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Invariant Subspaces, Dilation Theory, and the Structure of the Predual of a Dual Algebra, I

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0. INTRODUCTION

In the past five years some remarkable new ideas and techniques concerning the structure of the preduals of certain operator algebras have entered the theory of operators on Hilbert space. This has led to new invariant subspace theorems, a new dilation theory, and new techniques for proving the reflexivity of operators. (See the bibliography for a list of pertinent articles.)

In the present paper, which is a natural successor to [11] and [9], this theory of the structure of preduals of operator algebras is carried forward, and several additional interesting results are found. In particular, as corollaries of our main theorems in Section 2 we obtain a new invariant subspace theorem (Theorem 3.8), and some new and quite surprising propositions (Corollaries 3.3 and 3.4) concerning the invariant-subspace lattice of the much studied Bergman shift operator. The results herein were

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the subject matter of most of the lectures (given by the fourth author) at the CBMS/NSF regional conference in Tempe, Arizona in May 1984.

Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. It is well known (cf. [20, p. 40]) that $\mathscr{L}(\mathscr{H})$ is the dual space of the ideal $\mathscr{C}_1(\mathscr{H})$ of trace-class operators under the bilinear functional

$$\langle T, L\rangle = \operatorname{tr}(TL), \qquad T \in \mathscr{L}(\mathscr{H}),\ L \in \mathscr{C}_1(\mathscr{H}). \tag{1}$$

A subalgebra $\mathcal{A}$ of $\mathscr{L}(\mathscr{H})$ that contains $1_{\mathscr{H}}$ and is closed in the weak* topology on $\mathscr{L}(\mathscr{H})$ is called a *dual algebra*. It follows from general principles (cf. [18]) that if $\mathcal{A}$ is a dual algebra, then $\mathcal{A}$ is the dual space of the quotient space $Q_{\mathcal{A}} = \mathscr{C}_1(\mathscr{H})/{}^{\perp}\mathcal{A}$, where ${}^{\perp}\mathcal{A}$ is the pre-annihilator in $\mathscr{C}_1(\mathscr{H})$ of $\mathcal{A}$, under the pairing

$$\langle T, [L]\rangle = \operatorname{tr}(TL), \qquad T \in \mathcal{A},\ [L] \in Q_{\mathcal{A}}. \tag{2}$$

The first new idea in the sequence of developments under consideration was due to Scott Brown [17], who showed that information about the structure of $Q_{\mathcal{A}}$ for certain dual algebras $\mathcal{A}$ led to important new invariant subspace theorems. In particular, if x and y are vectors in $\mathscr{H}$, then the associated rank-one operator $x \otimes y$, defined by $(x \otimes y)(u) = (u, y)\,x$, $u \in \mathscr{H}$, belongs to $\mathscr{C}_1(\mathscr{H})$, so if $\mathcal{A}$ is any dual algebra, we may denote by $[x \otimes y]_{Q_{\mathcal{A}}}$, or simply $[x \otimes y]$ when no confusion will result, the image of $x \otimes y$ in the quotient space $Q_{\mathcal{A}}$. Since every operator L in $\mathscr{C}_1(\mathscr{H})$ can be written as $L = \sum_{i=1}^{\infty} x_i \otimes y_i$ for certain square summable sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ from $\mathscr{H}$ (with convergence in the trace norm, of course), it follows easily that every element of $Q_{\mathcal{A}}$ has the form $\sum_{i=1}^{\infty} [x_i \otimes y_i]$. Brown showed in [17] that for certain subnormal operators T in $\mathscr{L}(\mathscr{H})$, the dual algebra $\mathcal{A}_T$ generated by T has the property that its predual $Q_T = Q_{\mathcal{A}_T}$ consists entirely of elements of the form $[L] = [x \otimes y]$, and thereby solved the difficult invariant subspace problem for subnormal operators. We say that a dual algebra $\mathcal{A}$ has property $(\mathbb{A}_1)$ if this "rank-one" phenomenon occurs in which every $[L]$ in $Q_{\mathcal{A}}$ has the form $[L] = [x \otimes y]$ for certain vectors x and y in $\mathscr{H}$. Over the past five years, many papers have been written showing that the dual algebras $\mathcal{A}_T$ corresponding to various operators T have property $(\mathbb{A}_1)$, and, as mentioned above, this has led to a large number of new invariant subspace theorems. (In particular, most of the papers in the bibliography are of this nature.)

A second new idea in this development occurred in [11] and [12], where the single equation $[L] = [x \otimes y]$ was replaced by an $n \times n$ system of simultaneous equations

$$[L_{ij}] = [x_i \otimes y_j], \qquad 0 \leqslant i, j < n$$

(where the $[L_{ij}]$ belong to $Q_{\mathcal{A}}$ and n is some cardinal number not exceeding $\aleph_0$). Dual algebras $\mathcal{A}$ for which all systems of this form are solvable are said to have property $(\mathbb{A}_n)$. In [11] a very useful dilation theory was developed for a class of contractions T such that $\mathcal{A}_T$ has some property $(\mathbb{A}_n)$, and in [12] it was shown that dual algebras generated by (BCP) operators (first studied in [18]) have property $(\mathbb{A}_{\aleph_0})$.

In this paper we continue to investigate these properties $(\mathbb{A}_n)$. In Section 1 we consider a calculus of these properties and their relations to various other important properties of dual algebras. In Section 2 we specialize to singly generated dual algebras $\mathcal{A}_T$ isomorphic to $H^\infty(\mathbb{T})$, and we prove four basic structure theorems (Theorems 2.3, 2.4, 2.5, and 2.7) which show, among other things, that the class of operators T such that $\mathcal{A}_T$ has property $(\mathbb{A}_{\aleph_0})$ is much larger than one might have expected. (In particular, the Bergman shift belongs to this class.) Finally, in Section 3 we apply these structure theorems from Section 2 to obtain new results on invariant subspaces, dilation theory, and reflexivity for various classes of operators. In particular, we are able to improve various results set forth by Shields [34] about weighted shift operators.

1. General Dual Algebras

In this section we define various properties that general dual algebras $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ may have, and we study certain tensor products and ampliations of dual algebras, paying particular attention to how these properties behave under these constructions. Several structure theorems result. As noted above, if $\mathcal{A}$ is an arbitrary dual algebra and $x, y \in \mathcal{H}$, then $[x \otimes y]$ denotes the image of the rank one operator $x \otimes y$ in $Q_{\mathcal{A}}$. It is an easy consequence of (2) that

$$\langle A, [x \otimes y] \rangle = (Ax, y), \quad A \in \mathcal{A},\ x, y \in \mathcal{H}. \tag{3}$$

Definition 1.1. Let $\mathcal{A}$ be a dual algebra, and let n be any cardinal number satisfying $1 \leqslant n \leqslant \aleph_0$. Then $\mathcal{A}$ will be said to have property $(\mathbb{A}_n)$ provided every $n \times n$ system of simultaneous equations of the form

$$[x_i \otimes y_j] = [L_{ij}], \qquad 0 \leqslant i, j < n \tag{4}$$

(where the $[L_{ij}]$ are arbitrary elements from $Q_{\mathcal{A}}$) has a solution consisting of a pair of sequences $\{x_i\}_{0 \leqslant i < n}$, $\{y_j\}_{0 \leqslant j < n}$ of vectors from $\mathcal{H}$.

This definition applies, in particular, to singly generated dual algebras. If $T \in \mathcal{L}(\mathcal{H})$ and $\mathcal{A}_T$ denotes the smallest subalgebra of $\mathcal{L}(\mathcal{H})$ that contains T and $1_{\mathcal{H}}$ and is closed in the weak* topology, then $\mathcal{A}_T$ is the *dual algebra* *generated by T.* If, in the terminology of [11], $T \in \mathbb{A}_n(\mathscr{H})$ for some cardinal number n, $1 \leqslant n \leqslant \aleph_0$, then, according to the definition just given, $\mathcal{A}_T$ has property $(\mathbb{A}_n)$; see Section 2.

Suppose now that n is a positive integer, and let $\tilde{\mathscr{H}}_n$ denote the Hilbert space consisting of the direct sum of n copies of $\mathscr{H}$. Then, of course, $\mathscr{L}(\tilde{\mathscr{H}}_n)$ can be identified with the algebra $M_n(\mathscr{L}(\mathscr{H}))$ consisting of all $n \times n$ operator matrices (A_{ij}) with entries from $\mathscr{L}(\mathscr{H})$, where a matrix (A_{ij}) operates on $\tilde{\mathscr{H}}_n$ in the obvious way (cf. [15, Prob. 12H]). If $\mathcal{A}$ is a subalgebra of $\mathscr{L}(\mathscr{H})$, we denote by $M_n(\mathcal{A})$ that subalgebra of $\mathscr{L}(\tilde{\mathscr{H}}_n)$ consisting of all those $n \times n$ matrices with entries from $\mathcal{A}$. It is easy to see that if $\mathcal{A}$ is a dual algebra, then so is $M_n(\mathcal{A})$, since it necessarily contains $1_{\tilde{\mathscr{H}}_n}$ and is ultraweakly closed. Thus $M_n(\mathcal{A})$ is the dual space of $Q_{M_n(\mathcal{A})} = \mathscr{C}_1(\tilde{\mathscr{H}}_n)/{}^{\perp}M_n(\mathcal{A})$. The following lemma provides a convenient identification of $Q_{M_n(\mathcal{A})}$.

LEMMA 1.2. *If $\mathcal{A} \subset \mathscr{L}(\mathscr{H})$ is a dual algebra, then the predual $Q_{M_n(\mathcal{A})}$ of $M_n(\mathcal{A})$ is identifiable, as a Banach space, with the Banach space $M_n(Q_{\mathcal{A}})$ consisting of all $n \times n$ matrices with entries from $Q_{\mathcal{A}}$. Under this identification, the duality between $M_n(\mathcal{A})$ and $M_n(Q_{\mathcal{A}})$ is given by*

$$\langle (A_{ij}), ([L_{ij}]) \rangle = \sum_{i,j=1}^{n} \langle A_{ij}, [L_{ij}] \rangle, \qquad (A_{ij}) \in M_n(\mathcal{A}), ([L_{ij}]) \in M_n(Q_{\mathcal{A}}) \tag{5}$$

and the norm on $M_n(Q_{\mathcal{A}})$ is the norm that accrues to it as a linear manifold in $M_n(\mathcal{A})^*$. In particular, if $\tilde{x} = (x_1, ..., x_n)$ and $\tilde{y} = (y_1, ..., y_n)$ belong to $\tilde{\mathscr{H}}_n$, then $[\tilde{x} \otimes \tilde{y}]_{Q_{M_n(\mathcal{A})}}$ is identified with the $n \times n$ matrix $([x_j \otimes y_i]_{Q_{\mathcal{A}}})$.

Proof. Suppose first that $\{[L_{ij}]\}_{i,j=1}^{n}$ is a doubly indexed collection of n^2 elements of $Q_{\mathcal{A}}$ and $(A_{ij}) \in M_n(\mathcal{A})$. Since

$$\begin{aligned}
\sum_{i,j=1}^{n} |\langle A_{ij}, [L_{ij}] \rangle| &\leqslant \sum_{i,j=1}^{n} \|A_{ij}\| \, \|[L_{ij}]\| \\
&\leqslant \left(\sum_{i,j=1}^{n} \|A_{ij}\| \right) \max_{i,j} \|[L_{i,j}]\| \\
&\leqslant n^2 (\max_{i,j} \|A_{ij}\|)(\max_{i,j} \|[L_{ij}]\|) \\
&\leqslant n^2 (\max_{i,j} \|[L_{ij}]\|) \, \|(A_{ij})\|,
\end{aligned}$$

it is clear that (5) defines a bounded linear functional on $M_n(\mathcal{A})$, which we may denote by $([L_{ij}])$. We define $\|([L_{ij}])\|$ to be the norm of this linear functional. It is easy to see that if $\{(A_{ij}^{(\lambda)})\}$ is a net in $M_n(\mathcal{A})$, then $\{(A_{ij}^{(\lambda)})\}$ converges ultraweakly to an oprator (A_{ij}^0) in $M_n(\mathcal{A})$ if and only if for each fixed i and j, the net $\{A_{ij}^{(\lambda)}\}$ converges ultraweakly to A_{ij}^0. It follows that the functional induced on $M_n(\mathcal{A})$ by $([L_{ij}])$ via (5) is also ultraweakly continuous, and thus (cf. [15, Prob. 15J]) corresponds to an element of the predual $Q_{M_n(\mathcal{A})}$. On the other hand, if $[L] \in Q_{M_n(\mathcal{A})}$, then by restricting the ultraweakly continuous functional induced on $M_n(\mathcal{A})$ by $[L]$ to those matrices (A_{ij}) all of whose entries except the (i_0, j_0) entry are equal to zero, we obviously obtain an ultraweakly continuous linear functional on $\mathcal{A}$, which may be denoted by $[L_{i_0 j_0}]$, since it is induced by an element of $Q_{\mathcal{A}}$. Letting i_0 and j_0 range over the set $\{1, ..., n\}$, we obtain a matrix $([L_{ij}])$ corresponding to $[L]$, and $\langle (A_{ij}), [L] \rangle_{M_n(\mathcal{A})}$ obviously agrees with $\langle (A_{ij}), ([L_{ij}]) \rangle$ as defined by (5). That $[\tilde{x} \otimes \tilde{y}]_{Q_{M_n(\mathcal{A})}}$ is identified with the matrix $([x_j \otimes y_i]_{Q_{\mathcal{A}}})$ follows immediately from the fact that if $(A_{ij}) \in M_n(\mathcal{A})$ and satisfies $A_{ij} = 0$ whenever $i \neq i_0$ and $j \neq j_0$, then

$$\begin{aligned}\langle (A_{ij}), [\tilde{x} \otimes \tilde{y}] \rangle &= ((A_{ij})\, \tilde{x}, \tilde{y})_{\tilde{\mathcal{H}}_n} \\ &= (A_{i_0 j_0} x_{j_0}, y_{i_0})_{\mathcal{H}} = \langle A_{i_0 j_0}, [x_{j_0} \otimes y_{i_0}] \rangle.\end{aligned}$$

Since it is clear from the definition above of the norm on $M_n(Q_{\mathcal{A}})$ that the mapping $[L] \to ([L_{ij}])$ of $Q_{M_n(\mathcal{A})}$ onto $M_n(Q_{\mathcal{A}})$ is norm-preserving, the proof is complete.

The next proposition will frequently be used in what follows.

PROPOSITION 1.3. *Suppose $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra and n is a positive integer. Then $\mathcal{A}$ has property $(\mathbb{A}_n)$ if and only if the dual algebra $M_n(\mathcal{A})$ has property $(\mathbb{A}_1)$.*

Proof. According to Lemma 1.2, if $[L] \in Q_{M_n(\mathcal{A})}$, then we may identify $[L]$ with an $n \times n$ matrix $([L_{ji}])$, where the $[L_{ji}]$ belong to $Q_{\mathcal{A}}$. Furthermore if $\tilde{x} = (x_1, ..., x_n)$ and $\tilde{y} = (y_1, ..., y_n)$ are (unknown) vectors in $\tilde{\mathcal{H}}_n$, then $[\tilde{x} \otimes \tilde{y}] = ([x_j \otimes y_i])$. Thus the solvability of the equation $[\tilde{x} \otimes \tilde{y}] = [L]$ is equivalent to the solvability of $([x_j \otimes y_i]) = ([L_{ji}])$, which is obviously equivalent to the solvability of the system (4). Thus the proof is complete.

The following definitions lead to an approach for showing that a dual algebra $\mathcal{A}$ has property $(\mathbb{A}_1)$.

DEFINITION 1.4. Suppose $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra and $0 \leqslant \theta < +\infty$. We denote by $\mathcal{X}_\theta(\mathcal{A})$ the set of all $[L]$ in $Q_{\mathcal{A}}$ such that there exist sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ in $\mathscr{H}$ satisfying the following conditions:

$$\|[x_i \otimes z]\| \to 0, \ \|[z \otimes x_i]\| \to 0,$$

$$\|[y_i \otimes z]\| \to 0, \ \|[z \otimes y_i]\| \to 0, \qquad z \in \mathscr{H}, \tag{6}$$

$$\|x_i\| \leqslant 1, \qquad \|y_i\| \leqslant 1, \qquad 1 \leqslant i < \infty, \tag{7}$$

and

$$\limsup_{i \to \infty} \|[L] - [x_i \otimes y_i]\| \leqslant \theta. \tag{8}$$

Note that it follows from (6) and (3) (with $A = 1$) that the sequences $\{x_i\}$ and $\{y_i\}$ converge weakly to zero in $\mathscr{H}$.

DEFINITION 1.5. Suppose $\mathfrak{A} \subset \mathscr{L}(\mathscr{H})$ is a dual algebra and $0 \leqslant \theta < \gamma < +\infty$. We say that $\mathfrak{A}$ has property $X_{\theta,\gamma}$ if the closed absolutely convex hull of the set $\mathscr{X}_\theta(\mathfrak{A})$ in Definition 1.4 contains the closed ball $B_{0,\gamma}$ of radius γ centered at the origin in $Q_{\mathfrak{A}}$,

$$\overline{\text{aco}}\, \mathscr{X}_\theta(\mathfrak{A}) \supset \{[L] \in Q_{\mathfrak{A}} : \|[L]\| \leqslant \gamma\} = B_{0,\gamma}.$$

The following proposition shows how property $X_{0,\gamma}$ transfers from a dual algebra $\mathfrak{A}$ to $M_n(\mathfrak{A})$.

PROPOSITION 1.6. *Suppose $\mathfrak{A} \subset \mathscr{L}(\mathscr{H})$ is a dual algebra that has property $X_{0,\gamma}$ for some $\gamma > 0$. Then for every positive integer n, $M_n(\mathfrak{A})$ has property $X_{0,\gamma/n^2}$.*

Proof. One knows (cf. [19, Proposition 2.2]) that if E is a subset of a Banach space $\mathscr{B}$, then $\overline{\text{aco}}(E)$ contains the closed ball of radius ρ centered at the origin in $\mathscr{B}$ if and only if

$$\sup_{x \in E} |\langle x, \phi \rangle| \geqslant \rho \, \|\phi\|, \qquad \phi \in \mathscr{B}^*. \tag{9}$$

Thus what we must show, using Lemma 1.2 to identify $Q_{M_n(\mathfrak{A})}$ with $M_n(Q_{\mathfrak{A}})$, is that

$$\sup_{([L_{ij}]) \in \mathscr{X}_0(M_n(\mathfrak{A}))} |\langle (A_{ij}), ([L_{ij}]) \rangle| \geqslant (\gamma/n^2) \|(A_{ij})\| \tag{10}$$

for every matrix (A_{ij}) in $M_n(\mathfrak{A})$. Of course, to do this we must first identify some elements $([L_{ij}])$ of $\mathscr{X}_0(M_n(\mathfrak{A}))$. To this end, suppose that $[L] \in \mathscr{X}_0(\mathfrak{A})$. Then, according to Definition 1.4, there exist sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ in $\mathscr{H}$ satisfying (6), (7), and (8) with $\theta = 0$. For any fixed i_0 and j_0, $1 \leqslant i_0, j_0 \leqslant n$, and every $i \in \mathbb{N}$, we define $\tilde{x}_{i,i_0}$ to be the vector in $\tilde{\mathscr{H}}_n$ that has x_i as the i_0-component and zeros elsewhere, and we define $\tilde{y}_{i,j_0}$ similarly. Then $\|\tilde{x}_{i,i_0}\| \leqslant 1$, $\|\tilde{y}_{i,j_0}\| \leqslant 1$, for all i, and it is obvious from Lemma 1.2 and (6) that the sequences $\{\tilde{x}_{i,i_0}\}_{i=1}^{\infty}$ and $\{\tilde{y}_{i,j_0}\}$ satisfy $\|[\tilde{z} \otimes \tilde{x}_{i,i_0}]\| \to 0$, $\tilde{z} \in \tilde{\mathscr{H}}_n$, and the other counterparts of (6). Furthermore, if we define $[L^{j_0,i_0}] \in M_n(Q_{\mathfrak{A}})$ to be that $n \times n$ matrix $([L_{ij}])$ whose j_0, i_0 entry is equal to $[L]$ and whose other entries are equal to zero, then it is obvious that

$$\|[\tilde{x}_{i,i_0} \otimes \tilde{y}_{i,j_0}] - [L^{j_0,i_0}]\| = \|[x_i \otimes y_i] - [L]\| \to 0,$$

so we conclude that the element $[L^{j_0,i_0}] \in \mathscr{X}_0(M_n(\mathfrak{A}))$, and this is true for arbitrary $1 \leqslant i_0, j_0 \leqslant n$. In other words, we conclude that $\mathscr{X}_0(M_n(\mathfrak{A}))$ contains all those $n \times n$ matrices in $M_n(Q_{\mathfrak{A}})$ with the properties that all but one of the entries is zero and the nonzero entry belongs to $\mathscr{X}_0(\mathfrak{A})$. It is clear from this and the inequality $\sup_{[L] \in \mathscr{X}_0(\mathfrak{A})} |\langle A, [L]\rangle| \geqslant \gamma \|A\|$, valid for all A in $\mathfrak{A}$, that if $(A_{ij}) \in M_n(\mathfrak{A})$, then

$$\sup_{([L_{ij}]) \in \mathscr{X}_0(M_n(\mathfrak{A}))} |\langle (A_{ij}), ([L_{ij}])\rangle| \geqslant \gamma \|A_{i_0 j_0}\|, \quad 1 \leqslant i_0, j_0 \leqslant n. \tag{11}$$

Thus n^2 multiplied by the left hand side of (11) dominates $\gamma(\sum_{i,j=1}^{n} \|A_{ij}\|)$, which in turn dominates $\gamma \|(A_{ij})\|$, and this is the desired inequality (10).

We are now ready to prove our first structure theorem about dual algebras. The following lemma contains the essence of what is to be proved.

LEMMA 1.7. *Suppose that* $\mathfrak{A} \subset \mathscr{L}(\mathscr{H})$ *is a dual algebra with property* $X_{\theta,\gamma}$ *for some* $0 \leqslant \theta < \gamma < +\infty$, *that* $[L] \in Q_{\mathfrak{A}}$, *and that* $\varepsilon > 0$. *Suppose also that vectors* x_0, y_0, *and a finite set* $\{z_1, ..., z_t\}$ *from* $\mathscr{H}$ *are given. Then there exist vectors* x *and* y *in* $\mathscr{H}$ *satisfying the following conditions*:

$$\|[L] - [x \otimes y]\| \leqslant \left(\frac{\theta + \varepsilon}{\gamma}\right) \|[L] - [x_0 \otimes y_0]\|, \tag{12}$$

$$\max\{\|x_0 - x\|, \|y_0 - y\|\} \leqslant \left(\frac{1+\varepsilon}{\gamma}\right)^{1/2} \|[L] - [x_0 \otimes y_0]\|^{1/2}, \tag{13}$$

$$\begin{aligned} \|x\| &\leqslant \left(\|x_0\|^2 + \left(\frac{1}{\gamma}\right)\|[L] - [x_0 \otimes y_0]\| + \varepsilon\right)^{1/2}, \\ \|y\| &\leqslant \left(\|y_0\|^2 + \left(\frac{1}{\gamma}\right)\|[L] - [x_0 \otimes y_0]\| + \varepsilon\right)^{1/2}, \end{aligned} \tag{14}$$

and

$$\begin{aligned} &\|[(x_0-x)\otimes z_j]\|<\varepsilon, \qquad \|[z_j\otimes(x_0-x)]\|<\varepsilon, \quad 1\leqslant j\leqslant t,\\ &\|[(y_0-y)\otimes z_j]\|<\varepsilon, \qquad \|[z_j\otimes(y_0-y)]\|<\varepsilon, \quad 1\leqslant j\leqslant t. \end{aligned} \tag{15}$$

Proof. If $[L]=[x_0\otimes y_0]$ we set $x=x_0$ and $y=y_0$. Thus we may suppose that $\|[L]-[x_0\otimes y_0]\|=\eta>0$, and we choose $\delta>0$ so small that

$$\frac{\theta+\varepsilon}{\gamma}-\left(\delta+\frac{\theta+\delta}{\gamma}\right)=\tau>0. \tag{16}$$

Since $\mathcal{A}$ has property $X_{\theta,\gamma}$, we know that $\overline{\mathrm{aco}}\,\mathcal{X}_\theta(\mathcal{A})$ contains the closed ball with radius γ centered at the origin in $Q_{\mathcal{A}}$. Thus with $[L]$, x_0, and y_0 as given, we can choose elements $[L_1],\dots,[L_m]$ in $\mathcal{X}_\theta(\mathcal{A})$ and scalars $\alpha_1,\dots,\alpha_m$ satisfying

$$\left\|[L]-[x_0\otimes y_0]-\sum_{j=1}^m \alpha_j[L_j]\right\|<\delta\eta \text{ and } \sum_{j=1}^m |\alpha_j|\leqslant\frac{\eta}{\gamma}. \tag{17}$$

Furthermore, by definition of $\mathcal{X}_\theta(\mathcal{A})$, for each fixed $1\leqslant j\leqslant m$, there exist sequences $\{x_i^{(j)}\}_{i=1}^\infty$ and $\{y_i^{(j)}\}_{i=1}^\infty$ in $\mathcal{H}$ satisfying (6) for every z in $\mathcal{H}$, (7), and (8) with $[L_j]$ replacing $[L]$. Thus there exists an integer i_0 sufficiently large that

$$\|[L_j]-[x_i^{(j)}\otimes y_i^{(j)}]\|<\theta+\delta, \qquad i\geqslant i_0,\ 1\leqslant j\leqslant m. \tag{18}$$

Thus, combining (17) and (18) we obtain

$$\begin{gathered}\left\|[L]-[x_0\otimes y_0]-\sum_{j=1}^m \alpha_j[x_{i_j}^{(j)}\otimes y_{i_j}^{(j)}]\right\|<\left(\delta+\frac{\theta+\delta}{\gamma}\right)\eta,\\ i_0\leqslant i_j,\ 1\leqslant j\leqslant m.\end{gathered} \tag{19}$$

Choose β_j so that $\beta_j^2=\alpha_j$, $1\leqslant j\leqslant m$. We set

$$x=x_0+\sum_{j=1}^m \beta_j x_{i_j}^{(j)}, \qquad y=y_0+\sum_{j=1}^m \beta_j y_{i_j}^{(j)}, \tag{20}$$

where the increasing sequence $\{i_1,\dots,i_m\}$ consists of positive integers greater than i_0 to be chosen, one-by-one, in the order indicated, so that (12), (13), (14), and (15) are valid. From (20) and (17) we obtain

$$\|x-x_0\|^2\leqslant\frac{\eta}{\gamma}+\sum_{\substack{j,k=1\\ j\neq k}}^m |\beta_j\beta_k|\,|(x_{i_j}^{(j)},x_{i_k}^{(k)})|, \tag{21}$$

$$\|x\|^2 = \|x_0 + (x - x_0)\|^2 \leqslant \|x_0\|^2 + \frac{\eta}{\gamma} + \sum_{\substack{j,k=1\\ j\neq k}}^{m} |\beta_j\beta_k|\,|(x_{i_j}^{(j)}, x_{i_k}^{(k)})| + 2\sum_{j=1}^{m} |\beta_j|\,|(x_{i_j}^{(j)}, x_0)|, \tag{22}$$

and similar upper bounds for $\|y - y_0\|^2$ and $\|y\|^2$. Moreover, again from (20), we obtain

$$\|[(x - x_0)\otimes z_k]\| \leqslant \sum_{j=1}^{m} |\beta_j|\,\|[x_{i_j}^{(j)}\otimes z_k]\|, \qquad 1 \leqslant k \leqslant t, \tag{23}$$

and similar upper bounds for the other three expressions on the left-hand side of (15). Finally, from (20) and (19) we have

$$\|[L] - [x\otimes y]\| \leqslant \left(\delta + \frac{\theta+\delta}{\gamma}\right)\eta + \sum_{\substack{j,k=1\\ j\neq k}}^{m} |\beta_j\beta_k|\,\|[x_{i_j}^{(j)}\otimes y_{i_k}^{(k)}]\| + \sum_{j=1}^{m} |\beta_j|\,\|[x_0\otimes y_{i_j}^{(j)}]\| + \sum_{j=1}^{m} |\beta_j|\,\|[x_{i_j}^{(j)}\otimes y_0]\|. \tag{24}$$

Thus we see from (21), (22), (23), and (24), that for x and y to satisfy (12), (13), (14), and (15), it suffices to choose the increasing sequence $\{i_1,..., i_m\}$ so that the following are valid:

$$\sum_{\substack{j,k=1\\ j\neq k}}^{m} |\beta_j\beta_k|\,|(x_{i_j}^{(j)}, x_{i_k}^{(k)})| \leqslant \frac{\varepsilon\eta}{\gamma}, \tag{25}$$

$$\sum_{\substack{j,k=1\\ j\neq k}}^{m} |\beta_j\beta_k|\,|(x_{i_j}^{(j)}, x_{i_k}^{(k)})| + 2\sum_{j=1}^{m} |\beta_j|\,|(x_{i_j}^{(j)}, x_0)| \leqslant \varepsilon, \tag{26}$$

$$\sum_{j=1}^{m} |\beta_j|\,\|[x_{i_j}^{(j)}\otimes z_k]\| \leqslant \varepsilon, \qquad 1 \leqslant k \leqslant t, \tag{27}$$

$$\sum_{\substack{j,k=1\\ j\neq k}}^{m} |\beta_j\beta_k|\,\|[x_{i_j}^{(j)}\otimes y_{i_k}^{(k)}]\| + \sum_{j=1}^{m} |\beta_j|\,\|[x_0\otimes y_{i_j}^{(j)}]\| + \sum_{j=1}^{m} |\beta_j|\,\|[x_{i_j}^{(j)}\otimes y_0]\| \leqslant \tau\eta, \tag{28}$$

and similar inequalities involving the $y_{i_j}^{(j)}$. That it is possible to choose the sequence $\{i_1,\ldots, i_m\}$ so that (25), (26), (27), (28), and the corresponding inequalities involving the $y_{i_j}^{(j)}$ are satisfied now follows easily by a finite selection process, using the fact (from (6)) that for any fixed w in $\mathscr{H}$, we have

$$\lim_i \|[w\otimes x_i^{(j)}]\| = \lim_i \|[x_i^{(j)}\otimes w]\| = 0, \qquad 1\leqslant j\leqslant m, \tag{29}$$

and the corresponding relations for the sequence $\{y_i^{(j)}\}_{i=1}^{\infty}$. (One uses here also the elementary fact (deduced from (2)) that (29) implies that $\lim_i |(w, x_i^{(j)})| = 0$, $1\leqslant j\leqslant m$.) In other words, we choose $i_1\geqslant i_0$ so that all the terms involving $x_{i_1}^{(1)}$ or $y_{i_1}^{(1)}$ and vectors already fixed are appropriately small. Next we choose $i_2\geqslant i_1$ so that all the terms involving $x_{i_2}^{(2)}$ or $y_{i_2}^{(2)}$ and vectors already fixed (including the $x_{i_1}^{(1)}$ and $y_{i_1}^{(1)}$) are appropriately small, etc. Further details of this selection are left to the reader.

From Lemma 1.7 we easily deduce the following lemma.

LEMMA 1.8. *Suppose that $\mathfrak{A}\subset\mathscr{L}(\mathscr{H})$ is a dual algebra with property $X_{\theta,\gamma}$ for some $\gamma>\theta\geqslant 0$, that $[L]\in Q_{\mathfrak{A}}$, and that $\varepsilon>0$. Suppose also that a finite set of vectors $\{z_1,\ldots, z_t\}$ from $\mathscr{H}$ is given. Then there exist vectors x and y in $\mathscr{H}$ such that*

$$[L] = [x\otimes y], \tag{30}$$

$$\max\{\|x\|, \|y\|\} \leqslant (1+\varepsilon)\left(\frac{1}{\gamma-\theta}\right)^{1/2} \|[L]\|^{1/2}, \tag{31}$$

and

$$\begin{aligned} &\|[x\otimes z_j]\|<\varepsilon, \qquad \|[z_j\otimes x]\|<\varepsilon, \quad 1\leqslant j\leqslant t,\\ &\|[y\otimes z_j]\|<\varepsilon, \qquad \|[z_j\otimes y]\|<\varepsilon, \quad 1\leqslant j\leqslant t. \end{aligned} \tag{32}$$

Proof. We choose $\varepsilon_1>0$ so small that $\varepsilon_1<\varepsilon$, $\varepsilon_1<1$, $\mu=(\theta+\varepsilon_1)/\gamma<1$, and

$$\left(\frac{\|[L]\|}{\gamma-\theta-\varepsilon_1}+\frac{\varepsilon_1}{1-\varepsilon_1}\right)^{1/2} < (1+\varepsilon)\left(\frac{\|[L]\|}{\gamma-\theta}\right)^{1/2}. \tag{33}$$

We next define sequences $\{x_k\}_{k=0}^{\infty}$ and $\{y_k\}_{k=0}^{\infty}$ from $\mathscr{H}$ by induction. Set $x_{-1}=y_{-1}=x_0=y_0=0$, and suppose that, for some nonnegative integer n, sequences $\{x_0,\ldots, x_n\}$ and $\{y_0,\ldots, y_n\}$ have been chosen to satisfy

$$\|[L]-[x_k\otimes y_k]\| \leqslant \mu^k \|[L]\|, \qquad 0\leqslant k\leqslant n, \tag{34}$$

$$\max\{\|x_k - x_{k-1}\|, \|y_k - y_{k-1}\|\} \leqslant \left(\frac{1+\varepsilon_1}{\gamma}\right)^{1/2} \mu^{(k-1)/2} \|[L]\|^{1/2}, \quad 0 \leqslant k \leqslant n, \tag{35}$$

$$\max\{\|x_k\|, \|y_k\|\} \leqslant \left(\sum_{j=0}^{k-1} \left\{\frac{\|[L]\|}{\gamma} \mu^j + \varepsilon_1^{j+1}\right\}\right)^{1/2}, \quad 0 \leqslant k \leqslant n, \tag{36}$$

and

$$\begin{aligned} &\|[(x_k - x_{k-1}) \otimes z_j\| < \frac{\varepsilon_1}{2^k}, \qquad \|[z_j \otimes (x_k - x_{k-1})]\| < \frac{\varepsilon_1}{2^k}, \\ &\qquad\qquad 1 \leqslant j \leqslant t, 0 \leqslant k \leqslant n, \\ &\|[(y_k - y_{k-1}) \otimes z_j]\| < \frac{\varepsilon_1}{2^k}, \qquad \|[z_j \otimes (y_k - y_{k-1})]\| < \frac{\varepsilon_1}{2^k}, \\ &\qquad\qquad 1 \leqslant j \leqslant t, 0 < k \leqslant n. \end{aligned} \tag{37}$$

Then we may apply Lemma 1.7 (with x_0, y_0, and ε of that lemma taken to be x_n, y_n, and $\varepsilon_2 = \min\{\varepsilon_1^{n+1}, \varepsilon_1/2^{n+1}\}$, respectively) to conclude the existence of vectors x_{n+1} and y_{n+1} in $\mathscr{H}$ such that

$$\|[L] - [x_{n+1} \otimes y_{n+1}]\| \leqslant \mu \cdot \|[L] - [x_n \otimes y_n]\| \leqslant \mu^{n+1} \|[L]\|, \tag{38}$$

$$\begin{aligned} \max\{\|x_{n+1} - x_n\|, \|y_{n+1} - y_n\|\} &\leqslant \left(\frac{1+\varepsilon_2}{\gamma}\right)^{1/2} \|[L] - [x_n \otimes y_n]\|^{1/2} \\ &\leqslant \left(\frac{1+\varepsilon_1}{\gamma}\right)^{1/2} \mu^{n/2} \|[L]\|^{1/2}, \end{aligned} \tag{39}$$

$$\begin{aligned} \max\{\|x_{n+1}\|, \|y_{n+1}\|\} &\leqslant \Big(\max\{\|x_n\|^2, \|y_n\|^2\} \\ &\qquad + \frac{1}{\gamma} \|[L] - [x_n \otimes y_n]\| + \varepsilon_2\Big)^{1/2} \\ &\leqslant \left(\sum_{j=0}^{n-1} \left\{\frac{\|[L]\|}{\gamma} \mu^j + \varepsilon_1^{j+1}\right\} + \frac{\|[L]\|}{\gamma} \mu^n + \varepsilon_1^{n+1}\right)^{1/2} \\ &= \left(\sum_{j=0}^{n} \left\{\frac{\|[L]\|}{\gamma} \mu^j + \varepsilon_1^{j+1}\right\}\right)^{1/2}, \end{aligned} \tag{40}$$

and

$$\|[(x_{n+1}-x_n)\otimes z_j]\| < \frac{\varepsilon_1}{2^{n+1}}, \quad \|[z_j \otimes (x_{n+1}-x_n)]\| < \frac{\varepsilon_1}{2^{n+1}}, \quad 1 \leqslant j \leqslant t,$$
$$\|[(y_{n+1}-y_n)\otimes z_j]\| < \frac{\varepsilon_1}{2^{n+1}}, \quad \|[(z_j \otimes (y_{n+1}-y_n)]\| < \frac{\varepsilon_1}{2^{n+1}}, \quad 1 \leqslant j \leqslant t. \tag{41}$$

Thus, by induction, there exist sequences $\{x_k\}_{k=0}^{\infty}$ and $\{y_k\}_{k=0}^{\infty}$ in $\mathscr{H}$ that satisfy (34), (35), (36), and (37) for $k=0, 1, 2,\ldots$. Since $\mu < 1$, it follows easily from (35) that these sequences are Cauchy, and hence converge; we set $x = \lim x_k$ and $y = \lim y_k$. It is obvious from (34) and the inequality $\|[u \otimes v]\| \leqslant \|u\| \, \|v\|$ that (30) is satisfied. Furthermore, for any $1 \leqslant j \leqslant t$, we have from (37) that

$$\begin{aligned}\|[x\otimes z_j]\| = \|[(x-x_0)\otimes z_j]\| &= \left\| \left[\sum_{k=1}^{\infty} (x_k - x_{k-1})\otimes z_j \right] \right\| \\ &\leqslant \sum_{k=1}^{\infty} \|[(x_k - x_{k-1})\otimes z_j]\| \leqslant \sum_{k=1}^{\infty} \frac{\varepsilon_1}{2^k} = \varepsilon_1 < \varepsilon,\end{aligned}$$

and the other inequalities in (32) are proved similarly. Finally, to see that (31) is valid, we use (36), (33) and the definition of μ:

$$\begin{aligned}\|x\| = \lim_k \|x_{k+1}\| &\leqslant \lim_k \left(\sum_{j=0}^{k} \left\{ \frac{\|[L]\|}{\gamma} \mu^j + \varepsilon_1^{j+1} \right\} \right)^{1/2} \\ &= \left(\frac{\|[L]\|}{\gamma} \left(\frac{1}{1-\mu} \right) + \frac{\varepsilon_1}{1-\varepsilon_1} \right)^{1/2} \\ &= \left(\frac{\|[L]\|}{\gamma - \theta - \varepsilon_1} + \frac{\varepsilon_1}{1-\varepsilon_1} \right)^{1/2} \\ &\leqslant (1+\varepsilon)\left(\frac{\|[L]\|}{\gamma - \theta} \right)^{1/2},\end{aligned}$$

and the corresponding inequality for $\|y\|$ is obtained similarly. Thus the proof is complete.

Our first structure theorem is an easy corollary of Lemma 1.8.

THEOREM 1.9. *Suppose $\mathfrak{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra with property $X_{\theta,\gamma}$ for some $\gamma > \theta \geqslant 0$. Then for every $[L]$ in $Q_{\mathfrak{A}}$, there exist sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ such that*

(a) $[L] = [x_i \otimes y_i]$, $i \in \mathbb{N}$,

(b) $\limsup_i \|x_i\| \leqslant (1/(\gamma-\theta))^{1/2} \, \|[L]\|^{1/2}$,
$\limsup_i \|y_i\| \leqslant (1/(\gamma-\theta))^{1/2} \, \|[L]\|^{1/2}$,

and

(c) $\lim_i(\|[x_i \otimes z]\| + \|[z \otimes x_i]\| + \|[y_i \otimes z]\| + \|[z \otimes y_i]\|) = 0, \quad z \in \mathcal{H}.$

In particular, $\mathfrak{A}$ has property $(\mathbb{A}_1)$ and also property $X_{0,\gamma-\theta}$.

Proof. Let $[L] \in Q_{\mathfrak{A}}$ and let $\{z_n\}_{n=1}^{\infty}$ be a dense set in $\mathcal{H}$. For each positive integer i, we choose by Lemma 1.8 vectors x_i and y_i in $\mathcal{H}$ such that (30), (31), and (32) are valid with $x = x_i$, $y = y_i$, $\varepsilon = (\frac{1}{2})^i$, and $t = i$. The sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ clearly satisfy (a) and (b). For each fixed positive integer n we have $\|[x_i \otimes z_n]\| \leqslant (\frac{1}{2})^i$ for all $i \geqslant n$, which implies that $\lim_i \|[x_i \otimes z_n]\| = 0$ for all positive integers n. Since the sequence $\{z_n\}$ is dense in $\mathcal{H}$ and the sequence $\{x_i\}$ is bounded, it follows easily that $\|[x_i \otimes z]\| \to 0$ for every z in $\mathcal{H}$, and since the analogous inequalities hold for the sequences $\{\|[z \otimes x_i]\|\}$, $\{\|[y_i \otimes z]\|\}$, and $\{\|[z \otimes y_i]\|\}$, we see that (c) is valid. That $\mathfrak{A}$ has property $(\mathbb{A}_1)$ now follows from (a). Finally, if $\|[L]\| < \gamma - \theta$, (a), (b), and (c) show that $[L] \in \mathcal{X}_0(\mathfrak{A})$, and since $\overline{\mathrm{aco}}\, \mathcal{X}_0(\mathfrak{A}) \supset \mathcal{X}_0(\mathfrak{A})^-$, $\mathfrak{A}$ has property $X_{0,\gamma-\theta}$ as asserted. Thus the proof is complete.

It is worth noting that $\mathcal{X}_0(\mathfrak{A})$ is always contained in the closed unit ball of $Q_{\mathfrak{A}}$, and thus it follows from this theorem that if $\mathfrak{A}$ is a dual algebra with property $X_{\theta,\gamma}$ (for some $\theta < \gamma$), then $\gamma - \theta \leqslant 1$. We also note that if $\mathfrak{A}$ is a dual algebra with property $X_{0,\gamma-\theta}$, then $\mathfrak{A}$ has property $X_{\theta,\gamma}$. (Indeed, if $[L] \in \mathcal{X}_0(\mathfrak{A})$ and $\|[L']\| \leqslant \theta$, then clearly $[L] + [L'] \in \mathcal{X}_\theta(\mathfrak{A})$. If $\|[A]\| < \gamma$, then $((\gamma-\theta)/\gamma)[A] \in \overline{\mathrm{aco}}\, \mathcal{X}_0(\mathfrak{A})$ if $\mathfrak{A}$ has property $X_{0,\gamma-\theta}$, so $((\gamma-\theta)/\gamma)[A]$ can be approximated arbitrarily closely by sums of the form $\sum_{j=1}^n c_j[L_j]$, where $[L_j] \in \mathcal{X}_0(\mathfrak{A})$, $c_j \geqslant 0$, and $\sum_{j=1}^n c_j = 1$. Thus $[A] = ((\gamma-\theta)/\gamma)[A] + (\theta/\gamma)[A]$ can be approximated arbitrarily closely by sums of the form $\sum_{j=1}^n c_j([L_j] + (\theta/\gamma)[A])$, and $[L_j] + (\theta/\gamma)[A] \in \mathcal{X}_\theta(\mathfrak{A})$.) This shows that the preceding theorem is sharp.

Theorem 1.9, together with Propositions 1.3 and 1.6, implies the following result which is essential for subsequent developments.

THEOREM 1.10. *Suppose $\mathfrak{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra that has property $X_{\theta,\gamma}$ for some $\gamma > \theta \geqslant 0$. Then $\mathfrak{A}$ has property $(\mathbb{A}_n)$ for every positive integer n.*

Proof. By Theorem 1.9, $\mathfrak{A}$ has property $X_{0,\gamma-\theta}$. Let n be any positive integer. Then, by Proposition 1.6, $M_n(\mathfrak{A})$ has property $X_{0,(\gamma-\theta)/n^2}$, and this implies, by Theorem 1.9, that $M_n(\mathfrak{A})$ has property $(\mathbb{A}_1)$. But, according to Proposition 1.3, this, in turn, implies that $\mathfrak{A}$ has property $(\mathbb{A}_n)$, so the theorem is proved.

We turn now to discuss another construction involving dual algebras. Let $\mathfrak{A} \subset \mathscr{L}(\mathscr{H})$ be a dual algebra, and let n be a cardinal number satisfying $1 \leqslant n \leqslant \aleph_0$. We write, as before, $\tilde{\mathscr{H}}_n$ for the direct sum of n copies of $\mathscr{H}$, and we define

$$\mathfrak{A}^{(n)} = \{\underbrace{A \oplus A \oplus \cdots}_{n \text{ copies}} : A \in \mathfrak{A}\}. \tag{42}$$

It is obvious that $\mathfrak{A}^{(n)}$ is a subalgebra of $\mathscr{L}(\tilde{\mathscr{H}}_n)$ containing $1_{\tilde{\mathscr{H}}_n}$ and that the mapping $\Phi: A \to A \oplus A \oplus \cdots$, is an isometric algebra isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}^{(n)}$. Furthermore, since $Q_{\mathfrak{A}} = \mathscr{C}_1(\mathscr{H})/{}^{\perp}\mathfrak{A}$ is separable as a Banach space, to show that Φ is weak* continuous (from $\mathfrak{A}$ into $\mathscr{L}(\tilde{\mathscr{H}}_n)$), it suffices to show (cf. [18, Theorem 2.3]) that if $\{A_n\}$ is a sequence in $\mathfrak{A}$ converging weak* to zero, then the sequence $\{\Phi(A_n)\}$ converges weak* to zero also. Since such a sequence $\{A_n\}$ must be bounded, a straightforward computation using the definition of the ultraweak topology shows that Φ is weak* continuous. Moreover, since Φ is an isometry, it follows (cf. [18, Theorem 2.7]) that $\mathfrak{A}^{(n)}$ is weak* closed in $\mathscr{L}(\tilde{\mathscr{H}}_n)$ and that Φ is a weak* homeomorphism between $\mathfrak{A}$ and $\mathfrak{A}^{(n)}$. In particular, then, $\mathfrak{A}^{(n)}$ is also a dual algebra, and has a predual $Q_{\mathfrak{A}^{(n)}}$ which is isometrically isomorphic to $Q_{\mathfrak{A}}$ via a linear transformation $\phi: Q_{\mathfrak{A}^{(n)}} \to Q_{\mathfrak{A}}$ that satisfies $\phi^* = \Phi$ (cf. [18, Proposition 2.5]). Thus, if $[L] \in Q_{\mathfrak{A}}$, and if we write $\widetilde{[L]} = \phi^{-1}([L])$ and $\tilde{A} = \Phi(A)$, we have the following equation:

$$\langle \tilde{A}, \widetilde{[L]} \rangle = \langle \Phi(A), \widetilde{[L]} \rangle = \langle A, \phi(\widetilde{[L]}) \rangle = \langle A, [L] \rangle, \quad A \in \mathfrak{A},\ [L] \in Q_{\mathfrak{A}}. \tag{43}$$

In particular, if $\tilde{x} = (x_0, x_1, \ldots)$ and $\tilde{y} = (y_0, y_1, \ldots)$ belong to $\tilde{\mathscr{H}}_n$, then $[\tilde{x} \otimes \tilde{y}] \in Q_{\mathfrak{A}^{(n)}}$, and using (43) and the equation $\langle \tilde{A}, [\tilde{x} \otimes \tilde{y}] \rangle = \sum_{0 \leqslant i < n} (Ax_i, y_i)$, we conclude easily that

$$\phi([\tilde{x} \otimes \tilde{y}]) = \sum_{0 \leqslant i < n} [x_i \otimes y_i]. \tag{44}$$

The following two propositions, which connect the algebras $\mathfrak{A}^{(n)}$ with the foregoing material, will be quite useful in Section 2.

PROPOSITION 1.11. *Suppose $\mathfrak{A} \subset \mathscr{L}(\mathscr{H})$ is a dual algebra and n is a positive integer such that $\mathfrak{A}^{(n)}$ has property $X_{\theta,\gamma}$ for some $\gamma > \theta \geqslant 0$ satisfying*

$\gamma-\theta>(n-1)/n$. Then $\mathfrak{A}$ has property $X_{(n-1)/n,\gamma-\theta}$ and consequently also has property $(\mathbb{A}_m)$ for every positive integer m.

Proof. It suffices to prove that $\mathscr{X}_{(n-1)/n}(\mathfrak{A})$ contains the open ball in $Q_{\mathfrak{A}}$ of radius $\gamma-\theta$ centered at the origin. Thus let $[L]\in Q_{\mathfrak{A}}$ satisfy $\|[L]\|<\gamma-\theta$, and write $\widetilde{[L]}=\phi^{-1}([L])$, so $\|[L]\|<\gamma-\theta$. From Theorem 1.9 we deduce that there exist sequences $\{\tilde{x}^{(i)}\}_{i=1}^{\infty}$ and $\{\tilde{y}^{(i)}\}_{i=1}^{\infty}$ in $\tilde{\mathscr{H}}_n$ such that

$$[\tilde{x}^{(i)}\otimes\tilde{y}^{(i)}]=\widetilde{[L]},\qquad i=1,2,..., \tag{45}$$

$$\|\tilde{x}^{(i)}\|<1,\qquad \|\tilde{y}^{(i)}\|<1,\qquad i=1,2,..., \tag{46}$$

and

$$\begin{aligned}&\|[\tilde{z}\otimes\tilde{x}^{(i)}]\|\to 0,\ \|[\tilde{x}^{(i)}\otimes\tilde{z}]\|\to 0,\\ &\|[\tilde{z}\otimes\tilde{y}^{(i)}]\|\to 0,\ \|[\tilde{y}^{(i)}\otimes\tilde{z}]\|\to 0,\quad \tilde{z}\in\tilde{\mathscr{H}}_n.\end{aligned} \tag{47}$$

For each $i\in\mathbb{N}$ we write

$$\tilde{x}^{(i)}=(x_1^{(i)},...,x_n^{(i)})\qquad\text{and}\qquad \tilde{y}^{(i)}=(y_1^{(i)},...,y_n^{(i)}),$$

and from (45) and (44) we have $[L]=\sum_{j=1}^{n}[x_j^{(i)}\otimes y_j^{(i)}]$, $i=1,2,...$. Next we choose j_i to satisfy $1\leqslant j_i\leqslant n$ and

$$\|[x_{j_i}^{(i)}\otimes y_{j_i}^{(i)}]\|=\max_{1\leqslant j\leqslant n}\|[x_j^{(i)}\otimes y_j^{(i)}]\|,\ i=1,2,.... \tag{48}$$

Since $\|[x_j^{(i)}\otimes y_j^{(i)}]\|\leqslant\|x_j^{(i)}\|\ \|y_j^{(i)}\|$ for all i and j, the Schwarz inequality yields

$$\sum_{j=1}^{n}\|[x_j^{(i)}\otimes y_j^{(i)}]\|\leqslant\sum_{j=1}^{n}\|x_j^{(i)}\|\ \|y_j^{(i)}\|\leqslant\|\tilde{x}^{(i)}\|\ \|\tilde{y}^{(i)}\|<1,$$

Considering the two cases $\|[x_{j_i}^{(i)}\otimes y_{j_i}^{(i)}]\|\leqslant 1/n$ and $\|[x_{j_i}^{(i)}\otimes y_{j_i}^{(i)}]\|>1/n$ separately, we see easily from (48) in either case that

$$\|[L]-[x_{j_i}^{(i)}\otimes y_{j_i}^{(i)}]\|\leqslant\sum_{\substack{j=1\\ j\neq j_i}}^{n}\|[x_j^{(i)}\otimes y_j^{(i)}]\|\leqslant\frac{n-1}{n},\qquad i=1,2,....$$

Since for each $i\in\mathbb{N}$, $1\leqslant j_i\leqslant n$, it follows that there exists an integer j_0 such that $j_i=j_0$ for infinitely many values of i. Thus we may drop down to a subsequence $\{[x_{j_0}^{(i_k)}\otimes y_{j_0}^{(i_k)}]\}_{k=1}^{\infty}$ such that $\lim_k\|[L]-[x_{j_0}^{(i_k)}\otimes y_{j_0}^{(i_k)}]\|$ exists and is less than or equal to $(n-1)/n$. Furthermore, it is immediate from (46) that $\|x_{j_0}^{(i_k)}\|<1$ and $\|y_{j_0}^{(i_k)}\|<1$ for all k. Finally, suppose $z\in\mathscr{H}$, and let $\tilde{z}$ in $\tilde{\mathscr{H}}_n$ be the vector with z as its only nonzero component, sitting in the j_0th slot. Then it follows easily from (47) and the equation

$\phi([\tilde{x}^{(i_k)}\otimes\tilde{z}])=[x_{j_0}^{(i_k)}\otimes z]$ that $\|[x_{j_0}^{(i_k)}\otimes z]\|\to 0$. Since the convergence to zero of the other three sequences analogous to (6) follows similarly, we have shown that $[L]\in\mathcal{X}_{(n-1)/n}(\mathcal{A})$, so $\mathcal{A}$ has property $X_{(n-1)/n,\gamma-\theta}$. That $\mathcal{A}$ also has property $(\mathbb{A}_m)$ for every $m\in\mathbb{N}$ now follows from Theorem 1.10.

PROPOSITION 1.12. *Suppose $\mathcal{A}\subset\mathcal{L}(\mathcal{H})$ is a dual algebra. Then $\mathcal{A}^{(\aleph_0)}$ is a dual algebra with property $X_{0,1}$, and consequently has property $(\mathbb{A}_n)$ for every positive integer n.*

Proof. We have seen that $\mathcal{A}^{(\aleph_0)}$ is a dual algebra, and since we know from Theorem 1.10 that dual algebras with property $X_{0,1}$ also have property $(\mathbb{A}_n)$ for every n in $\mathbb{N}$, it suffices to show that $\mathcal{A}^{(\aleph_0)}$ has property $X_{0,1}$. Furthermore, to accomplish this it is enough to show that $\mathcal{X}_0(\mathcal{A}^{(\aleph_0)})$ contains the open unit ball in $Q_{\mathcal{A}^{(\aleph_0)}}$. Thus let $\widetilde{[L]}\in Q_{\mathcal{A}^{(\aleph_0)}}$ satisfy $\|\widetilde{[L]}\|<1$, and set $[L]=\phi(\widetilde{[L]})$ in the notation introduced above. Since ϕ is an isometry, $\|[L]\|=\|\widetilde{[L]}\|<1$, and since $[L]$ induces an ultraweakly continuous linear functional on $\mathcal{A}$, one knows (cf. [20, p. 40]) that there exist sequences of vectors $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ in $\mathcal{H}$ such that

$$[L]=\sum_{i=1}^\infty [x_i\otimes y_i],\qquad \sum_{i=1}^\infty \|x_i\|^2<1,\qquad \sum_{i=1}^\infty \|y_i\|^2<1. \tag{49}$$

We next partition the set $\mathbb{N}$ of positive integers as $\mathbb{N}=\bigcup_{n=1}^\infty P_n$, where each P_n is an infinite set, and we use this partition to define sequences $\{\tilde{x}_i\}_{i=1}^\infty$ and $\{\tilde{y}_i\}_{i=1}^\infty$ of vectors in $\tilde{\mathcal{H}}_{\aleph_0}$ as follows. For $i=1, 2,...$, let $\tilde{x}_i$ be the vector in $\tilde{\mathcal{H}}_{\aleph_0}$ such that the only nonzero components of $\tilde{x}_i$ belong to the slots corresponding to the positive integers in P_i and such that in these slots are placed the vectors $x_1, x_2,...,$ in that order. (Thus if P_1 happens to be the set $\{1, 3, 5, 7,...,\}$, then $\tilde{x}_1=(x_1, 0, x_2, 0, x_3, 0,...)$.) We define the $\tilde{y}_i$ similarly, and note that for each $\tilde{A}\in\mathcal{A}^{(\aleph_0)}$ and each positive integer i, we have from (49) that

$$\begin{aligned}\langle\tilde{A}, [\tilde{x}_i\otimes\tilde{y}_i]\rangle &= \sum_{i=1}^\infty (Ax_i, y_i)=\left\langle A, \sum_{i=1}^\infty [x_i\otimes y_i]\right\rangle\\ &= \langle A, [L]\rangle=\langle\tilde{A}, \widetilde{[L]}\rangle,\end{aligned}$$

which proves that $\widetilde{[L]}=[\tilde{x}_i\otimes\tilde{y}_i]$, $i=1, 2,...$. Since from (49) we know that $\|\tilde{x}_i\|$, $\|\tilde{y}_i\|<1$, to show that $\mathcal{X}_0(\mathcal{A}^{(\aleph_0)})$ contains the open unit ball in $Q_{\mathcal{A}^{(\aleph_0)}}$, it suffices to show that for each fixed $\tilde{z}\in\tilde{\mathcal{H}}_{\aleph_0}$, the four sequences corresponding to (6) converge to zero. But by the way the $\tilde{x}_i$ and $\tilde{y}_i$ were defined, there exists a sequence $\{m_i\}$ of positive integers with $m_i\to+\infty$ such that the first m_i components of $\tilde{x}_i$ and $\tilde{y}_i$ are equal to zero, and the desired convergence now follows easily from (44) and the inequality $\|[\tilde{u}\otimes\tilde{v}]\| \leqslant \|\tilde{u}\|\,\|\tilde{v}\|$. Thus the proof is complete.

We close this section with some additional facts about dual algebras that will be useful in Section 3. If $r\geqslant 1$ we will say that a dual algebra $\mathcal{A}\subset\mathcal{L}(\mathcal{H})$ (with property $(\mathbb{A}_1)$) has property $(\mathbb{A}_1(r))$ if for each $s>r$ and each $[L]\in Q_{\mathcal{A}}$, there exist vectors x, $y\in\mathcal{H}$ such that $[L]=[x\otimes y]$ and

$$\|x\|\cdot\|y\|\leqslant s\,\|[L]\|.$$

(This property was called property $D_\sigma(r)$ in [26].)

PROPOSITION 1.13. *Suppose $\mathcal{A}\subset\mathcal{L}(\mathcal{H})$ is a dual algebra with property $(\mathbb{A}_1(r))$ for some $r\geqslant 1$. Then $\mathcal{A}$ is closed in the weak operator topology, and the weak* and weak operator topologies coincide on $\mathcal{A}$. Moreover, if X is an invertible operator in $\mathcal{L}(\mathcal{H})$ and we set*

$$X^{-1}\mathcal{A}X=\{X^{-1}AX: A\in\mathcal{A}\},$$

then $X^{-1}\mathcal{A}X$ is a dual algebra that has property $(\mathbb{A}_1(r'))$ for some $r'\geqslant 1$.

Proof. The proof of the first assertion is virtually identical with that of [30, Theorem 2] or [10, Corollary 1], so we say no more about it. As for the second assertion, it is clear that the mapping $\Psi: A\to X^{-1}AX$ is a weak* homeomorphism of $\mathcal{L}(\mathcal{H})$ onto itself, and therefore $X^{-1}\mathcal{A}X$ is a dual algebra. Furthermore, if we set $\Phi=\Psi|\mathcal{A}$, then Φ is a bounded linear transformation of $\mathcal{A}$ onto $X^{-1}\mathcal{A}X$, and thus the invertible $\phi: Q_{X^{-1}\mathcal{A}X}\to Q_{\mathcal{A}}$ with $\phi^*=\Phi$ given by [16, Proposition 2.5] is easily seen to satisfy

$$\phi([L]_{Q_{X^{-1}\mathcal{A}X}})=[XLX^{-1}]_{Q_{\mathcal{A}}},\qquad L\in\mathcal{C}_1(\mathcal{H}).$$

Moreover, the following calculation shows that $\|\phi\|\leqslant\|X\|\,\|X^{-1}\|$:

$$\begin{aligned}\|[XLX^{-1}]\|_{Q_{\mathcal{A}}}&=\sup_{\substack{A\in\mathcal{A}\\ \|A\|=1}}|\mathrm{tr}(AXLX^{-1})|=\sup_{\substack{A\in\mathcal{A}\\ \|A\|=1}}|\langle X^{-1}AX,[L]\rangle|\\ &\leqslant\|X\|\,\|X^{-1}\|\sup_{\substack{A\in\mathcal{A}\\ \|X^{-1}AX\|=1}}|\langle X^{-1}AX,[L]\rangle|\\ &=\|X\|\,\|X^{-1}\|\,\|[L]\|_{Q_{X^{-1}\mathcal{A}X}}.\end{aligned}$$

Now suppose $[L]\in Q_{X^{-1}\mathcal{A}X}$, and consider $\phi([L])=[XLX^{-1}]_{Q_{\mathcal{A}}}$. By hypothesis, if $s>r$, there exist vectors u, v in $\mathcal{H}$ such that

$[XLX^{-1}]_{Q\mathcal{A}} = [u \otimes v]_{Q\mathcal{A}}$ and $\max\{\|u\|, \|v\|\} \leqslant (s\,\|[XLX^{-1}]\|_{Q\mathcal{A}})^{1/2}$. We define $x = X^{-1}u$ and $y = X^*v$. Then

$$\max\{\|x\|, \|y\|\} \leqslant \max\{\|X\|, \|X^{-1}\|\}(s\,\|X\|\,\|X^{-1}\|\,\|[L]\|_{Q_{X^{-1}\mathcal{A}X}})^{1/2}.$$

Also, $[L]_{Q_{X^{-1}\mathcal{A}X}} = [x \otimes y]_{Q_{X^{-1}\mathcal{A}X}}$ since

$$\begin{aligned}\langle X^{-1}AX, [x\otimes y]\rangle &= (X^{-1}AXx, y) = (Au, v)\\ &= \langle A, [u\otimes v]\rangle = \langle A, [XLX^{-1}]\rangle\\ &= \operatorname{tr}(AXLX^{-1}) = \operatorname{tr}(X^{-1}AXL) = \langle X^{-1}AX, [L]\rangle\end{aligned}$$

for every A in $\mathcal{A}$. Thus if we set

$$r' = r(\max\{\|X\|, \|X^{-1}\|\})^2\,\|X\|\,\|X^{-1}\|,$$

then $X^{-1}\mathcal{A}X$ has property $(\mathbb{A}_1(r'))$, and the proof is complete.

REMARK 1.14. The preceding proof also implies that a dual algebra $\mathcal{A}$ has property $(\mathbb{A}_n)$ for some $1 \leqslant n \leqslant \aleph_0$ if and only if the dual algebra $X^{-1}\mathcal{A}X$ has property $(\mathbb{A}_n)$. Indeed, given a system $([L_{ij}])_{0\leqslant i,j<n}$ in $Q_{X^{-1}\mathcal{A}X}$, one has the corresponding system $(\phi([L_{ij}]))_{0\leqslant i,j<n}$ in $Q_{\mathcal{A}}$, and sequences $\{u_i\}_{0\leqslant i<n}$ and $\{v_i\}_{0\leqslant i<n}$ satisfy $\phi([L_{ij}]) = [u_i \otimes v_j]$, $0 \leqslant i, j < n$, if and only if the sequences $\{x_i = X^{-1}u_i\}_{0\leqslant i<n}$ and $\{y_i = X^*v_i\}_{0\leqslant i<n}$ satisfy $[L_{ij}] = [x_i \otimes y_i]$, $0 \leqslant i,j < n$. This is easily verified by a computation like that in the proof of Proposition 1.13.

2. SINGLY GENERATED DUAL ALGEBRAS.

In this section, we restrict our attention, as in [11], to dual algebras $\mathcal{A}_T$ generated by an absolutely continuous contraction T (i.e., a contraction T whose unitary part is absolutely continuous or acts on the space (0)). The notation and terminology will agree with that in [11]. For the convenience of the reader we recall a few pertinent definitions. The open unit disc in $\mathbb{C}$ is denoted by $\mathbb{D}$, and we write $\mathbb{T} = \partial\mathbb{D}$. The class $\mathbb{A}(\mathscr{H})$ consists of all those absolutely continuous contractions T in $\mathscr{L}(\mathscr{H})$ for which the Sz.-Nagy–Foias functional calculus Φ_T: $H^\infty(\mathbb{T}) \to \mathcal{A}_T$ is an isometry. If $T \in \mathbb{A}(\mathscr{H})$, then Φ_T is also a weak* homeomorphism of $H^\infty(\mathbb{T})$ onto $\mathcal{A}_T$ (cf. [16, Theorem 2.7]), and thus Φ_T is the adjoint of an isometry ϕ_T of Q_T (the predual of $\mathcal{A}_T$) onto $L^1(\mathbb{T})/H_0^1(\mathbb{T})$ (the predual of $H^\infty(\mathbb{T})$) (cf. [16]). Via the pair of mappings $\{\phi_T, \Phi_T\}$ the pair of spaces $\{L^1(\mathbb{T})/H_0^1(\mathbb{T}), H^\infty(\mathbb{T})\}$ can be identified with the pair $\{Q_T, \mathcal{A}_T\}$. If n is any cardinal number satisfying $1 \leqslant n \leqslant \aleph_0$, we denote by $\mathbb{A}_n(\mathscr{H})$ the set of all those T in $\mathbb{A}(\mathscr{H})$ such that the algebra $\mathcal{A}_T$ has property $(\mathbb{A}_n)$. When no confusion will result, we write simply $\mathbb{A}_n$ for $\mathbb{A}_n(\mathscr{H})$. In [11] we began the structure theory of the classes $\mathbb{A}_n(\mathscr{H})$, and, in particular, the dilation theory of the class $\mathbb{A}_{\aleph_0}(\mathscr{H})$. In [9] we showed that every operator in $\mathbb{A}_{\aleph_0}(\mathscr{H})$ is reflexive. One motivation for the introduction of these classes was as follows. Let $(\mathrm{BCP})(\mathscr{H})$ denote the set of all those completely nonunitary contractions in $\mathscr{L}(\mathscr{H})$ for which the intersection $\sigma_e(T)\cap\mathbb{D}$ of the essential spectrum of T with $\mathbb{D}$ is sufficiently large that almost every point of $\mathbb{T}$ is a nontangential limit point of $\sigma_e(T)\cap\mathbb{D}$ (such sets are said to be dominating for $\mathbb{T}$). It was shown in [12] (and also, by different techniques, in [33]) that $(\mathrm{BCP})(\mathscr{H})\subset\mathbb{A}_{\aleph_0}(\mathscr{H})$, so all of the results in [11] apply, in particular, to (BCP)-operators. (In fact, in [12] an increasing family $\{(\mathrm{BCP})_\theta\}_{0\leqslant\theta<1}$ of classes of contractions was introduced, with $(\mathrm{BCP})=(\mathrm{BCP})_0$, and it was shown that $\bigcup_{0\leqslant\theta<1}(\mathrm{BCP})_\theta\subset\mathbb{A}_{\aleph_0}$.)

In this section, we continue our study of the classes $\mathbb{A}_n$, and, in particular, we obtain a remarkable relation between $\mathbb{A}$, $\mathbb{A}_1$, and $\mathbb{A}_{\aleph_0}$. This leads, in Section 3, to some new dilation theorems and some new invariant subspace theorems.

We write, as usual, C_{00} or $C_{00}(\mathscr{H})$ for the set of all (completely nonunitary) contractions T in $\mathscr{L}(\mathscr{H})$ for which both sequences $\{T^n\}$ and $\{T^{*n}\}$ converge to 0 in the strong operator topology. We will also write $\mathrm{Lat}(T)$ for the lattice of invariant subspaces of an operator T. If $\mathscr{M}$, $\mathscr{N}\in\mathrm{Lat}(T)$ with $\mathscr{M}\supset\mathscr{N}$, so $\mathscr{M}\ominus\mathscr{N}$ is a semi-invariant subspace for T, we write $T_{\mathscr{M}\ominus\mathscr{N}}$ for the compression of T to this semi-invariant subspace. We also write $P_{\mathscr{M}}$ for the (orthogonal) projection whose range is $\mathscr{M}$. If $T\in\mathbb{A}$ and $\lambda\in\mathbb{D}$, we denote by $[C_\lambda]$ that element of Q_T with the property that

$$\langle f(T),[C_\lambda]\rangle=f(\lambda),\qquad f\in H^\infty(\mathbb{T}).\tag{50}$$

(Here, of course, $f(T)$ is $\Phi_T(f)$. See [11] for more details of the Sz.-Nagy–Foias functional calculus.) One knows from [18] that if $T\in\mathbb{A}$ and $\Lambda\subset\mathbb{D}$, then Λ is dominating for $\mathbb{T}$ if and only if $\overline{\mathrm{aco}}\{[C_\lambda]\in Q_T:\lambda\in\Lambda\}$ contains the closed unit ball in Q_T. It is also easy to see that if $T\in\mathbb{A}$, then $\sigma(T)\supset\mathbb{T}$.

Our first result shows that for operators T in C_{00}, the sets $\mathscr{X}_\theta(\mathfrak{A}_T)$ from Definition 1.4 can be defined somewhat more simply.

PROPOSITION 2.1. *Suppose that $T\in\mathbb{A}(\mathscr{H})$. Then $T\in C_{00}$ if and only if, for every sequence $\{x_i\}_{i=1}^\infty$ in $\mathscr{H}$ converging weakly to zero, we have*

$$\lim_i\|[x_i\otimes z]\|=\lim_i\|[z\otimes x_i]\|=0,\qquad z\in\mathscr{H}.\tag{51}$$

Consequently, if $T\in C_{00}\cap\mathbb{A}(\mathscr{H})$ and $\theta\geqslant0$, then $\mathscr{X}_\theta(\mathfrak{A}_T)$ consists of all those $[L]$ in Q_T for which there exist sequences $\{x_i\}$ and $\{y_i\}$ from $\mathscr{H}$ converging weakly to zero and satisfying (7) *and* (8).

Proof. If $T \in C_{00} \cap \mathbb{A}$, the validity of (51) follows from [18, Lemma 4.5] together with the fact that $\|[z \otimes x_i]\|_{Q_T} = \|[x_i \otimes z]\|_{Q_{T^*}}$. To prove the converse, let us suppose, to the contrary, that $T \notin C_{00}$, and, without loss of generality, let us suppose that there exists a vector z in $\mathscr{H}$ such that $\|T^{*n}z\| \not\to 0$. We know that the sequence $\{T^{*n}z\}$ converges weakly to zero. (In fact, for any fixed y in $\mathscr{H}$, the sequence $\{(T^{*n}z, y)\}$ is the sequence of negative Fourier coefficients of the element $\phi_{T^*}([z \otimes y]_{Q_{T^*}})$ in $L^1(\mathbb{T})/H_0^1(\mathbb{T})$; cf. [11, Lemma 3.1].) Furthermore, we have from (3) that

$$\begin{aligned} \|T^{*n}z\|^2 &= (T^n T^{*n}z, z) = \langle T^n, [T^{*n}z \otimes z]_{Q_T} \rangle \\ &\leqslant \|[T^{*n}z \otimes z]\|, \end{aligned}$$

so (51) is false if we set $x_i = T^{*i}z$. Thus, the first statement of the theorem is proved, and the second is an immediate consequence of Definition 1.4.

The next lemma leads to two of our basic structure theorems.

LEMMA 2.2. *If $T \in \mathbb{A}_1(\mathscr{H}) \cap C_{00}$, then $\mathfrak{A}_T$ has property $X_{0,1}$.*

Proof. According to the comment above Proposition 2.1, it is enough to show that

$$\{[C_\lambda] : \lambda \in \mathbb{D}\} \subset \mathscr{X}_0(\mathfrak{A}_T) \tag{52}$$

(where the $[C_\lambda]$ are the elements of Q_T given by (50)). Furthermore, since $T \in C_{00}$ we see from Proposition 2.1 that to establish (52), it certainly suffices to show that for each fixed λ_0 in $\mathbb{D}$, there exists a sequence $\{f_i\}_{i=1}^\infty$ of unit vectors in $\mathscr{H}$ converging weakly to zero such that $[f_i \otimes f_i] = [C_{\lambda_0}]$ for every i. Let m be any fixed positive integer, and let J_m be a strict contraction acting on a finite-dimensional Hilbert space of dimension m with the properties that J_m has a cyclic vector and $\sigma(J_m) = \{\lambda_0\}$. (For example, one could take J_m to be similar to a single $m \times m$ Jordan block operator with eigenvalue λ_0.) Then, since $T \in \mathbb{A}_1$, we see via [11, Theorem 3.4] that there exist invariant subspaces $\mathscr{M} \supset \mathscr{N}$ for T such that $\dim(\mathscr{M} \ominus \mathscr{N}) = m$ and $T_{\mathscr{M} \ominus \mathscr{N}}$ is similar to J_m. In particular, we may choose an orthonormal basis $\{e_1^{(m)}, ..., e_m^{(m)}\}$ for the semi-invariant subspace $\mathscr{M} \ominus \mathscr{N}$ so that the matrix for $T_{\mathscr{M} \ominus \mathscr{N}}$ relative to this basis is in upper triangular form (and thus has λ_0 as each diagonal entry). It follows easily from (3) and (50) that

$$[e_j^{(m)} \otimes e_j^{(m)}] = [C_{\lambda_0}], \qquad 1 \leqslant j \leqslant m.$$

We now extract from these finite orthonormal families $\{e_j^{(m)}\}_{j=1}^m$, $m = 1, 2, ...$, a sequence $\{f_i\}$ that converges weakly to zero and satisfies $[f_i \otimes f_i] = [C_{\lambda_0}]$, $i = 1, 2, ...$, as follows. Let $\{w_k\}_{k=1}^\infty$ be an orthonormal basis for $\mathscr{H}$. Then it is obvious that it suffices to choose the sequence $\{f_i\}$ from among the $e_j^{(m)}$ in such a way that $|(w_k, f_i)| \leqslant 1/\sqrt{i}$, $1 \leqslant k \leqslant i$. (For, this will imply $\lim_i (w_k, f_i) = 0$ for every k.) Consider, for a given m, the rectangular array

$$\begin{array}{l} |(w_1, e_1^{(m)})|^2 \; |(w_1, e_2^{(m)})|^2 \; |(w_1, e_3^{(m)})|^2 \cdots |(w_1, e_m^{(m)})|^2 \\ |(w_2, e_1^{(m)})|^2 \; |(w_2, e_2^{(m)})|^2 \; |(w_2, e_3^{(m)})|^2 \cdots |(w_2, e_m^{(m)})|^2 \\ |(w_3, e_1^{(m)})|^2 \cdots \end{array}$$

which has the obvious property that the sum of every row is less than or equal to 1 and the sum of every column is 1. Thus to choose f_i as desired it suffices to write down this array when $m = i^2$ and to observe that there must exist some column (say the jth) with the property that the sum of its first i entries is less than or equal to $1/i$. (For otherwise, the sum of the first i rows exceeds i, a contradiction). Upon setting $f_i = e_j^{(m)}$, we have the desired inequalities, and the proof is complete.

The following is our first basic structure theorem for singly generated dual algebras.

THEOREM 2.3. $\mathbb{A}_1(\mathscr{H}) \cap C_{00} = \mathbb{A}_{\aleph_0}(\mathscr{H}) \cap C_{00}$.

Proof. Since $\mathbb{A}_{\aleph_0} \subset \mathbb{A}_1$, it suffices to show that if $T \in \mathbb{A}_1 \cap C_{00}$, then $T \in \mathbb{A}_{\aleph_0}$. Furthermore, Exner [22] has shown that

$$\left(\bigcap_{n=1}^{\infty} \mathbb{A}_n \right) \cap C_{00} = \mathbb{A}_{\aleph_0} \cap C_{00},$$

so it suffices to show that $T \in \bigcap_{n=1}^{\infty} \mathbb{A}_n$, or, equivalently, that $\mathcal{A}_T$ has property $(\mathbb{A}_n)$ for every positive integer n. By Theorem 1.10, it suffices to show that $\mathcal{A}_T$ has property $X_{0,1}$, and that this is true follows from Lemma 2.2. Thus the proof is complete.

If $T \in \mathscr{L}(\mathscr{H})$ and n is a cardinal number satisfying $1 \leqslant n \leqslant \aleph_0$, we denote by $T^{(n)}$ the direct sum of n copies of T acting on $\tilde{\mathscr{H}}_n$. Of course the dual algebra generated by $T^{(n)}$ is $(\mathcal{A}_T)^{(n)}$. The following is our second basic structure theorem.

THEOREM 2.4. *If* $T \in \mathbb{A}(\mathscr{H})$, *then* $T^{(\aleph_0)} \in \mathbb{A}_{\aleph_0}$. *On the other hand, if* $T \in C_{00}$ *and* $T^{(n)} \in \mathbb{A}_1$ *for some positive integer* n, *then* $T \in \mathbb{A}_{\aleph_0}$.

Proof. Suppose $T \in \mathbb{A}$. Then, according to Proposition 1.12, $\tilde{T} = T^{(\aleph_0)} \in \mathbb{A}_1$, and by [11, Proposition 4.5] we have $\tilde{T}^{(\aleph_0)} \in \mathbb{A}_{\aleph_0}$. But obviously $\tilde{T}^{(\aleph_0)}$ is unitarily equivalent to $\tilde{T}$, so the first statement is proved.

Now suppose that $T \in C_{00}$ and there exists some positive integer n such that $T^{(n)} \in \mathbb{A}_1$. Since $T^{(n)} \in C_{00}$ also, we conclude from Lemma 2.2 that $\mathfrak{A}_T^{(n)}$ has property $X_{0,1}$ and from Proposition 1.11 that $\mathfrak{A}_T$ has property $X_{(n-1)/n,1}$. Then applying Theorem 1.9 we deduce that $T \in \mathbb{A}_1$, and finally, using Theorem 2.3 we see that $T \in \mathbb{A}_{\aleph_0}$, which completes the proof.

As usual, we write $\sigma(T)$ for the spectrum of an operator T. Our third basic structure theorem is

THEOREM 2.5. *Suppose $T \in C_{00}(\mathscr{H})$ and $\sigma(T) \cap \mathbb{D}$ contains a set Λ that is dominating for $\mathbb{T}$ and has the property that each $\lambda \in \Lambda$ belongs either to $\sigma_e(T)$ or to the derived set of $\sigma(T)$. Then $T \in \mathbb{A}_{\aleph_0}$.*

Proof. Since $\sigma(T) \cap \mathbb{D}$ is dominating for $\mathbb{T}$, one knows from [18] that $T \in \mathbb{A}$. Thus, according to Theorem 2.3 it suffices to show that $\mathfrak{A}_T$ has property $(\mathbb{A}_1)$, and to accomplish this, it is enough, by Theorem 1.9, to show that $\mathfrak{A}_T$ has property $X_{0,1}$. Furthermore, by Proposition 2.1 and the remark preceding it, we have only to show that for every $\lambda \in \Lambda$, there exists an orthonormal sequence $\{e_n\}_{n=1}^{\infty}$ in $\mathscr{H}$ satisfying

$$\|[C_\lambda] - [e_n \otimes e_n]\|_{Q_T} \to 0. \tag{53}$$

Accordingly, suppose $\lambda \in \Lambda$. If $\lambda \in \sigma_{le}(T)$, then one knows from [18, Lemma 4.3] that there exists an orthonormal sequence $\{e_n\}$ satisfying (53). Furthermore if $\lambda \in \sigma_{re}(T)$, the same proof (with $\{e_n\}$ taken to be an orthonormal sequence such that $\|(T^* - \bar{\lambda})\, e_n\| \to 0$) shows that (53) is valid. Thus we may suppose that $\lambda \in \sigma(T) \backslash \sigma_e(T)$, and hence that $T - \lambda$ is a Fredholm operator.

We note next that to complete the argument, it suffices to exhibit a semi-invariant subspace $\mathscr{M} \ominus \mathscr{N}$ for T and an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for $\mathscr{M} \ominus \mathscr{N}$ such that the matrix for the compression $T_{\mathscr{M} \ominus \mathscr{N}}$ relative to $\{e_n\}$ is either in upper or lower triangular form and has λ in every position on the main diagonal. (For, this being so, an easy calculation shows that for every polynomial p and every $n \in \mathbb{N}$, $(p(T)\, e_n, e_n) = p(\lambda)$, from which it follows immediately that $[C_\lambda] = [e_n \otimes e_n]$, $n \in \mathbb{N}$, so (53) is established.) Let $d = \text{index}\,(T - \lambda)$. We treat the case $d \geqslant 0$; the case $d < 0$ is dealt with similarly by consideration of T^*. Since $d \geqslant 0$, we must have $\ker(T - \lambda) \neq (0)$, and we denote by e_1 a unit vector belonging to this kernel. Now suppose, by induction, that we have found an orthonormal set $\{e_1, ..., e_n\}$ in $\mathscr{H}$ such that the subspace

$$\mathscr{K}_n = \bigvee \{e_1, ..., e_n\} \tag{54}$$

is invariant for T and the $n \times n$ matrix for $T \mid \mathscr{K}_n$ relative to the basis is in upper triangular form and has λ in every position on the diagonal. The argument will be completed by showing that the compression $(T - \lambda)_{\mathscr{K}_n^\perp}$

also has nontrivial kernel, and therefore there exists a unit vector $e_{n+1} \in \mathscr{H}_n^\perp$ such that $T_{\mathscr{H}_n^\perp} e_{n+1} = \lambda e_{n+1}$ (which clearly shows that the induction process can be continued, and leads to the desired orthonormal set $\{e_n\}_{n=1}^\infty$). Since $T-\lambda$ differs by a finite-rank operator from $0_{\mathscr{H}_n} \oplus (T-\lambda)_{\mathscr{H}_n^\perp}$, it follows, first, that $0_{\mathscr{H}_n} \oplus (T-\lambda)_{\mathscr{H}_n^\perp}$ is a Fredholm operator of index d, and, secondly, that $(T-\lambda)_{\mathscr{H}_n^\perp}$ is a Fredholm operator (on $\mathscr{H}_n^\perp$) of index d. Now suppose, to the contrary, that $(T-\lambda)_{\mathscr{H}_n^\perp}$ has trivial kernel. Then the inequality $d \geqslant 0$ forces $(T-\lambda)_{\mathscr{H}_n^\perp}$ to be invertible, and therefore $(T-\mu)_{\mathscr{H}_n^\perp}$ is invertible for all μ sufficiently close to λ. Since $(T-\mu)|\mathscr{H}_n$ is obviously invertible for all $\mu \neq \lambda$, this implies that $T-\mu$ is invertible for all μ sufficiently close to λ, contrary to hypothesis. Thus $(T-\lambda)_{\mathscr{H}_n^\perp}$ has nontrivial kernel, and the proof is complete.

We remark that the second half of the above proof, which simplifies somewhat the authors' original argument, is due to Derek Westwood. We also note that this theorem improves considerably the basic theorem of [18] for operators in C_{00}.

COROLLARY 2.6. *If $T \in C_{00}$ and $\sigma(T) = \mathbb{D}^-$, then $T \in \mathbb{A}_{\aleph_0}$.*

We are now prepared to prove our fourth and final structure theorem for singly generated dual algebras. If $T \in \mathscr{L}(\mathscr{H})$, let us write $\mathscr{W}_T$ for the smallest subalgebra of $\mathscr{L}(\mathscr{H})$ that contains T and $1_{\mathscr{H}}$ and is closed in the weak operator topology. Recall that it was shown in [9] that if $T \in \mathbb{A}_{\aleph_0}$, then T is reflexive, $\mathscr{A}_T = \mathscr{W}_T$, and the weak* and weak operator topologies coincide on this algebra.

THEOREM 2.7. *Suppose $T \in C_{00} \cap \mathbb{A}(\mathscr{H})$, and the weak* and weak operator topologies coincide on $\mathscr{A}_T$. Then $T \in \mathbb{A}_{\aleph_0}$.*

Proof. Let $[L_0] \in Q_T$. Then $[L_0]$ induces, as usual, a weak* continuous linear functional ψ on $\mathscr{A}_T$, and, by hypothesis, ψ is continuous if $\mathscr{A}_T$ is given the weak operator topology. Thus, ψ extends to a linear functional $\hat{\psi}$ on $\mathscr{L}(\mathscr{H})$ that is continuous in the weak operator topology (cf. [15, Proposition 14.13]), and one therefore knows that there exist finite sequences of vectors $\{x_i\}_{i=1}^p$ and $\{y_i\}_{i=1}^p$ in $\mathscr{H}$ such that $\hat{\psi}(A) = \sum_{i=1}^p (Ax_i, y_i)$ for each A in $\mathscr{L}(\mathscr{H})$ (cf. [20, p. 40]). It follows easily that $[L_0]$ has the form $[L_0] = \sum_{i=1}^p [x_i \otimes y_i]$, and since there exists some $n \geqslant p$ such that

$$\sum_{i=1}^p \|x_i\|^2 \leqslant n^2 \quad \text{and} \quad \sum_{i=1}^p \|y_i\|^2 \leqslant n^2, \tag{55}$$

we can also write $[L_0] = \sum_{i=1}^n [x_i \otimes y_i]$ by setting, if necessary, $x_i = y_i = 0$

for $p < i \leqslant n$. The point of this observation is that we now define, for each positive integer n, C_n to be the set of all $[L]$ in the open unit ball of Q_T such that there exist finite sequences $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ from $\mathscr{H}$ such that

$$[L] = \sum_{i=1}^{n} [x_i \otimes y_i], \qquad \sum_{i=1}^{n} \|x_i\|^2 \leqslant n^2, \qquad \sum_{i=1}^{n} \|y_i\|^2 \leqslant n^2. \tag{56}$$

Thus $\bigcup_{n=1}^\infty C_n$ is the open unit ball in Q_T, so $\bigcup_{n=1}^\infty C_n^-$ contains this open set. Hence it follows from the Baire category theorem that some C_N^- contains an open ball $B_{[L_1],\delta}$ in Q_T centered at $[L_1]$ with radius δ. We show next that the open ball $B_{0,\delta}$ in Q_T centered at the origin with radius δ is contained in C_{2N}^-. To this end, suppose $[K] \in B_{0,\delta}$ so $\|[K]\| < \delta$. Then, of course, $[L_1] + [K] \in B_{[L_1],\delta} \subset C_N^-$, so there exists a sequence $\{[M_m]\}_{m=1}^\infty$ in C_N such that $[L_1] + [K] = \lim_m [M_m]$. Since $[L_1] \in C_N^-$ there exists another sequence $\{[P_m]\}_{m=1}^\infty$ in C_N such that $[L_1] = \lim_m [P_m]$, so we have $[K] = \lim_m [M_m - P_m]$. Since $M_m - P_m$ clearly belongs to C_{2N} for all $m \in \mathbb{N}$, we see that $B_{0,\delta} \subset C_{2N}^-$. We next form the ampliated operator $T^{(2N)}$, and note from Theorem 2.4 that to complete the proof, it suffices to show that $T^{(2N)} \in \mathbb{A}_1$. For this purpose, let ϕ be the isometry of $Q_{\mathfrak{A}_{T^{(2N)}}}$ onto $Q_{\mathfrak{A}_T}$ described above (43), so $\phi(\widetilde{[L]}) = [L]$ for every $\widetilde{[L]}$ in $Q_{\mathfrak{A}_{T^{(2N)}}}$. If $\|\widetilde{[L]}\| < \delta$, then since $B_{0,\delta} \subset C_{2N}^-$, there exist sequences

$$\{\tilde{x}_i = (x_1^{(i)}, ..., x_{2N}^{(i)})\}_{i=1}^\infty \qquad \text{and} \qquad \{\tilde{y}_i = (y_1^{(i)}, ..., y_{2N}^{(i)})\}_{i=1}^\infty$$

in $\tilde{\mathscr{H}}_{2N}$ such that $\|[L] - \sum_{k=1}^{2N} [x_k^{(i)} \otimes y_k^{(i)}]\| \to 0$ and such that $\|\tilde{x}_i\|$, $\|\tilde{y}_i\| \leqslant 2N$ for all i in $\mathbb{N}$. By virtue of (44) we have $\|\widetilde{[L]} - [\tilde{x}_i \otimes \tilde{y}_i]\| \to 0$, which proves that the set

$$\{[\tilde{x} \otimes \tilde{y}] \in Q_{\mathfrak{A}_{T^{(2N)}}} : \|\tilde{x}\|, \|\tilde{y}\| \leqslant 1\}$$

is dense in the open ball in $Q_{\mathfrak{A}_{T^{(2N)}}}$ centered at the origin with radius $\delta/4N^2$. The following lemma (applied with $S = T^{(2N)}$) completes the proof of the theorem.

LEMMA 2.8. *Suppose* $S \in \mathbb{A}(\mathscr{H}) \cap C_{00}$ *and the set*

$$\{[x \otimes y] \in Q_S : \|x\|, \|y\| \leqslant 1\}^- \tag{57}$$

contains some open ball about the origin in Q_S. *Then* $S \in \mathbb{A}_1$.

Proof. Suppose the set in (57) contains the open ball $B_{0,\gamma}$ in Q_S centered at the origin with radius γ. We show that $S \in \mathbb{A}_1$ by showing that $\mathfrak{A}_S$ has property $X_{0,\gamma}$. To this end, let $[L] \in Q_S$ with $\|[L]\| < \gamma$. By Proposition 2.1, to show that $\mathfrak{A}_S$ has property $X_{0,\gamma}$ it suffices to construct sequences $\{x_n\}$ and $\{y_n\}$ in the unit ball of $\mathscr{H}$ that converge weakly to zero and satisfy $\|[L]-[x_n \otimes y_n]\| \to 0$. Let $[l] = \phi_S([L])$, and choose a function $h \in H_0^1(\mathbb{T})$ such that the L^1-function $l_1 = l + h$ satisfies $\|l_1\|_{L^1} < \gamma$ (which is possible since ϕ_S is an isometry). Then, of course, if n is any positive integer and we set $[L^{(2n)}] = \phi_S^{-1}([e^{-2nit} l_1])$, we have

$$\|[L^{(2n)}]\|_{Q_S} \leqslant \|e^{-2nit} l_1\|_{L^1} = \|l_1\|_{L^1} < \gamma.$$

Thus, by hypothesis, for each $n \in \mathbb{N}$ we may choose vectors u_n and v_n in the unit ball of $\mathscr{H}$ such that

$$\|[L^{(2n)}] - [u_n \otimes v_n]\| < 1/n. \tag{58}$$

We define $x_n = S^n u_n$, $y_n = S^{*n} v_n$, $n \in \mathbb{N}$, and note that for any fixed $z \in \mathscr{H}$,

$$(x_n, z) = (S^n u_n, z) = (u_n, S^{*n} z) \to 0$$

since $S \in C_{00}$. Thus $\{x_n\}$ converges weakly to zero, and, similarly, so does the sequence $\{y_n\}$. Furthermore, for each $n \in \mathbb{N}$ there exists a function f_n in $H^\infty(\mathbb{T})$ with $\|f_n\| = 1$ such that

$$\begin{aligned}
&\|[L]-[x_n \otimes y_n]\| \\
&\quad = \langle f_n(S), [L]-[x_n \otimes y_n]\rangle_{Q_S} \\
&\quad = \langle f_n(S), [L]\rangle_{Q_S} - \langle f_n(S), [S^n u_n \otimes S^{*n} v_n]\rangle_{Q_S} \\
&\quad = \langle f_n, [l]\rangle_{L^1/H_0^1} - (f_n(S)\, S^{2n} u_n, v_n) \\
&\quad = \langle f_n, l\rangle_{L^1} - \langle f_n(S)\, S^{2n}, [u_n \otimes v_n]\rangle \\
&\quad \leqslant |\langle f_n, l\rangle_{L^1} - \langle f_n(S)\, S^{2n}, [L^{(2n)}]\rangle| \\
&\qquad + |\langle f_n(S)\, S^{2n}, [L^{(2n)}] - [u_n \otimes v_n]\rangle| \\
&\quad \leqslant |\langle f_n, l\rangle_{L^1} - \langle f_n e^{2nit}, e^{-2nit} l_1\rangle_{L^1}| + \frac{1}{n} \\
&\quad = \frac{1}{n}
\end{aligned}$$

since (58) is valid and $\langle f_n, l\rangle_{L^1} = \langle f_n, l_1\rangle_{L^1}$. Thus $\|[L]-[x_n \otimes y_n]\| \to 0$ and the proofs of the lemma and the theorem are complete.

We turn next to some results about lattices of invariant subspaces for operators in $\mathbb{A}_{\aleph_0}$. These are analogs of [11, Theorem 3.8 and Corollary 3.9] and will be of interest in Section 3. We write $\operatorname{Lat}(\mathscr{H})$ for the lattice of all subspaces of $\mathscr{H}$.

THEOREM 2.9. *Suppose $T \in \mathbb{A}_{\aleph_0}(\mathscr{H})$. Then there is a one-to-one mapping*

η: Lat$(\mathscr{H}) \to$ Lat(T) that is increasing, preserves closed spans, and has the property that if $\{\mathscr{K}_i\}_{i \in I}$ is any family of subspaces of $\mathscr{H}$ such that $\bigcap_{i \in I} \mathscr{K}_i = (0)$, then $\bigcap_{i \in I} \eta(\mathscr{K}_i) = \bigcap_{i \in I} \{T\eta(\mathscr{K}_i)\}^-$.

Proof. It follows from [11, Proposition 4.2] that there exist invariant subspaces $\mathscr{M} \supset \mathscr{N}$ for T such that $\mathscr{H}' = \mathscr{M} \ominus \mathscr{N}$ is infinite dimensional and the compression $T_{\mathscr{H}'} = 0$. It clearly suffices to construct a mapping η: Lat$(\mathscr{H}') \to$ Lat(T) with the prescribed properties. For any subspace $\mathscr{K}$ of $\mathscr{H}'$, define $\eta(\mathscr{K})$ to be the smallest invariant subspace of T that contains $\mathscr{K}$. That η is increasing and preserves closed spans follows immediately from the definition, and that η is one-to-one is a consequence of the obvious relation $\eta(\mathscr{K}) \cap \mathscr{H}' = \mathscr{K}$. Now suppose that $\{\mathscr{K}_i\}_{i \in I}$ is an arbitrary family of subspaces of $\mathscr{H}'$ such that $\bigcap_{i \in I} \mathscr{K}_i = (0)$. Clearly $\bigcap_{i \in I} \{T\eta(\mathscr{K}_i)\}^- \subset \bigcap_{i \in I} \eta(\mathscr{K}_i)$, so we prove the reverse inclusion. Suppose $x \in \bigcap_{i \in I} \eta(\mathscr{K}_i)$ and let j be any fixed index in I. Then there exist sequences $\{p_k^{(n)}\}_{k=1}^{m_n}$ of polynomials and sequences $\{x_k^{(n)}\}_{k=1}^{m_n}$ of vectors in $\mathscr{K}_j$ such that

$$\lim_{n \to \infty} \left\| x - \sum_{k=1}^{m_n} p_k^{(n)}(T)\, x_k^{(n)} \right\| = 0. \tag{59}$$

Next suppose $y \in \mathscr{H}' \ominus \mathscr{K}_j$. Then, since $T\mathscr{K}_j \subset \mathscr{N} \subset (\mathscr{H}')^\perp$, it follows easily that $(x, y) = 0$. Thus, x is orthogonal to $\mathscr{H}' \ominus \mathscr{K}_j$, and since j is an arbitrary element of I, x is orthogonal to

$$\bigvee_{i \in I} (\mathscr{H}' \ominus \mathscr{K}_j) = \mathscr{H}' \ominus \bigcap_{j \in I} \mathscr{K}_j = \mathscr{H}'.$$

In particular, using (59) we obtain

$$0 = P_{\mathscr{K}_j} x = \lim_{n \to \infty} \sum_{k=1}^{m_n} P_{\mathscr{K}_j}(p_k^{(n)}(T)\, x_k^{(n)}) = \lim_{n \to \infty} \sum_{k=1}^{m_n} p_k^{(n)}(0)\, x_k^{(n)}.$$

Thus,

$$x = \lim_n \sum_{k=1}^{m_n} [p_k^{(n)}(T) - p_k^{(n)}(0)]\, x_k^{(n)},$$

from which it follows immediately that $x \in \{T\eta(\mathscr{K}_j)\}^-$. Once again, since j is arbitrary in I, we have

$$x \in \bigcap_{j \in I} \{T\eta(\mathscr{K}_j)\}^-,$$

and the proof is complete.

COROLLARY 2.10. *Suppose $T \in \mathbb{A}_{\aleph_0}(\mathscr{H})$ and has trivial kernel and closed range. If η is as in Theorem 2.9 and $\{\mathscr{K}_i\}_{i \in I}$ is any collection of subspaces of $\mathscr{H}$ such that $\bigcap_{i \in I} \mathscr{K}_i = (0)$, then $T(\bigcap_{i \in I} \eta(\mathscr{K}_i)) = \bigcap_{i \in I} \eta(\mathscr{K}_i)$. Moreover, if T also satisfies $\bigcap_{n=1}^{\infty} T^n \mathscr{H} = (0)$, then $\bigcap_{i \in I} \eta(\mathscr{K}_i) = (0)$ whenever $\bigcap_{i \in I} \mathscr{K}_i = (0)$.*

Proof. Since T is bounded below, the restriction of T to any invariant subspace has closed range, so $(T\eta(\mathscr{K}_i))^- = T\eta(\mathscr{K}_i)$ for all i. Furthermore, using Theorem 2.9 and the fact that T is one-to-one, we obtain

$$T\left(\bigcap_{i \in I} \eta(\mathscr{K}_i)\right) = \bigcap_{i \in I} T\eta(\mathscr{K}_i) = \bigcap_{i \in I} \eta(\mathscr{K}_i),$$

so T maps $\bigcap_{i \in I} \eta(\mathscr{K}_i)$ onto itself. Thus if $\bigcap_{n=1}^{\infty} T^n \mathscr{H} = (0)$, we have

$$\bigcap_{i \in I} \eta(\mathscr{K}_i) = \bigcap_{n=1}^{\infty} T^n \left(\bigcap_{i \in I} \eta(\mathscr{K}_i)\right) \subset \bigcap_{n=1}^{\infty} T^n \mathscr{H} = (0). \tag{60}$$

Recall that we sometimes say that two subspaces $\mathscr{M}$ and $\mathscr{N}$ of $\mathscr{H}$ are *disjoint* when $\mathscr{M} \cap \mathscr{N} = (0)$. Corollary 2.10 can be used to construct large families of disjoint invariant subspaces for certain operators in $\mathbb{A}_{\aleph_0}$.

COROLLARY 2.11. *Suppose $T \in \mathbb{A}_{\aleph_0}(\mathscr{H})$, T has trivial kernel and closed range, and $\bigcap_{n=1}^{\infty} T^n \mathscr{H} = (0)$. Then there exists a family $\{\mathscr{M}_A\}_{A \in 2^{\mathbb{N}}}$ of proper subspaces in* Lat(T), *indexed by subsets A of the positive integers, such that $A_1 \subset A_2$ implies $\mathscr{M}_{A_1} \subset \mathscr{M}_{A_2}$ and $A_1 \cap A_2 = \emptyset$ implies $\mathscr{M}_{A_1} \cap \mathscr{M}_{A_2} = (0)$. Moreover, there exists another family $\{\mathscr{N}_\lambda\}_{\lambda \in \mathbb{C}}$ of proper subspaces in* Lat(T), *indexed by $\mathbb{C}$, such that whenever λ, $\mu \in \mathbb{C}$ with $\lambda \neq \mu$, then $\mathscr{N}_\lambda \cap \mathscr{N}_\mu = (0)$.*

Proof. Let η be as in Theorem 2.9 and Corollary 2.10 and let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal system in $\mathscr{H}$. If A is any subset of $\mathbb{N}$, let $\mathscr{M}_A = \eta(\bigvee_{i \in A} e_i)$. Then, whenever A_1 and A_2 are disjoint subsets of $\mathbb{N}$, we have

$$\mathscr{M}_{A_1} \cap \mathscr{M}_{A_2} = \eta\left(\bigvee_{i \in A_1} e_i\right) \cap \eta\left(\bigvee_{j \in A_2} e_j\right) = (0)$$

from Corollary 2.10. Moreover, for $\lambda \in \mathbb{C}$, let us set $\mathscr{N}_\lambda = \eta(\bigvee\{e_1 + \lambda e_2\})$. Then, for $\lambda \neq \mu$, the one-dimensional subspaces spanned by $e_1 + \lambda e_2$ and $e_1 + \mu e_2$ are obviously disjoint, so again by Corollary 2.10 we have $\mathscr{N}_\lambda \cap \mathscr{N}_\mu = (0)$.

CONJECTURE 2.12. The authors conjecture that if $T \in \mathbb{A}_{\aleph_0}$, then Lat$(T)$ must contain two nonzero disjoint subspaces. In the special case of subnormal operators, this was established in [31]. Note that the following weaker result, interesting in its own right, is a consequence of Corollary 2.10: *If T*

is an invertible operator in $\mathbb{A}_{\aleph_0}$, *then either* T^{-1} *has a nontrivial invariant subspace or there exist* $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{Lat}(T)$ *with* $\mathcal{M}_1, \mathcal{M}_2 \neq (0)$ *but* $\mathcal{M}_1 \cap \mathcal{M}_2 = (0)$. (Proof. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be nonzero subspaces of $\mathcal{H}$ such that $\mathcal{K}_1 \cap \mathcal{K}_2 = (0)$. Then, according to Corollary 2.10, T maps $\eta(\mathcal{K}_1) \cap \eta(\mathcal{K}_2)$ onto itself and the result follows.)

We close the present discussion of properties of Lat(T) when $T \in \mathbb{A}_{\aleph_0}$ with one further interesting result. Recall that the *cyclic multiplicity* μ_T of an operator T in $\mathcal{L}(\mathcal{H})$ is the minimum cardinality of a set $\Lambda \subset \mathcal{H}$ with the property that $\bigvee_{n=0}^{\infty} T^n \Lambda = \mathcal{H}$.

COROLLARY 2.13. *For every* T *in* $\mathbb{A}_{\aleph_0}$ *and every cardinal number* n, $1 \leqslant n \leqslant \aleph_0$, *there exists an invariant subspace* $\mathcal{L}_n$ *for* T *such that the cyclic multiplicity of* $T | \mathcal{L}_n$ *equals* n.

Proof. Let η be the mapping constructed in the proof of Theorem 2.9. Let $\mathcal{K}_n$ be an n-dimensional subspace of $\mathcal{H}'$ and set $\mathcal{L}_n = \eta(\mathcal{K}_n)$. It is obvious that the cyclic multiplicity $\mu_{T|\mathcal{L}_n}$ of $T | \mathcal{L}_n$ is less than or equal to n. Assume that $\mu_{T|\mathcal{L}_n} < n$ so that we can find vectors $f_1, f_2, ..., f_p \in \mathcal{L}_n$ with $p < n$ and

$$\bigvee_{n \geqslant 0} \left(\bigvee_{1 \leqslant j \leqslant p} T^n f_j \right) = \mathcal{L}_n.$$

Then we can find a vector $e \in \mathcal{K}_n$ such that $(e, f_j) = 0$, $1 \leqslant j \leqslant p$. But, since $e \in \mathcal{M} \ominus T\mathcal{M}$, this clearly implies that e is orthogonal to $\mathcal{L}_n$, which contradicts $e \in \mathcal{K}_n \subset \mathcal{L}_n$.

The results of this section, especially Theorem 2.7, together with results that will appear in a future paper, have led us to

CONJECTURE 2.14. $\mathbb{A}(\mathcal{H}) = \mathbb{A}_1(\mathcal{H})$.

The reason for the importance of this conjecture is made clear by the following proposition.

PROPOSITION 2.15. *If Conjecture* 2.14 *is true, then every contraction* T *in* $\mathcal{L}(\mathcal{H})$ *such that* $\sigma(T) \supset \mathbb{T}$ *has a nontrivial invariant subspace.*

Proof. Let T be a contraction in $\mathcal{L}(\mathcal{H})$ such that $\sigma(T) \supset \mathbb{T}$. If T is not completely nonunitary, then it has a nontrivial unitary direct summand, and thus, as is easy to see, has nontrivial hyperinvariant subspaces. Thus we may suppose that T is a completely nonunitary contraction. By [2, Theorem 2.2], either T has a nontrivial hyperinvariant subspace or $T \in \mathbb{A}$, so we may suppose that $T \in \mathbb{A}$, and assuming the validity of the conjecture, we may suppose that $T \in \mathbb{A}_1$. Let $[C_0]$ be point evaluation at the origin in

Q_T, so (50) is satisfied, and choose x and y from $\mathscr{H}$ to satisfy $[x \otimes y] = [C_0]$. Then $1 = \langle 1, [x \otimes y] \rangle = (x, y)$ so neither x nor y is zero, and furthermore $0 = \langle T^n, [x \otimes y] \rangle = (T^n x, y)$ for $n \in \mathbb{N}$, so Tx is not a cyclic vector for T, and the proof is complete.

3. Applications: Dilation Theory, Reflexivity, and Invariant Subspaces

In this section we apply the structure theorems from Section 2 to obtain several new results in the directions indicated by the title. The following theorem summarizes some known results from [9] and [11]. Recall that an operator T in $\mathscr{L}(\mathscr{H})$ is *reflexive* if $\operatorname{Alg}\operatorname{Lat}(T) = \mathscr{W}_T$, where $\mathscr{W}_T$ is as defined below Corollary 2.6 and $\operatorname{Alg}\operatorname{Lat}(T)$ is the algebra of all operators A in $\mathscr{L}(\mathscr{H})$ such that $\operatorname{Lat}(T) \subset \operatorname{Lat}(A)$. (Reflexive operators thus have large invariant subspace lattices.)

THEOREM 3.1. *Suppose $T \in \mathbb{A}_{\aleph_0}(\mathscr{H})$. Then T is reflexive, $\mathscr{A}_T = \mathscr{W}_T$, and the weak* and weak operator topologies coincide on $\mathscr{A}_T$. Moreover* $\operatorname{Lat}(T)$ *contains a lattice that is isomorphic to* $\operatorname{Lat}(\mathscr{H})$. *Furthermore, if A is any countable direct sum of strict contractions, or any unilateral or bilateral weighted shift operator in C_{00} with weight sequence $\{\alpha_n\}$ satisfying $|\alpha_n| < 1$ for all n, then there exist invariant subspaces $\mathscr{M} \supset \mathscr{N}$ for T such that the compression $T_{\mathscr{M} \ominus \mathscr{N}}$ is unitarily equivalent to A.*

Proof. The first statement of the theorem is [9, Corollary 1.6 and Theorem 1.7]. The second statement follows immediately from [11, Corollary 4.3], or, alternatively, from Theorem 2.9 above. The last statement of the theorem is contained in [11, Theorems 4.8, 4.12, and 4.13].

It follows from this theorem that if we can prove that an operator belongs to $\mathbb{A}_{\aleph_0}$, then we can draw many conclusions about its dilation theory and invariant subspace lattice. Our first result along these lines concerns a particular remarkable operator.

Let $A_2(\mathbb{D})$ denote the Hilbert space of all holomorphic functions on $\mathbb{D}$ that are square integrable with respect to normalized planar Lebesgue measure on $\mathbb{D}$. Furthermore, let $B = M_\zeta$ be the operator of multiplication by the position function on $A_2(\mathbb{D})$. The operator B is called the *Bergman shift*; it is indeed a weighted unilateral shift (as defined below) with weight sequence $\{\sqrt{(n+1)/(n+2)}\}_{n=0}^{\infty}$ relative to the orthonormal basis $\{e_n(\zeta) = (1/\sqrt{n+1})\,\zeta^n\}_{n=0}^{\infty}$ for $A_2(\mathbb{D})$. It is obvious and well known that B belongs to C_{00} and satisfies $\sigma(B) = \mathbb{D}^-$. Thus Corollary 2.6 is applicable to

B, and this allows us to obtain considerable new information about this much studied operator.

PROPOSITION 3.2. *The Bergman shift B belongs to $\mathbb{A}_{\aleph_0}$.*

This gives a new proof, via Theorem 3.1, of the results from [30] that B is reflexive, $\mathcal{A}_B = \mathcal{W}_B$, and the weak* and weak operator topologies agree on $\mathcal{A}_B$. It was an unsolved problem for some time whether $\operatorname{Lat}(B)$ contains two nonzero subspaces $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \cap \mathcal{N} = (0)$. An affirmative answer was eventually given by C. Horowitz (see [30, pp. 86 and 122a]). Since obviously B has trivial kernel, is bounded below, and satisfies $\bigcap_{n=1}^{\infty} B^n \mathcal{H} = (0)$, we can apply Corollary 2.11 and Theorem 3.1 again to obtain these conclusions about $\operatorname{Lat}(B)$.

COROLLARY 3.3. *The lattice* $\operatorname{Lat}(B)$ *is so large that it contains a lattice isomorphic to* $\operatorname{Lat}(\mathcal{H})$. *Moreover, there is a family $\{\mathcal{N}_\lambda\}_{\lambda \in \mathbb{C}}$ of nonzero subspaces in* $\operatorname{Lat}(B)$ *indexed by $\mathbb{C}$ such that $\mathcal{N}_{\lambda_1} \cap \mathcal{N}_{\lambda_2} = (0)$ whenever $\lambda_1 \neq \lambda_2$. Furthermore, there is another family $\{\mathcal{M}_A\}_{A \in 2^{\mathbb{N}}}$ of nonzero subspaces in* $\operatorname{Lat}(B)$ *indexed by subsets A of $\mathbb{N}$ such that $A_1 \subset A_2$ implies $\mathcal{M}_{A_1} \subset \mathcal{M}_{A_2}$ and $A_1 \cap A_2 = \varnothing$ implies $\mathcal{M}_{A_1} \cap \mathcal{M}_{A_2} = (0)$.*

This corollary makes it clear why attempts to classify the invariant subspaces of B have not met with great success.

It is interesting to note that Corollary 2.6 also applies to the restriction of B to any nonzero invariant subspace. Indeed, if $\mathcal{M} \neq (0)$ belongs to $\operatorname{Lat}(B)$ and $\lambda \in \mathbb{D}$, then

$$\bigcap_{n=1}^{\infty} ((\lambda - B)|\mathcal{M})^n \mathcal{M} \subset \bigcap_{n=1}^{\infty} (\lambda - B)^n A_2(\mathbb{D}) = (0),$$

and consequently $\lambda \in \sigma(B|\mathcal{M})$. Thus $\sigma(B|\mathcal{M}) = \mathbb{D}^-$, and using also Corollary 2.13, we give the following answers to questions posed to us by S. Axler and R. Olin.

COROLLARY 3.4. *If $\mathcal{M}$ is any nonzero subspace in* $\operatorname{Lat}(B)$, *then $B|\mathcal{M} \in \mathbb{A}_{\aleph_0}(\mathcal{M})$ so the conclusions of Corollary* 3.3 *are also valid for $B|\mathcal{M}$. Furthermore, if n is any cardinal number such that $1 \leqslant n \leqslant \aleph_0$, then there exists a subspace $\mathcal{M}_n \subset \mathcal{M}$ in* $\operatorname{Lat}(B)$ *such that the cyclic multiplicity of $\mathcal{M}_n$ for $B|\mathcal{M}$ is n.*

The Bergman shift B is both a weighted shift and a subnormal operator. Thus it is natural to inquire about generalizations of these results concerning B to those two classes of operators. We begin with the following result that characterizes the subnormal operators in $\mathbb{A}_{\aleph_0} \cap C_{00}$.

THEOREM 3.5. *If S is a subnormal operator in $C_{00}(\mathscr{H})$, then the following three conditions are equivalent*:

(a) $S \in \mathbb{A}$,

(b) $S \in \mathbb{A}_{\aleph_0}$,

(c) $\sigma(S) \cap \mathbb{D}$ *is dominating for* $\mathbb{T}$.

Proof. That a subnormal operator in $\mathbb{A}$ belongs to $\mathbb{A}_1$ was proved by Olin and Thomson in [30], so (a) implies (b) by Theorem 2.3. On the other hand, (b) implies (a) by definition, so (a) and (b) are equivalent. That (c) implies (a) is [11, Proposition 2.3], so it remains to show that (a) implies (c). Thus suppose that $S \in \mathbb{A}$ and let N be the minimal normal extension of S acting on a Hilbert space $\mathscr{K} \supset \mathscr{H}$. It follows from the minimality of N that $\mathscr{K} = \bigvee_{n=0}^{\infty} N^{*n}\mathscr{H}$ and that $\sigma(N) \subset \sigma(S)$ (cf. [27]). Thus N is a contraction, and it follows easily from the relation $S = N|\mathscr{H}$ and the above characterization of $\mathscr{K}$ that N belongs to $C_{00} \cap \mathbb{A}(\mathscr{K})$. In particular, N must be a completely nonunitary contraction satisfying $\|f(N)\| = \|f\|_{\infty}$ for all f in $H^{\infty}(\mathbb{T})$. Thus it suffices to show that $\sigma(N) \cap \mathbb{D}$ is dominating for $\mathbb{T}$, or, equivalently (cf. [16]), that

$$\|f(N)\| = \sup_{\lambda \in \sigma(N) \cap \mathbb{D}} |\hat{f}(\lambda)|, \qquad f \in H^{\infty}, \tag{61}$$

where $\hat{f}$ is the analytic extension of f to $\mathbb{D}$. By [19, Lemma 3.1], $\hat{f}(\lambda) \in \sigma(f(N))$ whenever $\lambda \in \sigma(N) \cap \mathbb{D}$, so the left side of (61) certainly dominates the right side. On the other hand, if $N = \int_{\sigma(N)} \lambda dE$ is the spectral integral for N, then

$$f(N) = \int_{\sigma(N)} \hat{f}(\lambda)\, dE = \int_{\sigma(N) \cap \mathbb{D}} \hat{f}(\lambda)\, dE, \qquad f \in H^{\infty},$$

by [36, Theorem III.2.1], so we have

$$\begin{aligned} \|f(N)\| &= \sup_{\substack{x \in \mathscr{H} \\ \|x\| = 1}} |(f(N)\,x, x)| = \sup_{\substack{x \in \mathscr{H} \\ \|x\| = 1}} \left| \int_{\sigma(N) \cap \mathbb{D}} \hat{f}(\lambda)\, d(Ex, x) \right| \\ &\leqslant \sup_{\lambda \in \sigma(N) \cap \mathbb{D}} |\hat{f}(\lambda)|, \qquad f \in H^{\infty}, \end{aligned}$$

and the theorem is proved.

It is worth remarking that without the hypothesis that $S \in C_{00}$ in Theorem 3.2 (or something similar), one cannot hope to conclude the equivalence of (a) and (b). In particular, in [11] it is shown that the unilateral shift operator U_n of multiplicity n ($n < \aleph_0$) belongs to $\mathbb{A}_n$ but not to $\mathbb{A}_{n+1}$. Thus there are many subnormal operators in $\mathbb{A}$ that do not belong to $\mathbb{A}_{\aleph_0}$.

We now concern ourselves with generalizing the above results about B to a larger class of weighted shift operators. By a *forward unilateral weighted shift* in $\mathscr{L}(\mathscr{H})$ we mean an operator U_α with the property that there exists an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathscr{H}$ and a bounded sequence $\alpha=\{\alpha_n\}_{n=0}^{\infty}$ of complex numbers such that $U_\alpha e_n=\alpha_n e_{n+1}$, $n \geqslant 0$. The adjoint U_α^* of U_α is called a *backward weighted unilateral shift.* A *bilateral weighted shift* W_ω in $\mathscr{L}(\mathscr{H})$ is an operator with the property that there exists an orthonormal basis $\{f_n\}_{n=-\infty}^{\infty}$ for $\mathscr{H}$ and a bounded sequence $\omega=\{\omega_n\}_{n=-\infty}^{\infty}$ such that $W_\omega f_n=\omega_n f_{n+1}$, $-\infty<n<\infty$. Clearly W_ω^* is also a bilateral weighted shift. If U_α is a unilateral weighted shift, then $\|U_\alpha\|=\sup_n |\alpha_n|$. If U_α is a contraction, then clearly $\{U_\alpha^{*n}\}$ converges to zero in the strong operator topology, and it is easy to see that $\{U_\alpha^n\}$ converges strongly to zero (and thus belongs to C_{00}) if and only if the infinite product $\prod_{n=0}^{\infty}\alpha_n$ diverges to zero. (This means, as usual, that either infinitely many α_n equal 0 or $\lim_{k\to\infty}|\alpha_m\alpha_{m+1}\cdots\alpha_{m+k}|=0$, where α_{m-1} is the last zero weight.) We use the traditional notation $r(T)$ for the spectral radius of an operator T.

THEOREM 3.6. *Let T be either a unilateral or bilateral weighted shift in $\mathscr{L}(\mathscr{H})$. If $r(T)=\|T\|=1$, then $T\in\mathbb{A}_1$. If, in addition, T belongs to C_{00}, then $T\in\mathbb{A}_{\aleph_0}$.*

Proof. We note first that there is no generality lost by assuming that T is a unilateral shift. (Indeed, if $T=W_\omega$ is a weighted bilateral shift, and we define $\alpha_n=\omega_n$ and $\beta_n=\bar{\omega}_{-n-1}$ for $n\geqslant 0$, then T can be written as a matrix

$$T=\begin{pmatrix} U_\alpha & F \\ 0 & U_\beta^* \end{pmatrix}, \tag{62}$$

where F is an operator of rank one. Thus, by [11, Propositions 3.1 and 3.2], in order to show that T belongs to $\mathbb{A}_1$ [resp. $\mathbb{A}_{\aleph_0}$] it suffices to show that either U_α or U_β belongs to the corresponding class (these classes are self-adjoint). Since it is immediate from (62) that $r(T)=\max\{r(U_\alpha), r(U_\beta)\}$, one of the operators U_α or U_β has spectral radius one and hence satisfies the hypotheses of the theorem.) Therefore we may assume that $T=U_\alpha$ is a unilateral shift. It is known (cf. [34, Theorem 4]) that $\sigma(U_\alpha)=\mathbb{D}^-$. If $T\in C_{00}$, we conclude that $T\in\mathbb{A}_{\aleph_0}$ from Corollary 2.6. Thus we have only to treat the case in which $T\notin C_{00}$. This implies, as noted above, that there exists an m in $\mathbb{N}$ such that $\alpha_n\neq 0$ for $n\geqslant m$ and such that

$$\lim_k |\alpha_m\alpha_{m+1}\cdots\alpha_{m+k}|=\inf_k |\alpha_m\alpha_{m+1}\cdots\alpha_{m+k}|>0. \tag{63}$$

There is no loss of generality in assuming that $m=0$. (Indeed, if we set $\alpha'_n=\alpha_{m+n}$ for $n\geqslant 0$, we can write $T=U_{\alpha'}\oplus G$ where G acts on a finite dimensional space, and again by [11, Proposition 3.2] it is enough to show that $U_{\alpha'}\in\mathbb{A}_1$.) In this case it follows immediately from (63) and [34, Theorem 2′] that T is similar to the unilateral shift U_1 all of whose weights equal 1, and we know from [11, Theorem 3.7] that $U_1\in\mathbb{A}_1$. Since T is a contraction satisfying $\sigma(T)=\mathbb{D}^-$, $T\in\mathbb{A}_1$ by Remark 1.14. Thus the proof is complete.

The following corollary of the proof of Theorem 3.6 answers some questions raised in [34].

THEOREM 3.7. *Let T be either a unilateral or bilateral weighted shift in $\mathscr{L}(\mathscr{H})$ such that $\|T\|=r(T)$. Then $\mathfrak{A}_T=\mathscr{W}_T$, the weak* and weak operator topologies coincide on $\mathfrak{A}_T$, and T is reflexive.*

Proof. If $T=0$, the desired conclusions are obvious, so no generality is lost if we suppose $\|T\|=r(T)=1$. It follows easily that T is an absolutely continuous contraction. If $T\in C_{00}$, then $T\in\mathbb{A}_{\aleph_0}$ by Theorem 3.6 and the theorem then follows from Theorem 3.1. Thus we may suppose that $T\notin C_{00}$. To show that $\mathfrak{A}_T=\mathscr{W}_T$ and that the weak* and weak operator topologies coincide on $\mathfrak{A}_T$, one observes from Proposition 1.13 that it suffices to show that $\mathfrak{A}_T$ has property $(\mathbb{A}_1(r))$ for some $r\geqslant 1$. (Of course, we know from Theorem 3.6 that $\mathfrak{A}_T$ has property $(\mathbb{A}_1)$, but we must do a little better.) It is easy to see from the proof of [11, Proposition 3.2] that it is enough to know that some compression T' of T to a semi-invariant subspace belongs to $\mathbb{A}$ and that $\mathfrak{A}_{T'}$ has property $(\mathbb{A}_1(r))$. And, perusal of the proof of Theorem 3.6 shows that there is a compression of T to a semi-invariant subspace that is similar to either the unweighted unilateral shift U_1 or to its adjoint. Since the desired conclusions are valid for $\mathfrak{A}_T$ if and only if they are valid for $\mathfrak{A}_{T^*}$, we may suppose that T has a compression to a semi-invariant subspace that is similar to U_1. Thus, by Proposition 1.13, it suffices to show that $\mathfrak{A}_{U_1}$ has property $(\mathbb{A}_1(r'))$ for some $r'\geqslant 1$, and this follows, for example, from [30, Theorem 1]. This proves that $\mathfrak{A}_T=\mathscr{W}_T$ and that the weak* and weak operator topologies coincide on $\mathfrak{A}_T$.

Finally, we must show that our absolutely continuous contraction T (not in C_{00}) is reflexive. We consider first the case in which T is a unilateral shift U_α. Since $T\notin C_{00}$ the proof of Theorem 3.6 shows that we can write $T=U_{\alpha'}\oplus G$, where $U_{\alpha'}$ is similar to the unweighted shift U_1 and G is a nilpotent operator acting on a finite-dimensional space. Since similarity transformations and taking adjoints preserve reflexivity, it suffices to prove that $U_1^*\oplus J$ is reflexive, where J is a nilpotent operator acting on a finite-dimensional space. Let $X\in\operatorname{Alg}\operatorname{Lat}(U_1^*\oplus J)$. Obviously we can write $X=Y\oplus Z$ with Y in $\operatorname{Alg}\operatorname{Lat}(U_1^*)$ and Z in $\operatorname{Alg}\operatorname{Lat}(J)$. Since U_1^* is known to be reflexive (cf. [34]), it follows from what was proved already that $Y = u(U_1^*)$ for some $u \in H^\infty(\mathbb{T})$. For any fixed vector h in the Hilbert space of J, let $\mathcal{M}_h$ be the cyclic invariant subspace for J generated by h. Then $J | \mathcal{M}_h$ is a cyclic nilpotent operator acting on a finite-dimensional space, and hence there exists a subspace $\mathcal{N}_h \in \mathrm{Lat}(U_1^*)$ and an invertible operator W_h mapping $\mathcal{M}_h$ onto $\mathcal{N}_h$ such that $(U_1^* | \mathcal{N}_h) W_h = W_h(J | \mathcal{M}_h)$. The subspace $\mathcal{P}_h = \{W_h k \oplus k : k \in \mathcal{M}_h\}$ is clearly in $\mathrm{Lat}(U_1^* \oplus J)$, and thus $YW_h h \oplus Zh \in \mathcal{P}_h$. Therefore $YW_h h = W_h Zh$ and, on the other hand, $YW_h h = u(U_1^*) W_h h = W_h u(J) h$, so we conclude that $Zh = u(J) h$. Since h was arbitrary in the space of J, $X = Y \oplus Z = u(U_1^*) \oplus u(J) = u(U_1^* \oplus J)$, which belongs to $\mathcal{A}_{U_1^* \oplus J}$. Thus $U_1^* \oplus J$ is reflexive. We proceed now to the case in which $T = W_\omega$ is a bilateral shift. Just as in the proof of Theorem 3.6, T must have the form (62), where either $r(U_\alpha) = 1$ or $r(U_\beta) = 1$. Replacing T by T^* if necessary, we may suppose that $r(U_\alpha) = 1$ and that T shifts the orthonormal basis $\{f_n\}_{n=-\infty}^{\infty}$ of $\mathcal{H}$: $Tf_n = \omega_n f_{n+1}$. For each positive integer m, let $\mathcal{K}_m$ be the subspace spanned by the orthonormal set $\{f_n\}_{n=-m}^{\infty}$. Clearly $T\mathcal{K}_m \subset \mathcal{K}_m$ and $r(T | \mathcal{K}_m) \geqslant r((T | \mathcal{K}_m) | \mathcal{K}_0) = r(T | \mathcal{K}_0) = r(U_\alpha) = 1$ for every $m \in \mathbb{N}$, so $T | \mathcal{K}_m$ is reflexive by what was just proved about unilateral weighted shifts. Suppose now that $X \in \mathrm{Alg\,Lat}(T)$. Then $X\mathcal{K}_m \subset \mathcal{K}_m$ and $X | \mathcal{K}_m \in \mathrm{Alg\,Lat}(T | \mathcal{K}_m) = \mathcal{A}_{T|\mathcal{K}_m}$, $m \in \mathbb{N}$. Hence, for each $m \in \mathbb{N}$, there exists a function u_m in $H^\infty(\mathbb{T})$ such that $X | \mathcal{K}_m = u_m(T | \mathcal{K}_m) = u_m(T) | \mathcal{K}_m$. Of course the fact that $r(T | \mathcal{K}_m) = 1$ implies that $\sigma(T | \mathcal{K}_m) = \mathbb{D}^-$ and hence that $T | \mathcal{K}_m \in \mathbb{A}_1$ by Theorem 3.6. In particular, each of the functions u_m is unique. But

$$u_m(T) | \mathcal{K}_0 = u_m(T | \mathcal{K}_m) | \mathcal{K}_0 = (X | \mathcal{K}_m) | \mathcal{K}_0 = X | \mathcal{K}_0 = u_0(T) | \mathcal{K}_0,$$

so for each positive integer m, $u_m = u_0$, which implies that $X = u_0(T) \in \mathcal{A}_T$, so T is reflexive, and the theorem is proved.

We close this paper with a new invariant-subspace theorem for operators on Hilbert space that results from our structure theorems in Section 2.

THEOREM 3.8. *If T is a C_{00}-contraction in $\mathcal{L}(\mathcal{H})$, $\sigma(T) \supset \mathbb{T}$, and the weak* and weak operator topologies coincide on $\mathcal{A}_T$, then T has a nontrivial invariant subspace.*

Proof. If $T \notin \mathbb{A}$, then T has a nontrivial hyperinvariant subspace by [2, Theorem 2.2], so we may suppose that $T \in \mathbb{A}$. It thus follows from Theorem 2.7 that $T \in \mathbb{A}_{\aleph_0}$, so T not only has nontrivial invariant subspaces, but also $\mathrm{Lat}(T)$ contains a lattice isomorphic to $\mathrm{Lat}(\mathcal{H})$ by Theorem 3.1.

REMARK 3.9. We do not know of an example of an operator T in $\mathbb{A}$ for which the weak* and weak operator topologies on $\mathcal{A}_T$ are different, but

Westwood [38] produced an example of a contraction T in $\mathscr{L}(\mathscr{H})$ such that $\mathfrak{A}_T = \mathscr{W}_T$ but the weak* and weak* operator topologies on $\mathfrak{A}_T$ are different. The authors conjecture that if $T \in \mathbb{A}$, then the weak* and weak operator topologies on $\mathfrak{A}_T$ are identical; if this is true, then the special case of Conjecture 2.14 in which $T \in C_{00}$ would follow from Theorem 2.7.

Note added in proof. Since this paper was written, there have been some advances in the theory set forth herein that should be mentioned. First, Theorem 1.10 and Proposition 1.11 have been improved to the following: *A dual algebra $\mathfrak{A}$ has property $X_{\theta,\gamma}$ for some $\gamma > \theta \geqslant 0$ if and only if $\mathfrak{A}$ has property* $\mathbb{A}_{\aleph_0}$ (cf. Dual algebras with applications to invariant subspaces and dilation theory, CBMS Regional Conf. Ser. in Math. No. 56, Amer. Math. Soc., to appear). Second, there is a sequel to this paper, Invariant ... algebra, II, *Indiana Univ. Math. J.*, in press. Finally, in a paper entitled "Some new criteria for membership in $\mathbb{A}_{\aleph_0}$ with applications to invariant subspaces," by B. Chevreau and the fourth author, Theorem 3.8 has been improved by removing the hypothesis that $T \in C_{00}$.

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