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EXISTENCE THEOREMS FOR NONSELFADJOINT SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

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1. INTRODUCTION

IN THE present paper we present new existence theorems for weak and strong solutions of problems of the form

$$Ex = Nx,$$

where E is a real elliptic linear differential operator in a bounded domain G of $\mathbb{R}^n$ with a given system of linear homogeneous conditions, say, $Bx = 0$ on the boundary ∂G of G and where N is a Nemitsky type nonnecessarily linear operator.

We shall make use here of the alternative method, and particularly we shall make use for the elliptic case of new remarks. These remarks suggest that both the auxiliary and bifurcation equations can be analyzed under different topologies, and by a more specific construction of the operator S: $Y_0 \to X_0$.

Actually some of these remarks have been already used implicitly in previous papers on the semilinear wave equation in $\mathbb{R}^2$ (Cesari and Kannan [5], Cesari and Pucci [6]).

For selfadjoint elliptic problems, Landesman and Lazer [9] proved, also by the alternative method, a remarkable theorem which was then extended by Williams [14] by the same method, and by others by different arguments. Later, Shaw [12] proved, again by the alternative method, that Landesman's and Lazer's theorem extends even to nonselfadjoint problems with equal Fredholm indices and whose eigenfunctions share regions of positivity and negativity with their corresponding adjoint eigenfunctions.

In the present paper we definitely aim at elliptic problems which are not necessarily selfadjoint and do not necessarily satisfy Shaw's requirements. The sufficient conditions we obtain are more quantitative in character and concern the cases $Nx = f(t) + g(t, D^\alpha x)$ and $Nx = f(t) + g(t, x(t))$, $t \in G$. However, as we show by examples for the case $Nx = f(t) + g(t, x(t))$, our sufficient conditions for existence allow a great freedom on g, on which no monotonicity is required.

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2. GENERAL ASSUMPTIONS

Let G be a bounded connected open set in $\mathbb{R}^n$ with smooth boundary ∂G; in fact most of our results extend to a compact connected smooth Riemannian n-manifold with or without boundary. On the other hand, we shall consider situations where the smoothness assumptions on ∂G can be relaxed so as to allow, say, G to be an interval in $\mathbb{R}^n$.

For s a nonnegative integer we denote by H^s the Sobolev (Hilbert) space $W^{s,2}(G)$ of square integrable functions on G whose (distributional) derivatives of order $\leq s$ are also square integrable functions ($H^0 = L_2(G)$), with norm $\|x\|_{H^s} = \left(\sum_{0\leq|\alpha|\leq s} \|D^\alpha x\|_2^2\right)^{1/2}$. Here we denote by (x, y) and $\|x\|_2 = (x, x)^{1/2}$ the inner product and norm in $L_2(G)$. Then H_0^s is the linear subspace of H^s which is obtained by completion in the norm above of the set of the functions of class C^∞ and compact support in G.

Let E be a (not necessarily selfadjoint) uniformly elliptic real linear partial differential operator of order $2m$, i.e.

$$Ex = \sum_{|\alpha|,|\beta|\leq m} (-1)^{|\beta|} D^\beta (a_{\alpha\beta}(t) D^\alpha x),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, all α_s, β_s are nonnegative integers, with associated linear homogeneous boundary conditions, say $Bx = 0$ on ∂G; or more explicitly $B_j(t, D^\alpha x) = 0$, $j = 1, \ldots, M$, where $D^\alpha x$ denotes the set of the derivatives $D^\alpha x$ on the boundary ∂G, $0 \leq |\alpha| \leq K \leq m_0 - 1$ for some K (the derivatives being replaced by their traces for x in a Sobolev space H^{m_0}). We denote by $A[\phi, \psi]$ the usual bilinear form associated to E:

$$A[\phi, \psi] = \sum_{|\alpha|,|\beta|\leq m} \int_G a_{\alpha\beta}(t) D^\alpha \psi(t) D^\beta \phi(t)\, dt.$$

We denote by E^* the formal adjoint operator,

$$E^*y = \sum_{|\alpha|,|\beta|\leq m} (-1)^{|\beta|} D^\beta (a_{\beta\alpha}(t) D^\alpha y),$$

with the associated boundary conditions, say $Dy = 0$ on ∂G, or $D_s(t, D^\beta y) = 0$, $s = 1, \ldots, N$.

We assume that both Ker E and Ker E^* are finite dimensional, with $\infty > p \geq q \geq 0$, $p = \dim \operatorname{Ker} E$, $q = \dim \operatorname{Ker} E^*$. (However, we shall consider situations where $\infty = p > q \geq 0$).

A few words on the concept of solution x of the linear problem $Ex = f$, $f \in L_2(G)$, with $B_j x = 0$, $j = 1, \ldots, M$.

For weak solutions we need only to assume that the coefficients $a_{\alpha\beta}$ are measurable bounded functions on G. For instance, for the typical homogeneous Dirichlet boundary problem the boundary conditions are given by $x = 0$, $\partial x/\partial n = 0, \ldots, \partial^{m-1}x/\partial n^{m-1} = 0$, n the exterior normal to ∂G, and we say that x is a weak solution of $Ex = f$ provided $x \in H_0^m$ and $A[x, \psi] = (f, \psi)$ for all $\psi \in C_0^\infty(G)$. In other words, $W = D(E) = H_0^m$ is the domain of E. We refer to [2] for the concept of weak solution for more general boundary value problems.

For strong, or classical solutions, we need to assume that the coefficients $a_{\alpha\beta}$ are of class $C^{|\beta|}(\bar{G})$. Again, for the homogeneous Dirichlet boundary value problem above we say that x

is a strong solution of $Ex = f$ provided $x \in H_0^m \cap H^{2m}$ and $Ex = f$ a.e. in G. Thus, $W = D(E) = H_0^m \cap H^{2m}$ and W is equipped with the topology of H^{2m}, $W = D(E)$.

In general, let $W = D(E)$ be the subspace of all elements x of a Sobolev space H^{m_0} for which $Bx = 0$, and we assume that W is closed in H^{m_0}. Since B involves derivatives of orders $\leq K$ we shall assume $K \leq m_0 - 1$. For weak solutions of the Dirichlet problem above we have $K = m - 1$, $m_0 = m$. For strong solutions of the same problem we have $K = m - 1$, $m_0 = 2m$.

We assume that a Fredholm's alternative theorem holds, that is, for $f \in L_2(G)$, the problem $Ex = f$, with $B_j x = 0$, $j = 1, \ldots, M$, has a solution $x \in W$ if and only if f is orthogonal (in $L_2(G)$) to all elements ω of Ker E^*.

Let $X = Y = L_2(G)$. We assume that E is a closed operator with range $R(E)$ closed in Y.

All the assumptions above are generally true for reasonable boundary conditions B (cf., e.g., [10, pp. 148–154, 111–113]).

Let N be a (nonlinear) Nemitsky operator of the form $Nx = f(t) + g(t, D^\alpha x)$, where f is a given element in $L_2(G)$, where $D^\alpha x$ denotes the set of all derivatives of x of orders α, $0 \leq |\alpha| \leq k_0$, where k_0 is any integer $\leq 2m$ and in any case $k_0 \leq m_0$, and where $g(t, u)$, or g: $G \times \mathbb{R}^\mu \rightarrow \mathbb{R}$, is measurable in t for all $u \in \mathbb{R}^\mu$, and continuous in u for a.a. $t \in G$. Here $\mu = 1 + n + n(n+1)/2 + \ldots + n(n + k_0 - 1)/2$ is the number of different derivatives in $\mathbb{R}^n$ of orders α, $0 \leq |\alpha| \leq k_0$.

Thus, for g bounded in $G \times \mathbb{R}^\mu$, then since $0 \leq k_0 \leq m_0$, and $W \subset H^{m_0} \subset H^{k_0}$, then N: $W \rightarrow H^0$ is a bounded continuous operator from $W \subset H^{m_0}$ to $H^0 = L_2(G)$ in the topologies of H^{m_0} and $L_2(G)$ (cf. [13, p. 155; 8, p. 27]). Again, for g bounded, the range $R(N)$ of N is made up of functions on G which are in absolute value $\leq |f(t)| + \sup|g|$. If both f and g are bounded, then the range of N is a subset of $L_\infty(G)$, and N is a bounded continuous operator from W to $L_\infty(G)$.

If $g = g(t, u)$ is bounded in $G \times \mathbb{R}^\mu$ and uniformly Lipschitzian in u, then N: $W \rightarrow H^0$ is a Lipschitzian operator.

With these assumptions, we shall consider elliptic problems of the form:

$$\left.\begin{array}{ll} Ex = f(t) + g(t, D^\alpha x), & t \in G, \\ B_j(t, D^\alpha x) = 0 \quad \text{on } \partial G, & j = 1, \ldots, M. \end{array}\right\} \qquad (1)$$

3. SOME FURTHER PROPERTIES OF E

Let $\phi_1, \ldots, \phi_p$ be an orthonormal basis for Ker E, the elements ϕ_i being certainly in $X = L_2(G)$, and let P: $X \rightarrow X$ denote the orthogonal projection of X onto $X_0 = \text{Ker } E = \langle \phi_1, \ldots, \phi_p \rangle$ defined by $Px = \sum_{i=1}^{p} (x, \phi_i)\, \phi_i$ for $x \in X$. Then $PP = P$ and we take $X_1 = (\text{Ker } E)^\perp = (I - P)X$ so that we have the decomposition $X = X_0 + X_1$, $X_0 = \text{Ker } E$.

The restriction of E: $D(E) \subset X \rightarrow Y$ to the subspace $D(E) \cap (\text{Ker } E)^\perp$ is a $1-1$ closed linear operator, whose closed range is the complete range of E. Hence, by the closed graph theorem, the inverse map $H = [E|D(E) \cap (\text{Ker } E)^\perp]^{-1}$ is a $1-1$ continuous linear operator with domain $R(E)$ and range $D(E) \cap (\text{Ker } E)^\perp$, or $D(H) = R(E)$, $R(H) = D(E) \cap (\text{Ker } E)^\perp$.

Moreover,

$$\left.\begin{aligned} &EHy = y && \text{for all } y \in R(E), \\ &HEx = x - \sum_{i=1}^{p} (x, \phi_i)\phi_i && \text{for all } x \in D(E), \\ &\|Hy\|_W \leq L\|y\|_2 && \text{for all } y \in R(E) \text{ and some constant } L. \end{aligned}\right\} \quad (2)$$

Let $\omega_1, \ldots, \omega_q$ be an orthonormal basis for Ker E^*, the elements ω_s being certainly in $Y = L_2(G)$, and let Q: $Y \to Y$ denote the orthogonal projection of Y onto $Y_0 = \text{Ker } E^* = \langle \omega_1, \ldots, \omega_q \rangle$ defined by $Qy = \sum_{s=1}^{q} (y, \omega_s)\omega_s$ for $y \in Y$. Then, $QQ = Q$ and we take $Y_1 = (\text{Ker } E^*)^\perp = (I - Q)Y$ so that we have the decomposition $Y = Y_0 + Y_1$.

Thus, by the Fredholm alternative theorem, for every $f \in Y = L_2(G)$ there is a solution of $Ex = f$ if and only if $f \perp \text{Ker } E^*$, that is, if and only if $Qf = 0$, that is, if and only if $f \in Y_1$, with $Y_1 = (I - Q)Y = (\text{Ker } E^*)^\perp$.

Since $R(E) = Y_1$ and $QY_1 = 0$, then $QE = 0$; since $PX = \text{Ker } E$ we also have $EP = 0$. These two remarks and relations (2) yield now

$$HE = I - P, \quad QE = 0 = EP, \quad EH(I - Q) = I - Q,$$

and these are only particularizations of the usual relations for the alternative method [4], namely:

$$H(I - Q)E = I - P, \quad QE = EP, \quad EH(I - Q) = I - Q.$$

For $p > q$ we further decompose $X_0 = \langle \phi_1, \ldots, \phi_p \rangle$ into a space, say $X_{01} = \langle \phi_1, \ldots, \phi_q \rangle$ of dimension q, and a space $X_{02} = \langle \phi_{q+1}, \ldots, \phi_p \rangle$ of dimension $p - q$. For $p = q$ we take $X_{01} = X_0$ and $X_{02} = \{0\}$.

We make here the specific assumption:

$$\left.\begin{aligned} &\text{the decomposition } X_0 = X_{01} + X_{02} \text{ can be made in such a way that the } q \times q \text{ matrix} \\ &M = [(\omega_s, \phi_i), s, i = 1, \ldots q] \text{ is nonsingular.} \end{aligned}\right\} \quad (3)$$

For instance, if $\phi_i = \omega_i$, $i = 1, \ldots, q$, as in the selfadjoint case, then $M = I$ is the identity matrix. We shall use the notational convention to denote briefly by σ any given vector $\sigma = \text{col}(\sigma_1, \ldots, \sigma_q)$.

We now define the linear map S: $Y_0 \to X_{01}$. For any $y \in Y_0$ we have $y = \sum_{s=1}^{q} d_s^{\#}\omega_s$, with $d_s^{\#} = (y, \omega_s)$, since the ω_s form a basis, and we take $Sy = \sum_{i=1}^{q} d_i\phi_i$, with $d = M^{-1}d^{\#}$. Let us prove that $S^{-1}(0) = \{0\}$. Indeed, if $y \in S^{-1}(0)$, then $Sy = 0$, that is $d = 0$, hence $d^{\#} = 0$, since M is nonsingular, and $y = 0$.

We note that $X_{01} \subset X = Y$; hence Q: $Y \to Y$ is well defined on X_{01}. Let us prove that $SQx = x$ for every $x \in X_{01}$. In other words, SQ reduces to the identity map on X_{01}. Indeed, if $x \in X_{01}$, then $x = \sum_{i=1}^{q} c_i\phi_i$, $c_i = (x, \phi_i)$, and

$$Qx = \sum_{s=1}^{q} (x, \omega_s)\omega_s = \sum_{s=1}^{q} \left(\sum_{i=1}^{q} c_i\phi_i, \omega_s\right) \omega_s = \sum_{s=1}^{q} \left(\sum_{i=1}^{q} (\omega_s, \phi_i)c_i\right) \omega_s = \sum_{s=1}^{q} c_s^{\#}\omega_s,$$

where $c^{\#} = Mc$, and then $SQx = \sum_{i=1}^{q} c_i' \phi_i$, with $c' = M^{-1}(Mc) = c$, or $SQx = x$.

It may occur that a given element, say ω_1 in Y_0 is also an element, say ϕ_1 in X_{01}. Since both bases are orthonormal, the matrix $M = [m_{ij}]$ has $m_{11} = 1$, and all $m_{1i} = m_{i1} = 0$ for $i \neq 1$. In particular, if $\phi_i = \omega_i$, $i = 1, \ldots, q$, as in the selfadjoint case, then $M = I$ is the $q \times q$ identity matrix as already stated.

4. THE INTERMEDIATE TOPOLOGY

As we know from the alternative method (cf. [4]), the original problem $Ex = Nx$, with the conditions $Bx = 0$ on ∂G, is equivalent to the system of auxiliary and bifurcation equations:

$$x = Px + H(I - Q)Nx, \quad QNx = 0, \quad x \in W,$$

This system can be written as the problem of the fixed points of the transformation $\mathcal{T}$: $(x_{01}, x_{02}, x_1) \to (\bar{x}_{01}, \bar{x}_{02}, \bar{x}_1)$ defined by

$$\mathcal{T}: \begin{array}{l} \bar{x}_1 = \mathcal{T}_1 x = H(I - Q)N(x_{01} + x_{02} + x_1), \\ \bar{x}_{01} = \mathcal{T}_0 x = x_{01} - kSQN(x_{01} + x_{02} + x_1), \quad \bar{x}_{02} = x_{02}. \end{array} \tag{4}$$

where $x = x_{01} + x_{02} + x_1 \in W$, $\bar{x} = \bar{x}_{01} + \bar{x}_{02} + \bar{x}_1 \in W$, $x_{01}, \bar{x}_{01} \in X_{01} \subset W$, $x_{02}, \bar{x}_{02} \in X_{02} \subset W$, $x_1, \bar{x}_1 \in X_1 \cap W$, and the chain of maps is as follows

$$\mathcal{T}_1: W \xrightarrow{N} L_2(G) \xrightarrow{I-Q} Y_1 \xrightarrow{H} X_1 \cap W,$$

$$\mathcal{T}_0: W \xrightarrow{N} L_2(G) \xrightarrow{Q} Y_0 \xrightarrow{S} X_{01} \subset W,$$

where X_{01} is a finite dimensional subspace of W. Thus, $\mathcal{T}$ can be thought of as a map from W into W, $W \subset H^{m_0}$, W with the topology of H^{m_0}.

We shall introduce a different topology.

Indeed, we shall denote by Z a Banach space satisfying the following requirements:

$$\left.\begin{array}{l} W \subset H^{m_0} \subset Z \subset H^{k_0} \subset Y = L_2(G) \text{ with continuous imbeddings } j_1: W \to Z, \\ j_2: Z \to H^{k_0}, \text{ and } j_1 \text{ compact.} \end{array}\right\} \tag{5}$$

Since j_3: $H^{k_0} \to L_2(G)$ is certainly continuous, then $j_3 j_2$: $Z \to L_2(G)$ is also continuous. Note that for the continuity of j_1 and j_2 we require that for some constants γ, β we have $\|x\|_Z \leq \gamma\|x\|_W$ for all $x \in W$, and $\|x\|_{H^{k_0}} \leq \beta\|x\|_Z$ for all $x \in Z$.

Under assumptions (5) the chain of maps in the transformation $\mathcal{T}$ becomes

$$\left.\begin{array}{l} \mathcal{T}_1: Z \xrightarrow{j_2} H^{k_0} \xrightarrow{N} L_2(G) \xrightarrow{I-Q} Y_1 \xrightarrow{H} X_1 \cap W \xrightarrow{j_1} X_1 \cap Z, \\ \mathcal{T}_2: Z \xrightarrow{j_2} H^{k_0} \xrightarrow{N} L_2(G) \xrightarrow{Q} Y_0 \xrightarrow{S} X_{01} \xrightarrow{j_1} X_{01} \subset Z, \end{array}\right\} \tag{6}$$

where X_{01} is a finite dimensional subspace of W.

As a particular case we assume first that $2(m_0 - k_0) > n$ so that, from Sobolev's imbedding theorem (cf., e.g., [1, p. 97, Case C]), for any $x \in W \subset H^{m_0}$, the function x and all its distributional derivatives $D^\alpha x$, $0 \leq |\alpha| \leq k_0$, are bounded in G, that is, are in $L_\infty(G)$, and are continuous in the open set G. Then we can take for instance $Z = H^{k_0}$, or $Z = W^{k_0,\infty}(G)$, since in either case the imbeddings

$$j_1: W \subset H^{m_0} \to H^{k_0}, \quad j_2: H^{k_0} \to H^{k_0} \text{ the identity, or}$$
$$j_1: W \subset H^{m_0} \to W^{k_0,\infty}(G), \quad j_2: W^{k_0,\infty}(G) \to H^{k_0},$$

are continuous, and j_1 is compact.

In the further particular case in which $k_0 = 0$, $2m_0 > n$, then $g = g(t, x)$ depends only on $x = x(t)$ and not on the derivatives, and for any $x \in W \subset H^{m_0}$, x is a bounded function on G, or $x \in L_\infty(G)$, (hence, Ker $E \subset L_\infty(G)$), and we can take $Z = L_\infty(G)$. Moreover, we assumed g to be bounded, and for $f = f(t)$ in (1) also bounded, then N: $L_\infty(G) \to L_\infty(G)$. Furthermore, we assume that Ker E^* is made up of bounded functions. Now we may restrict $X = Y = L_2(G)$ to the space $X^* = Y^* = L_\infty(G)$ with the norm $\|x\|_\infty = \sup|x(t)|$ of $L_\infty(G)$. Let $X_0^* = PX^*$, $X_1^* = (I - P)X^*$, $Y_0^* = QY^*$, $Y_1^* = (I - Q)Y^*$, and now Ker E and Ker E^* are made up of bounded functions (all ϕ_i and ω_s are bounded in G), and thus $X_0^*, Y_0^* \subset L_\infty(G)$. In this situation then relations (5) become

$$\left.\begin{aligned} &W \subset H^{m_0} \to Z = Y = L_2(G) \text{ with continuous imbeddings } j_1: W \to Z = L_\infty(G),\\ &j_2: L_\infty(G) \to L_\infty(G) \text{ the identity, and } j_1 \text{ is compact.}\end{aligned}\right\} \qquad (5')$$

Moreover, the chain of maps in $\mathcal{T}$ becomes:

$$\left.\begin{aligned} &\mathcal{T}_1: L_\infty(G) \xrightarrow{N} L_\infty(G) \xrightarrow{I-Q} Y_1^* \xrightarrow{H} X_1^* \cap W \xrightarrow{j_1} X_1^* \cap L_\infty(G)\\ &\mathcal{T}_0: L_\infty(G) \xrightarrow{N} L_\infty(G) \xrightarrow{Q} Y_0^* \xrightarrow{S} X_{01}^* \cap W \xrightarrow{j_1} X_{01}^* \subset L_\infty(G),\end{aligned}\right\} \qquad (7)$$

and thus $\mathcal{T}$ is a map from $L_\infty(G)$ into $L_\infty(G)$.

On the other hand, the operators P: $X^* \to X^*$, Q: $Y^* \to Y^*$, H: $Y_0^* \to X_1^*$, SQ: $Y_0^* \to X_0^*$ should be thought of in the topology of $L_\infty(G)$, and the norms of P and Q may be ≥ 1, and the norm of H may be different from L. We may not need the exact value of these norms but estimates, say

$$\|P\| \leq c_1, \quad \|I - P\| \leq c_2, \quad \|Q\| \leq c_3, \quad \|I - Q\| \leq c_4, \quad \|H(I - Q)\| \leq L_0, \quad \|SQ\| \leq c_5,$$

and certainly $c_1, c_2, c_3, c_4, c_5 \geq 1$.

5. EXISTENCE THEOREMS

The general assumptions in Section 2 are typical of elliptic problems and we do not repeat them here.

Let us write $g = g(t, D^\alpha x, 0 \leq |\alpha| \leq \mu)$ in the form $g(t, x, D^\alpha x, 1 \leq |\alpha| \leq \mu)$, that is, $g = g(t, u)$, $u \in \mathbb{R}^\mu$, in the form $g = g(t, z, \zeta)$, $z \in \mathbb{R}$, $\zeta \in \mathbb{R}^{\mu-1}$, $t \in G$. We shall denote by L_0, c_5 positive constants so that

$$\|H(I - Q)y\|_Z \leq L_0\|y\|_2, \quad \|SQy\|_Z \leq c_5\|y\|_2 \quad \text{for all} \quad y \in L_2(G). \qquad (8)$$

Note that $L_0 \leq \gamma L$ since $\|H(I - Q)y\|_Z \leq \alpha\|H(I - Q)y\|_2 \leq \alpha L\|y\|$.

THEOREM 1. Under assumptions (3) and (5) with $W \subset Z \subset L_2(G)$, j_1: $W \to Z$, j_2: $Z \to H^{k_0}(G)$ continuous and j_1 compact, let $p = q$, $f \in L_2(G)$, g: $G \times \mathbb{R}^\mu \to \mathbb{R}$ and assume that for suitable positive constants c, C, D, R_0, r, ρ, k we have

$$\|f\|_2 \leq c, \quad |g(t,u)| \leq |G|^{-1/2}C \qquad \text{for } (t,u) \in G \times \mathbb{R}^\mu, \tag{9}$$
$$|g(t,u) - g(t,v)| \leq D|u - v| \qquad \text{for a.a. } t \in G \text{ and } u, v \in \mathbb{R}^\mu, \tag{10}$$
$$\|x(t) - kg(t, x(t), (D^\alpha x)(t), 1 \leq |\alpha| \leq k_0)\|_2 \leq \rho R_0 \quad \text{for } x \in Z \text{ and } \|x\|_Z \leq R_0, \tag{11}$$
$$L_0(c + C) \leq r, \quad \rho c_5 < 1, \tag{12}$$
$$kc_5(c + D\beta r) \leq (1 - \rho c_5)R_0. \tag{13}$$

Then, problem (1) has at least a solution $x \in W$ with $\|x\|_Z \leq R_0 + r$.

If $p > q$, then for every element $\xi_{02} \in X_{02} \subset \operatorname{Ker} E \subset W$ we take $x = \xi_{02} + y$ so that problem $Ex = f(t) + g(t, D^\alpha x)$, $Bx = 0$, is changed into $Ey = \tilde{f}(t) + \tilde{g}(t, Dy)$, $By = 0$, where $\tilde{f} = f - E\xi_{02} = f$ since $\xi_{02} \in \operatorname{Ker} E$, and $\tilde{g}(t, D^\alpha y) = g(t, D^\alpha \xi_{02} + D^\alpha y)$. Now theorem 1 can be completed as follows.

If $p > q$ and there are constants $R_2 > 0$ and c, C, D, R_0, r, ρ, k as above such that for every $\xi_{02} \in X_{02}$ with $\|\xi_{02}\|_W \leq R_2$ the new functions $\tilde{f}$ and $\tilde{g}$ satisfy relations (9)–(13), then problem (1) has ∞ − many solutions $x = x_{01} + \xi_{02} + x_1$, $x \in W \subset Z$, $\|x\|_Z \leq \|\xi_{02}\|_Z + R_0 + r$, namely at least one for every $\xi_{02} \in X_{02}$ with $\|\xi_{02}\|_W \leq R_2$.

Proof. Let $\Omega = S_0 \times S_1$ with $S_0 = \{x_{01} \in X_{01}: \|x_{01}\|_Z \leq R_0\}$, $S_1 = \{x_1 \in X_1: \|x_1\|_Z \leq r\}$, where we note that X_{01} is a finite dimensional subspace, $X_{01} \subset \operatorname{Ker} E \subset W \subset Z$, hence the topology in X_{01} does not depend on the norm we choose. We have taken the norm of Z in S_0 instead of the norm of W.

Now let us consider the transformation $\mathcal{T}$ in (4) with $x_{02} = 0$. Hence, $\mathcal{T}$ is a transformation $(x_{01}, x_1) \to (\bar{x}_{01}, \bar{x}_1)$ which we think of as defined in Ω. Now for $x = x_{01} + x_1$, $x_{01} \in S_0$, $x_1 \in S_1$, we have

$$\bar{x}_1 = \mathcal{T}_1 x = H(I - Q)[f(t) + g(t, (D^\alpha x)(t))]$$
$$\|\bar{x}_1\|_Z \leq L_0(\|f\|_2 + \|g(t, D^\alpha x)\|_2) \leq L_0(c + C) \leq r,$$

using hypotheses (8), (9) and (12).

Thus, $\mathcal{T}_1$ maps Ω into S_1. Moreover, by properties (6) and (8) we also have $\bar{x}_1 \in W$.

We have now

$$\begin{aligned}\bar{x}_{01} &= \mathcal{T}_0 x = x_{01}(t) - kSQ[f(t) + g(t, (D^\alpha x)(t))] \\ &= (x_{01} - SQx_{01})(t) + kSQf(t) + SQ[x_{01}(t) - kg(t, (D^\alpha x_{01})(t), 0 \leq |\alpha| \leq k_0)] \\ &\quad + kSQ[g(t, (D^\alpha x_{01})(t), 0 \leq |\alpha| \leq k_0) - g(t, (D^\alpha x_{01})(t) + (D^\alpha \bar{x}_1)(t), 0 \leq |\alpha| \leq k_0)],\end{aligned}$$

where the first term in the last expression is zero, since by using assumption (3) we have shown in Section 2 that SQ is the identity map on X_{01}. Hence, we obtain

$$\begin{aligned}\|\bar{x}_{01}\|_Z &\leq kc_5c + c_5\rho R_0 + kc_5D\beta\|\bar{x}_1\|_Z \\ &\leq kc_5c + c_5\rho R_0 + kc_5D\beta r \leq R_0,\end{aligned}$$

making use of assumptions (8), (9), (11), (10) and (13).

Thus, $\mathcal{T}_0$ maps Ω into S_0, and $\mathcal{T}$ maps Ω into itself.

Let us prove that $\mathcal{T}: \Omega \to \Omega$ is a compact map. Indeed, $\mathcal{T}_1\Omega$ is a bounded closed subset of W, and this set is then compact in Z because j_1 is a compact map by assumption (5). On the other hand, by relations (6) we note that $\mathcal{T}_0\Omega$ is a bounded closed subset of Ker E which is a finite dimensional space. Thus, $\mathcal{T}\Omega$ is a compact set in Z.

By Schauder's fixed point theorem, $\mathcal{T}: \Omega \to \Omega$ has a fixed point $x \in Z$ and actually $x \in W$ with $\|x\|_Z \leq R_0 + r$ satisfying both the auxiliary and the bifurcation equation, and x is a solution of the original problem (1).

In the particular case when $k_0 = 0$, $2m_0 > n$, then $g = g(t, x)$ depends only on the function $x = x(t)$ and not on its derivatives, and for $x \in W$, then $x = x(t)$ is bounded in G. As stated in Section 2, we can take $Z = L_\infty(G)$. In this situation, the following variant of theorem 1 is of interest.

We denote here by L_0, c_5 positive constants so that

$$\|H(I - Q)y\|_\infty \leq L_0\|y\|_\infty, \quad \|SQy\|_\infty \leq c_5\|y\|_\infty \quad \text{for all } y \in Y^* = L_\infty(G). \tag{14}$$

THEOREM 2. Under assumptions (3) and (5)′ with $W \subset Z = L_\infty(G) \subset L_2(G)$, $k_0 = 0$, $2m_0 > n$, let $p = q$, $f \in L_\infty(G)$, g: $G \times \mathbb{R} \to \mathbb{R}$, and assume that for suitable constants c, C, D, R_0, r, ρ, k we have

$$\|f\|_\infty \leq c, \quad |g(t, u)| \leq C \quad \text{for } (t, u) \in G \times \mathbb{R}; \tag{15}$$

$$|g(t, u) - g(t, v)| \leq D|u - v|, \quad \text{for a.a. } t \in G,\ u, v \in \mathbb{R} \text{ with } |u|, |v| \leq R_0 + r; \tag{16}$$

$$|u - kg(t, u)| \leq \rho R_0, \quad \text{for } t \in G \text{ and } |u| \leq R_0; \tag{17}$$

$$L_0(c + C) \leq r, \quad \rho c_5 < 1; \tag{18}$$

$$kc_5(c + Dr) \leq (1 - \rho c_5)R_0. \tag{19}$$

Then, problem $Ex = f(t) + g(t, x(t))$, $Bx = 0$, has at least a solution $x \in W$ with $\|x\|_\infty \leq R_0 + r$.

If $p > q$, then for every element $\xi_{02} \in X_{02} \subset \text{Ker } E \subset W$, we take $x = \xi_{02} + y$ so that problem $Ex = f(t) + g(t, x(t))$, $Bx = 0$, is changed into $Ey = \tilde{f}(t) + \tilde{g}(t, y(t))$, $By = 0$, where $\tilde{f} = f - E\xi_{02} = f$ since $\xi_{02} \in \text{Ker } E$, $By = 0$, since certainly $B\xi_{02} = 0$, and $\tilde{g}(t, y(t)) = g(t, \xi_{02}(t) + y(t))$. Now theorem 2 can be completed as follows:

If $p > q$ and there are constants $R_2 > 0$ and c, C, D, R_0, r, ρ, k as above such that, for every $\xi_{02} \in X_{02}$ with $\|\xi_{02}\|_\infty \leq R_2$ the new functions $\tilde{f}$ and $\tilde{g}$ satisfy relations (15)–(19), then problem $Ex = f(t) + g(t, x(t))$, $Bx = 0$, has ∞ − many solutions $x = x_{01}(t) + \xi_{02}(t) + x_1(t)$, $x \in W \subset L_\infty(G)$, $\|x\|_\infty \leq \|\xi_{02}\|_\infty + R_0 + r$, namely at least one for every $\xi_{02} \in X_{02}$ with $\|\xi_{02}\|_\infty \leq R_2$.

Proof. Let $\Omega = S_0 \times S_1$ with $S_0 = \{x_{01} \in X_{01}^* : \|x_{01}\|_\infty \leq R_0\}$, $S_1 = \{x_1 \in X_1^* : \|x_1\|_\infty \leq r\}$, with the same remarks as for theorem 1. Now let us consider the transformation $\mathcal{T}$ in (4) (cf. (7)) with $x_{02} = 0$. Hence, $\mathcal{T}$ is a transformation $(x_{01}, x_1) \to (\tilde{x}_{01}, \tilde{x}_1)$ which is defined in Ω. As for theorem 1 we have now

$$\bar{x}_1 = \mathcal{T}_1 x = H(I - Q)[f(t) + g(t, x(t))],$$

$$\|\bar{x}_1\|_\infty \leq L_0(\|f\|_\infty + \|g(t, x(t))\|_\infty) \leq L_0(c + C) \leq r,$$

by using assumptions (14), (15) and (18). Thus, $\mathcal{T}_1\colon \Omega \to S_1$ and analogously, by (7) and (14), we obtain $\bar{x}_1 \in W$ and $\|\bar{x}_1\|_W \leq L(c + C)$.

On the other hand,

$$\begin{aligned}\bar{x}_{01} &= \mathcal{T}_0 x = x_{01}(t) - kSQ[f(t) + g(t, x(t))] \\ &= (x_{01} - SQx_{01})(t) + kSQf(t) + SQ[x_{01}(t) - kg(t, x_{01}(t))] \\ &\quad + kSQ[g(t, x_{01}(t)) - g(t, x_{01}(t) + \bar{x}_1(t))]\end{aligned}$$

and

$$\begin{aligned}\|\bar{x}_{01}\|_\infty &\leq 0 + kc_5 c + c_5 \rho R_0 + kc_5 D\|\bar{x}_1\|_\infty \\ &\leq kc_5 c + c_5 \rho R_0 + kc_5 Dr \leq R_0,\end{aligned}$$

by making use of assumptions (3), (14), (15), (17), (16) and (19).

Thus, $\mathcal{T}_0$ maps Ω into S_0, and $\mathcal{T}$ maps Ω into itself. Here $j_1\colon W \to L_\infty(G)$ is a compact map and therefore the compactness argument is the same as for the previous theorem 1.

The case $p > q$ can be treated as before.

For problems of perturbation, that is, $Ex = \varepsilon[f(t) + g(t, x(t))]$, $Bx = 0$, where $\varepsilon > 0$ is a small parameter, the following corollary holds:

COROLLARY. Under assumptions (3) and (5) with $W \subset Z = L_\infty(G) \subset L_2(G)$, $k_0 = 0$, $2m_0 > n$, let $p = q$, $f \in L_\infty(G)$, $g\colon G \times \mathbb{R} \to \mathbb{R}$, and assume that for suitable constants C, D, R_0, r, ρ, k we have $|g(t, u)| \leq C$ for all $(t, u) \in G \times \mathbb{R}$; $|g(t, u) - g(t, v)| \leq D|u - v|$ for all $t \in G$, $u, v \in \mathbb{R}$ with $|u|, |v| \leq R_0 + r$; $|u - kg(t, u)| \leq \rho R_0$ for all $t \in G$ and $|u| \leq R_0$; $\rho c_5 < 1$. Then, there always are positive constants c, ε_0, r_0 such that for $\|f\|_\infty \leq c$, then εf and εg satisfy all relations (15)–(19) for $\varepsilon \leq \varepsilon_0$, and hence problem

$$Ex = \varepsilon[f(t) + g(t, x(t))], \quad Bx = 0,$$

has at least a solution $x \in W$, with $\|x\|_\infty \leq R_0 + r_0\varepsilon$, for every $\varepsilon \leq \varepsilon_0$.

For $p > q$ a statement analogous to the previous one holds.

Proof. First, εf, εg satisfy relations (15), (16) with c, C, D replaced by εc, εC, εD respectively and relation (17) with k replaced by $\varepsilon^{-1}k$. Now we take $r_0 = L_0(c + C)$ and we apply theorem 2 with r replaced by $r' = r_0\varepsilon = L_0(\varepsilon c + \varepsilon C)$, provided $r_0\varepsilon = L_0(\varepsilon c + \varepsilon C) \leq r$, that is, for $0 < \varepsilon \leq \varepsilon_1 = L_0^{-1}(c + C)^{-1}r$. Now relation (19) becomes

$$\varepsilon^{-1}kc_5[\varepsilon c + \varepsilon DL_0(\varepsilon c + \varepsilon C)] \leq (1 - \rho c_5)R_0,$$

or

$$kc_5(1 + \varepsilon DL_0)c + \varepsilon kDL_0Cc_5 \leq (1 - \rho c_5)R_0$$

and this relation can be satisfied by taking, say $\varepsilon \leq \varepsilon_2$ and

$$\varepsilon_2 \leq \min\{(2kDL_0Cc_5)^{-1}(1 - \rho c_5)R_0, (DL_0)^{-1}\}, c \leq (4kc_5)^{-1}(1 - \rho c_5)R_0.$$

We shall take now $\varepsilon_0 = \min[\varepsilon_1, \varepsilon_2]$.

6. EXAMPLES

It is clear that theorems 1 and 2 hold for both selfadjoint and nonselfadjoint problems, though for selfadjoint problems the stronger theorem of Landesman and Lazer [9] holds. For the sake of simplicity, let us consider first two selfadjoint problems.

Example 1. Let us consider the elliptic problem

$$\left.\begin{aligned} &u_{tt} + u_{ss} + (2\pi^2/T^2)u = f(t, s) + g(u(t, s)), \qquad (t, s) \in G = [0, T] \times [0, T], \\ &u(t, s) = 0 \qquad \text{on } \partial G. \end{aligned}\right\} \tag{20}$$

We know that the problem is strongly elliptic and selfadjoint. The operator E defined by $Eu = u_{tt} + u_{ss} + (2\pi^2/T^2)u$, with the homogeneous Dirichlet boundary conditions above, has eigenvalues and eigenfunctions

$$\lambda_{ab} = (\pi^2/T^2)(2 - a^2 - b^2), \qquad \phi_{ab}(t, s) = (2/T)\sin(a\pi t/T)\sin(b\pi s/T),$$
$$a, b = 1, 2, \ldots .$$

Also, Ker $E = \{c\phi_{11}\} =$ Ker E^*, and thus we can take $\phi_1 = \omega_1 = (2/T)\sin(\pi t/T)\sin(\pi s/T)$, $p = q = 1$. Thus, $Qf = (f, \omega_1)\omega_1$, and for $f = \sum_{(a,b)} c_{ab}\phi_{ab}$, we have

$$\|f\|_2 = \sum_{(a,b)} c_{ab}^2, \qquad (I - Q)f = \sum_{(a,b)\neq(1,1)} c_{ab}\phi_{ab}, \qquad H(I - Q)f = \sum_{(a,b)\neq(1,1)} \lambda_{ab}^{-1} c_{ab}\phi_{ab}.$$

To simplify notations, we take $\lambda_{ab}^* = 2 - a^2 - b^2$, and we note that for $(a, b) \neq (1, 1)$ these numbers have values $-3, -6, -8, -11, \ldots$. Hence

$$\|H(I - Q)f\|_2 = \Big(\sum_{(a,b)\neq(1,1)} \lambda_{ab}^{-2} c_{ab}^2\Big)^{1/2} = (T^2/\pi^2)\Big(\sum_{(a,b)\neq(1,1)} \lambda_{ab}^{*-2} c_{ab}^2\Big)^{1/2}$$
$$\leq (T^2/\pi^2)(1/3)\Big(\sum_{(a,b)\neq(1,1)} c_{ab}^2\Big)^{1/2} \leq (T^2/3\pi^2)\|f\|_2.$$

On the other hand

$$|H(I - Q)f(t, s)| \leq (T^2/\pi^2)\Big(\sum_{(a,b)\neq(1,1)} \lambda_{ab}^{*-2}\Big)^{1/2}\Big(\sum_{(a,b)\neq(1,1)} c_{ab}^2\phi_{ab}^2\Big)^{1/2}$$
$$\leq (T^2/\pi^2)(2/T)\Big(\sum_{(a,b)\neq(1,1)} \lambda_{ab}^{*-2}\Big)^{1/2}\|f\|_2$$
$$\leq (T^2/\pi^2)(2/T)\Big(\sum_{(a,b)\neq(1,1)} \lambda_{ab}^{*-2}\Big)^{1/2} T\|f\|_\infty.$$

Thus,

$$\|H(I - Q)f\|_\infty \leq (2T^2/\pi^2)\Big(\sum_{(a,b)\neq(1,1)} \lambda_{ab}^{*-2}\Big)^{1/2}\|f\|_\infty.$$

For these L_2- and L_∞-estimates (cf. Cesari [3]).

The sum of the series can be evaluated by separating the terms with $1 < a^2 + b^2 < 50$, from those with $a^2 + b^2 \geq 50$, or

$$\sum_{(a,b)\neq(1,1)} \lambda_{ab}^{*-2} = \Big(\sum_{1<a^2+b^2<50} + \sum_{a^2+b^2\geq 50}\Big)\lambda_{ab}^{*-2} = S_1 + S_2.$$

For $a^2 + b^2 \geqslant 50$ we have $(a^2 + b^2)/(a^2 + b^2 - 2) \leqslant 50/48$ (the equality holds for $a = b = 5$), and

$$S_2 = \sum_{a^2+b^2\geqslant 50} (a^2 + b^2 - 2)^{-2} \leqslant (50/48)^2 \sum_{a^2+b^2\geqslant 50} (a^2 + b^2)^{-2}$$
$$\leqslant (50/48)^2 \sum_{a^2+b^2\geqslant 50} \int_{a-1}^{a} \int_{b-1}^{b} (t^2 + s^2)^{-2} \, dt \, ds.$$

It is immediately seen that for $a^2 + b^2 \geqslant 50$ we have $(a-1)^2 + (b-1)^2 \geqslant 32$ (again the equality holds for $a = b = 5$), and

$$S_2 \leqslant (50/48)^2 \iint_{t^2+s^2\geqslant 32} (t^2 + s^2)^{-2} \, dt \, ds = (50/48)^2 \int_{\sqrt{32}}^{+\infty} \int_0^{\pi/2} \rho^{-4} \rho d\rho d\theta$$
$$= (50/48)^2 2^{-1} (32)^{-1} (\pi/2) = (1.085068)(0.0245436) = 0.026632.$$

There are only 29 terms in S_1 whose sum is 0.335210. Hence

$$S_1 + S_2 < 0.335210 + 0.026632 = 0.361842.$$

Thus $(0.361842)^{1/2} = 0.601533$, and

$$\|H(I-Q)f\|_\infty \leqslant (T^2/\pi^2)(2)(0.361842)^{1/2} \|f\|_\infty = (T^2/\pi^2)(1.203066) \|f\|_\infty.$$

We also have

$$|Qf(t,s)| = \left| \left(\int_0^T \int_0^T f(\alpha,\beta)\phi_{11}(\alpha,\beta) \, d\alpha d\beta \right) \phi_{11}(t,s) \right|$$
$$\leqslant (2/T) \left(\int_0^T \sin(\pi\alpha/T) \, d\alpha \right) \left(\int_0^T \sin(\pi\beta/T) \, d\beta \right) \|f\|_\infty (2/T)$$
$$= (2/T)(2/T)(2T/\pi)(2T/\pi) \|f\|_\infty = (16/\pi^2) \|f\|_\infty$$
$$= (1.621139) \|f\|_\infty$$
$$|(I-Q)f(t,s)| \leqslant (2.621139) \|f\|_\infty,$$

and if we take $S : Y_0 \to X_0$ the identity map, we have

$$|SQf(t,s)| \leqslant (1.621139) \|f\|_\infty.$$

Thus, we can take

$$L_0 = (T^2/\pi^2)(1.203066), \qquad c_4 = 2.621139, \qquad c_5 = 1.621139.$$

For instance, for $c = 0.1$, $C = 1$, $D = 2$, $T = 0.5$, $k = 1$, $R_0 = 1$, $\rho = 0.4$, relations (15)–(19) are satisfied since

$$L_0(c + C) = (T^2/\pi^2)(1.203066)(0.1 + 1) = 0.0335214 < 0.0336 = r$$
$$\rho c_5 = (0.4)(1.621139) = 0.648455 < 1$$
$$(1 - \rho c_5)R_0 = 0.351544$$
$$c_5(c + Dr) = (1.621139)(0.1 + 0.0672) = 0.271054 < 0.351544.$$

Thus, for any $f = f(t,s)$ measurable and bounded, $\|f\|_\infty \leqslant 0.1$, and any continuous $g = g(u)$ with $|g(u)| \leqslant 1$, $|g(u) - g(v)| \leqslant 2|u - v|$ for all $|u|, |v| \leqslant 1.0336$, $|u - g(u)| \leqslant 0.4$ for $|u| \leqslant 1$,

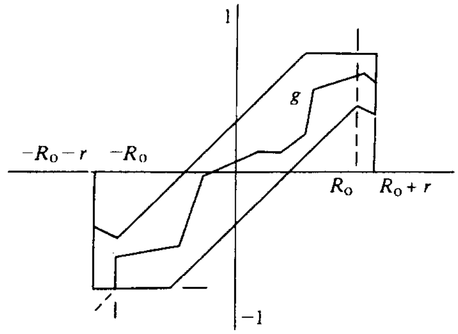

Fig. 1.

problem (20) has at least one solution. In other words, g can be any continuous function, Lipschitzian of constant two, whose graph is within the heavy lines of Fig. 1.

Example 2. Let us consider the strongly elliptic and selfadjoint problem

$$\left.\begin{array}{ll} u_{tt} + u_{ss} + 5(2\pi^2/T^2)u = f(t,s) + g(u(t,s)), & (t,s) \in G = [0,T] \times [0,T], \\ u(t,s) = 0 \quad \text{on} \quad \partial G. & \end{array}\right\} \tag{21}$$

The operator E defined by $Eu = u_{tt} + u_{ss} + 5(2\pi^2/T^2)u$ with the Dirichlet homogeneous boundary conditions above has eigenvalues and eigenfunctions:

$$\lambda_{ab} = (\pi^2/T^2)(5 - a^2 - b^2), \qquad \phi_{ab}(t,s) = (2/T)\sin(a\pi t/T)\sin(b\pi s/T),$$
$$a, b = 1, 2, \ldots .$$

Also, Ker $E = \{c_1\phi_{12} + c_2\phi_{21}\} =$ Ker E^* and thus we can take $\phi_1 = \omega_1 = \phi_{12}$, $\phi_2 = \omega_2 = \phi_{21}$, $p = q = 2$. Thus, $Qf = (f, \omega_1)\omega_1 + (f, \omega_2)\omega_2$, and for $f = \sum_{(a,b)} c_{ab}\phi_{ab}$, we have

$$\|f\|_2 = \sum_{(a,b)} c_{ab}^2, \qquad (I - Q)f = \sum{}' c_{ab}\,\phi_{ab},$$

where Σ' ranges over all (a, b) different from $(1, 2)$ and $(2, 1)$, and then

$$H(I - Q)f = \sum{}' \lambda_{ab}^{-1} c_{ab}\phi_{ab},$$

and the smallest $|\lambda_{ab}|$ with $(a, b) \neq (1, 2)$, and $(a, b) \neq (2.1)$ are $\lambda_{11} = (3\pi^2/T^2)$, $\lambda_{22} = (-3\pi^2/T^2)$. Hence

$$\|H(I - Q)f\|_2 = \left(\sum{}' \lambda_{ab}^{-2} c_{ab}^2\right)^{1/2} \|f\|_2 \leq (T^2/3\pi^2)\,\|f\|_2.$$

On the other hand, as before

$$\|H(I - Q)f\|_\infty \leq (2T^2/\pi^2)\left(\sum{}' \lambda^{*-2}_{ab}\right)^{1/2} \|f\|_\infty,$$

where here $\lambda^*_{ab} = 5 - a^2 - b^2$.

The estimate of the sum of the series can be made as for example 1. For $a^2 + b^2 \geq 50$ we have $(a^2 + b^2)/(a^2 + b^2 - 5) \leq 50/45 = 10/9$, and

$$\sum_{a^2+b^2\geq 50} (a^2 + b^2 - 5)^{-2} \leq (10/9)^2 \sum_{a^2+b^2\geq 50} (a^2 + b^2)^{-2} \leq (1.234567)(0.0007669) = 0.000947$$

There are other 28 terms with $a^2 + b^2 < 50$ and their sum is 0.385298. Then,

$$\sum{}' \lambda_{ab}^{*-2} \leq 0.385298 + 0.000947 = 0.386245.$$

Thus, $(0.386245)^{1/2} = 0.621486$, and

$$\|H(I-Q)f\|_\infty < (T^2/\pi^2)(2)(0.386245)^{1/2} \|f\|_\infty = (T^2/\pi^2)(1.242972) \|f\|_\infty.$$

Therefore, we can take

$$L_0 = (T^2/\pi^2)(1.242972), \qquad c_4 = 2.621139, \qquad c_5 = 1.621139.$$

For instance, for $c = 0.1$, $C = 1$, $D = 2$, $T = 0.5$, $k = 1$, $R_0 = 1$, $\rho = 0.4$, relations (15)–(19) are satisfied since

$$L_0(c + C) = (T^2/\pi^2)(1.242972)(1.1) = 0.0346331 < 0.035 = r$$

$$\rho c_5 = (0.4)(1.621139) = 0.648555 < 1$$

$$(1 - \rho c_5)R_0 = 0.351544$$

$$c_5(c + Dr) = (1.621139)(0.17) = 0.275593 < 0.351544.$$

Thus, for any $f = f(t, s)$ measurable and bounded, $\|f\|_\infty \leq 0.1$, and any continuous $g = g(u)$ with $|g(u)_\infty| \leq 1$, $|g(u) - g(v)| \leq 2|u - v|$ for $|u|, |v| \leq 1.035$, $|u - g(u)| \leq 0.4$ for $|z| \leq 1$, problem (21) has at least one solution. A geometric picture of a possible g is shown in Fig. 1.

Example 3. Let us consider the elliptic nonselfadjoint problem

$$Eu \equiv u_{tt} + u_{ss} + 2\lambda^2 u = f(t, s) + g(u(t, s)), \qquad (t, s) \in G = [0, T] \times [0, T/2], \tag{22}$$

$$\begin{aligned} u(0, s) &= -\sigma(u_t(0, s) - u_t(T, s)), && 0 \leq s \leq T/2, \\ u(T, s) &= \sigma(u_t(0, s) - u_t(T, s)), && 0 \leq s \leq T/2, \\ u(t, 0) &= u(t, T/2) = 0, && 0 \leq t \leq T, \end{aligned} \tag{23}$$

where $\lambda = 2\pi/T$, and $\sigma > 0$ is a given constant. For $\sigma = 0$ we have the Dirichlet conditions $u = 0$ on ∂G. The underlying linear problem is

$$Eu \equiv u_{tt} + u_{ss} + 2\lambda^2 u = 0 \tag{24}$$

with the same boundary conditions (23). The corresponding dual problem is then

$$\begin{aligned} E^*v &= v_{tt} + v_{ss} + 2\lambda^2 v = 0, && (t, s) \in G, \\ v(0, s) &= v(T, s) = -\sigma(v_t(0, s) + v_t(T, s)), && 0 \leq s \leq T/2, \\ v(t, 0) &= v(t, T/2) = 0, && 0 \leq t \leq T. \end{aligned} \tag{25}$$

The nonzero solutions of problem (24), (23) are all proportional to

$$\Phi(t, s) = 2^{3/2} T^{-1} \sin \lambda t \sin \lambda s, \qquad (t, s) \in G;$$

the nonzero solutions of (25), (26) are all proportional to

$$\omega(t, s) = 2^{3/2}T^{-1}(1 + 4\sigma^2\lambda^2)^{-1/2} (\sin \lambda t - 2\sigma\lambda \cos \lambda t) \sin \lambda s.$$

In other words, Ker $E = \{c\phi\}$, Ker $E^* = \{d\omega\}$, $p = q = 1$. Certainly problem (22), (23) is nonselfadjoint and ϕ and ω do not share regions of positivity and negativity in G.

However, an analysis similar to the one for the preceding examples has been possible, leading to an existence theorem for weak solutions of problem (22), (23) under analogous requirements on f and g. Also numerical examples have been exhibited. The intricate details, and the choice of topologies, will be presented elsewhere [11].

Example 4. Let us consider the elliptic nonselfadjoint problem of order $2m = 4$,

$$\left.\begin{aligned} &\Delta^2 u = \varepsilon[f(t, s) + g(u(t, s))], \qquad (t, s) \in G \\ &B : \partial u/\partial x = 0, \qquad \partial(\Delta u)/\partial n = 0 \qquad \text{on } \partial G, \end{aligned}\right\} \tag{26}$$

where $\Delta = \partial^2 u/\partial t^2 + \partial^2 u/\partial s^2$, G is a bounded region in the ts-plane with smooth boundary ∂G, n is the exterior normal to ∂G, and ε is a small parameter, $\varepsilon > 0$.

The linear operator $E = \Delta^2$ with the homogeneous boundary conditions above has Ker $E = \{c_1 + c_2 s + c_3 s^2\}$, while Ker $E^* = \{c\}$, c, c_1, c_2, c_3 constants, $p = 3$, $q = 1$ (cf. Hormander [7, pp. 265–266]). Here we have $2m = 4$, and for weak solutions $u \in H_0^2$ we have $m_0 = 2$, $n = 2$, $4 > n$, so that for $f \in L_2(G)$ the solutions of the linear problem $\Delta^2 u = f$, $Bu = 0$, are in $H_0^2 \cap L_\infty(G)$. The corollary applies.

Upon evaluation of the constants L_0 and c_5, if a real function $g : \mathbb{R} \to \mathbb{R}$ satisfies $|g(u)| \leq C$ for all $|u| \leq R_0 + r$, $|g(u) - g(v)| \leq D|u - v|$ for all $|u|, |v| \leq R_0 + r$, $|u - kg(u)| \leq \rho R_0$ for $|u| \leq R_0$, $\rho c_5 < 1$, for given constants C, R_0, r, D, ρ, k, then there are constants $\varepsilon_0 > 0$, $c > 0$, $r_0 > 0$, such that relations (15)–(19) of the corollary hold and problem (22) has at least one weak solution $u \in H_0^2 \cap L_\infty(G)$ with $\|u\|_\infty \leq R_0 + r\varepsilon_0$ for every $\varepsilon \leq \varepsilon_0$.

This example 4 is not selfadjoint, is not in the classes considered by Shaw in [12], and, for g arbitrary and not differentiable, the usual theorems for perturbation type problems do not apply.

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