(\tilde{\mathscr{R}} \cup \tilde{D})$ contains no critical points, solutions of $(21)_\varepsilon$ have similar behavior for small $\varepsilon > 0$. From the previous paragraph, we see that every such solution exits $\tilde{L} \times J_{K_1}$ in at least one time direction.

The case in which $|\tilde{v} - \sigma_2| \leqslant \rho$ is treated in a similar fashion. This completes the proof.

We can now continuously deform the regions $\tilde{\mathscr{R}}$ and $\tilde{\mathscr{S}}$ to $\tilde{T}$ without changing the maximal invariant set of $(21)_\varepsilon$. The index therefore remains invariant throughout the homotopy.

Next, consider the flow of $(21)_\varepsilon$ in $\tilde{L} \times J_{K_1}$, with $v = v_0$ and small $\varepsilon > 0$. We choose $K > 0$ so large that

$$K_2 \equiv (-\theta + O(v_0))K - \max |\bar{f}| > 0,$$

where the maximum is taken over the projection of $\tilde{L}$ on the $\tilde{U}$-plane. Then under the change of variables $\tilde{w} \to r$, the region $\tilde{L} \times J_{K_1}$ is mapped onto a region whose projection on the r-axis contains the interval $J_{K_2} = \{|r| < K_2\}$; its projection on the $(\tilde{u}, \tilde{v})$ plane is an annular region A about the origin. If v_0 is small, then σ_i will be within $\rho/4$ of σ_i', $i = 1, 2$. For such v_0, the annulus will have the form indicated in Fig. 14.

Consider the flow of $(23)_\varepsilon$ with $v = 0$. This is seen to be the Van der Pol equations, crossed with a "two-dimensional repelling critical point." It is well known that the Van der Pol equations admit a unique attracting limit cycle. For sufficiently small ε, say $0 < \varepsilon \leqslant \varepsilon_1$, this orbit will lie interior to the annulus of Fig. 13; indeed, the annulus in an attractor block for the periodic orbit. If we now let $v > 0$ be small, we find that $|r| = O(v)$ along any complete solution of $(23)_\varepsilon$ in $A \times J_{K_1} \times J_{K_2}$, since if $|r| > kv$, $r' > 0$ so the orbit does not stay in the neighborhood. Furthermore, we see that this latter set perturbs to an isolating neighborhood for $(23)_\varepsilon$ for $0 \leqslant v \leqslant v_1$, and $0 < \varepsilon \leqslant \varepsilon_0$. Finally, if we replace v_0 by $v_2 = \min(v_0, v_1)$, in the above arguments, we obtain that $A \times J_{K_1} \times J_{K_2}$ is an isolating neighborhood for $0 < v \leqslant v_2$ and $0 < \varepsilon \leqslant \min(\varepsilon_0, \varepsilon_1)$. Moreover, the index of $S(A \times J_{K_1} \times J_{K_2})$ with respect to $(21)_\varepsilon$ is the same as that of $S(\tilde{L} \times J_{K_1})$ with respect to $(20)_\varepsilon$, since any point for which $|r| = K_1$ or $|\tilde{w}| = K$, is an exit point under the respective flows.

The construction of the homotopy is now complete.

6. Existence of a Periodic Orbit

A. *Computation of the Index*

It follows from our constructions in Sections 4 and 5 that the index of $S_\varepsilon = S(N_\varepsilon)$ with respect to Eqs. $(2)_\varepsilon$ (and of $S(\tilde{N}_\varepsilon)$ with respect to $(3)_\varepsilon$) equals the index of $S(A \times J_{K_1} \times J_{K_2})$ with respect to the equations

$$\begin{aligned} u' &= \bar{f}(\tilde{u}, \tilde{v}), \qquad & p' &= -\theta\varepsilon^{-1}p, \\ \tilde{v}' &= \varepsilon\tilde{u}, & r' &= -\theta r, \end{aligned} \tag{24}$$

where $\bar{f}(\tilde{u}, \tilde{v}) = \tilde{u} - \tilde{u}^3 - \tilde{v}$. As we have noted above, the $(\tilde{u}, \tilde{v})$ equations are precisely the Van der Pol equations, and they admit a unique attracting periodic orbit lying in the projection of $A \times J_{K_1} \times J_{K_2}$ onto $\tilde{u}, \tilde{v}$ space.

The index of a hyperbolic periodic orbit with n positive Floquet exponents is $\Sigma^n \vee \Sigma^{n+1}$, where Σ^k denotes a pointed k-sphere, and "$\vee$" is the "wedge product" of pointed spaces (see [2]). Since the (p, r) equations have index Σ^2, we obtain

$$h(\tilde{S}_\varepsilon) = (\Sigma^0 \vee \Sigma^1) \wedge \Sigma^2 = \Sigma^2 \vee \Sigma^3,$$

where $\wedge$ denotes the "smash product;" see [2]. It follows that $h(\tilde{S}_\varepsilon) \neq \bar{0}$; whence $\tilde{S}_\varepsilon = S(\tilde{N}_\varepsilon) \neq \varnothing$.

B. *Existence of a Periodic Solution*

At each step of the above constructed homotopy, the isolating neighborhoods are all homeomorphic to $S^1 \times D^3$, where D^3 is a three-cell. Let λ denote the homotopy parameter, and let N_λ denote the corresponding isolating neighborhood. Now orbits in $S(N_\lambda)$ cannot "turn around," and in particular, if D_λ is a section of N_λ which is homeomorphic to D^3, then the Poincaré map (the "first return" map), T_λ, under the positive flow, is well-defined on the set $S(N_\lambda) \cap D_\lambda$. (Indeed, for the original equations $(3)_\varepsilon$, this follows from Lemma 1 of Section 4; for the Van der Pol equations, it is obvious, while for the flows generated in $(19)_\varepsilon$, it follows from Conley's construction in [3].) Thus T_λ is well-defined at all points sufficiently close to $S(N_\lambda) \cap D_\lambda$.

Let $O_\lambda \subset D_\lambda$ be a small open neighborhood of $S(N_\lambda) \cap D_\lambda$ in D, such that T_λ is defined on $\bar{O}_\lambda$. Then T_λ has no fixed points on ∂O_λ, since N_λ is an isolating neighborhood. Thus the degree, $\deg(I - T_\lambda, O_\lambda, 0)$, is well-defined and is independent of λ. If the flow with $\lambda = 0$ is the flow generated by (24), then the degree of $I - T_0$ is $(-1)^2 = 1$, since the periodic orbit has no positive Floquet exponents.

Now consider the two changes of variables $\tilde{w} \to r$ and $\tilde{z} \to p$. It is easily checked that the transformations $(\tilde{u}, \tilde{w}, \tilde{v}, \tilde{z}) \to (\tilde{u}, \tilde{w}, \tilde{v}, p)$ and $(\tilde{u}, \tilde{w}, \tilde{v}, p) \to$

$(\tilde{u}, r, \tilde{v}, p)$ are both orientation-preserving diffeomorphisms of $\mathbb{R}^4$. It follows then, that the degrees of the associated flows are all the same. Thus, the degree of $I - T_1$, where T_1 is the first return map for the original equations $(3)_\varepsilon$, is $+1$. It follows that $(3)_\varepsilon$ admits a periodic solution. This completes the proof of the theorem, (cf. Section 1E).

Acknowledgments

We are indebted to Charles Conley for discussing this problem with us, and in particular for suggesting the continuations described above. We would also like to thank the staff at the University of Warwick for their kind and generous hospitality.

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