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The Koszul Homology of an Ideal

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INTRODUCTION

Recently many researchers have worked on problems connected with various graded algebras associated to an ideal I in a local ring R. Two algebras in particular have received the most attention: the associated graded algebra of I, $\mathrm{gr}_I(R) = R/I \oplus I/I^2 \oplus \cdots$, and the Rees algebra of I, defined to be $R[It]$. Brodmann [2] and Goto and Shimoda [9] have studied the local cohomology of $R[It]$ for certain primary ideals I, Herzog has obtained new results in the case where I is the maximal ideal, Eisenbud and Huneke [7] studied the Cohen–Macaulayness of these algebras, while the concepts of d-sequences [11] and Hodge algebras [5] have been used to understand these graded algebras.

Recently Simis and Vasconcelos [20, 21] related the Cohen–Macaulayness, torsion-freeness, and normality of these algebras to the Koszul homology of I.

If I is an ideal, we let $H_j(I; R)$ denote the jth Koszul homology of the ideal I with respect to some *fixed* system of generators for I. We denote the symmetric algebra of a module M by $\mathrm{Sym}(M)$, and denote the jth graded piece of this algebra by $\mathrm{Sym}_j(M)$.

In [20] and [21] Simis and Vasconcelos construct a complex $\mathscr{M}(I)$ (or simply $\mathscr{M}$) with the following properties:

(1) $\mathscr{M}$ is graded, and if we let $\mathscr{M}^n$ be the nth graded piece of $\mathscr{M}$, then $\mathscr{M}^n$ is the complex,

$$\to H_j(I; R) \otimes \mathrm{Sym}_{n-j}(F/IF) \to H_{j-1}(I; R) \otimes \mathrm{Sym}_{n-j+1}(F/IF) \to \cdots \to \mathrm{Sym}_n(F/IF) \to 0,$$

where F is a free R-module of rank equal to the number of generators of I (which are fixed).

(2) There is an isomorphism $H_0(\mathscr{M}) \cong \mathrm{Sym}(I/I^2)$.

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(3) The homology $H_i(\mathscr{M})$ is independent of the generating set of I. This is rather important and quite nice since the Koszul homology of I is *not* independent of the generating set.

Using what is basically the acyclicity lemma of Peskine and Szpiro [17], Simis and Vasconcelos are able to show,

THEOREM A [21]. *Let R be a local ring and let I be an ideal of grade j generated by $n = j + s$ elements. Assume*:

(i) *For each prime ideal $I \subset P$, $j \leqslant ht\, P < n$, I_P can be generated by $ht\, P$ elements.*

(ii) *For each integer r and each prime ideal $I \subset P$,*

$$\operatorname{depth}(H_r(I; R))_P \geqslant \inf\{ht(P/I), r\},$$

where $H_r(I; R)$ is the Koszul homology of I with respect to the fixed n generators chosen. Then the complex $\mathscr{M}$ is exact.

The first condition can be more easily stated as follows. If R is a ring and P a prime ideal, let $v(I_P)$ denote the minimal number of generators of the ideal I_P. Then condition (1) is equivalent to the condition:

(1′) $\quad v(I_P) \leqslant ht\, P \quad$ for all prime ideals $\quad P \supseteq I$.

Simis and Vasconcelos further show if $v(I_P) \leqslant \max(ht\, I, ht\, P - 1)$ for all prime ideals P containing I, then $\operatorname{Sym}(I/I^2) = H_0(\mathscr{M})$ can be identified with $\operatorname{gr}_I(R)$. In Section two we will show this already holds under the assumptions of the theorem above.

Simis and Vasconcelos apply this result and obtain the very nice theorem,

THEOREM B. [21]. *Let R be a Cohen–Macaulay ring and let I be an ideal of height j. Assume*:

(a) *I is generically a complete intersection.*

(b) *The homology modules of the Koszul complex on a system of generators are Cohen–Macaulay modules.*

Then, the following are equivalent:

(i) *For every prime ideal $I \subset P$, $ht\, P > j$, $v(I_P) \leqslant ht\, P - 1$.*

(ii) $\operatorname{Sym}(I/I^2)$ *is Cohen–Macaulay and R/I-torsion-free. In addition, in these cases,* $\operatorname{Sym}(I/I^2) \simeq \operatorname{gr}_I(R)$.

Thus, the Cohen–Macaulayness of the Koszul homology is related to the Cohen–Macaulayness of the associated graded ring of I.

Our purpose in this paper is to relate the exactness of $\mathcal{M}$ and the Cohen–Macaulayness of the Koszul homology of I to the property of being generated by a d-sequence. We recall the definition of a d-sequence [11].

DEFINITION. A sequence of elements $x_1, ..., x_n$ in a ring R is said to be a d-sequence if $x_i \notin (x_1, ..., \hat{x}_i, ..., x_n)$ and for all $0 \leqslant i \leqslant n-1$, $k \geqslant i+1$, $((x_0, x_1, ..., x_i): x_{i+1} x_k) = ((x_0, x_1, ..., x_i): x_k)$, where we set $x_0 = 0$. The d-sequence is said to be Cohen–Macaulay if the rings $R/((x_0, ..., x_i): I)$ and $R/(((x_0, ..., x_i): I) + I)$ are Cohen–Macaulay for all $0 \leqslant i \leqslant n-1$.

As we will describe below, this approach provides a sharpening of the theorems stated above, and extends them to the symmetric algebra of I, $\mathrm{Sym}(I)$. In addition, this approach provides several new examples of ideals with the property that $\mathcal{M}$ is exact. In particular we are able to prove

THEOREM 2.3. *Let R be a Cohen–Macaulay local ring and suppose I is an ideal such that $v(I_P) \leqslant ht\, P$ for all prime ideals $I \subseteq P$, and suppose either*

(a) *$ht\, I = 2 = pd\, R/I$, or*

(b) *$ht\, I = 3 = pd\, R/I$, and both R and R/I are Gorenstein. Then I is generated by a Cohen–Macaulay d-sequence and the complex $\mathcal{M}(I)$ is exact.*

We show that if I is generated by a Cohen–Macaulay d-sequence then $\mathcal{M}$ is exact (Theorem 2.2). Conversely the conditions of Theorem B force I to be generated by a Cohen–Macaulay d-sequence (Theorem 2.4.). Finally we show that if I is an ideal generated by a Cohen–Macaulay d-sequence then $H_i(I; R)$ are Cohen–Macaulay if and only if depth $I^k/I^{k+1} \geqslant \mathrm{depth}(R/I) - k$ for $k \geqslant 0$ (Theorem 2.5). We apply this to the case where I is a perfect ideal of height two (Corollary 2.1). The proof of Theorem 2.5 rests heavily on the notion of syzygeticness developed by Simis [19] and also studied by Fiorentini [8].

If I is generated by a Cohen–Macaulay d-sequence then some properties of $\mathrm{gr}_I(R)$ and $R[It]$ are known. We state the relevant theorem of [13].

THEOREM C [13]. *Let R be Cohen–Macaulay local ring and suppose I is an ideal generated by a Cohen–Macaulay d-sequence. Then* $\mathrm{Sym}(I) \simeq R[It]$ *and* $\mathrm{gr}_I(R) \simeq \mathrm{Sym}(I/I^2)$, *and both of these algebras are Cohen–Macaulay.*

When $\mathrm{gr}_I(R)$ is Cohen–Macaulay, the torsion-freeness and normality follows from the behavior of the local analytic spreads of I. Recall that if P is a prime in R, then the analytic spread of I_P in R_P, written $l(I_P)$ is defined [16] by $l(I_P) = \dim R_p/PR_P \otimes R[It]$. The following result was proved in [13].

THEOREM D. [13]. *Let R be a Noetherian ring and I an ideal of R. Suppose* $\mathrm{gr}_I(R)$ *is Cohen–Macaulay.*

(1) $\mathrm{gr}_I(R)$ *is torsion-free over* R/I *if and only if*

$$l(I_P) \leqslant \max\{ht\,P-1, ht\,I\}$$

for all prime ideals P or R.

(2) *If* R/I *is normal, then* $\mathrm{gr}_I(R)$ *is normal if*

$$l(I_P) \leqslant \max\{ht\,P-2, ht\,I\}$$

for all prime ideals P or R.

When I is generated by a d-sequence, $l(I_P)=v(I_P)$ [11] and statements (1) and (2) immediately transpose into similar results found in the paper of Simis and Vasconcelos [20]. For our basic notation and terminology, we refer the reader to Matsumura's excellent book [15].

1

In this section we prove several elementary properties of the Koszul complex which we shall use throughout the paper, some of which are of independent interest. We begin by noting several well-known facts about Koszul homology. Some of the proofs we will sketch; for those we do not, we refer the reader to [16].

Remark 1.1. Let $I=(x_1,..., x_n)$. If $k=\mathrm{grade}\,I$, then the last nonvanishing $H_i(\mathbf{x}; R)$ is at $i=n-k$. If $x_1,..., x_k$ is maximal R-sequence in I, then $H_{n-k}(\mathbf{x}; R) \simeq ((x_1,..., x_k): I)/(x_1,..., x_k)$.

Remark 1.2. Let $I=(x_1,..., x_n)$ and set $S=R[T_1,..., T_n]$, the polynomial ring in n-variables over R. We may take R into an S-module by mapping T_i to x_i. In addition, we may consider R as an S-module via T_i goes to 0. We denote this second S-action by R_0 to distinguish it from the first action. Any R-module M becomes an S-module through the first homomorphism, sending $T_i \to x_i$. Since the S-module R_0 has a finite free resolution over S given by the Koszul complex of $(T_1,..., T_n)$, we may compute $\mathrm{Tor}_i^S(R_0, M)$ via this resolution. It is easy to see that

$$\mathrm{Tor}_i^S(R_0, M) \simeq H_i(\mathbf{x}; M).$$

Remark 2.3. Suppose z is an element of R, and I is an ideal of R. By "—" denote the surjection from R to R/Rz. If z is a nonzero divisor on R and on $H_{i-1}(I; R)$ (for some fixed generating set of I), and $i \geqslant 1$, then

$H_i(I; R)/zH_i(I; R) \simeq H_i(\bar{I}, \bar{R})$. This follows by the long exact sequence associated to Koszul homology. The exact sequence,

$$0 \to R \xrightarrow{z} R \to \bar{R} \to 0$$

gives a long exact sequence, (this follows from Remark 1.2)

$$H_i(I; R) \xrightarrow{z} H_i(I; R) \to H_i(I, \bar{R}) \to H_{i-1}(I; R) \xrightarrow{z} H_{i-1}(I; R).$$

Since z is a nonzero divisor on $H_{i-1}(I; R)$ we find

$$H_i(I; R) \xrightarrow{z} H_i(I; R) \to H_i(\bar{I}; \bar{R}) \to 0$$

is exact, which establish our claim.

Remark 1.4. Let $I = (x_1, \ldots, x_n)$. Then $H_i(\mathbf{x}, 0; R) \simeq H_i(\mathbf{x}; R) \oplus H_{i-1}(\mathbf{x}; R)$. If R is local and $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are two *minimal* generating set of I, then there is an isomorphism $H_i(\mathbf{x}; R) \simeq H_i(\mathbf{y}; R)$.

Remark 1.5. Suppose R is a Cohen–Macaulay local ring. Then $\dim H_i(I; R) = \dim R/I$ for every ideal I in R (independent of the generating set of I).

Proof. If for some generating set $(x_1, \ldots, x_n)$ of I, $\dim H_i(\mathbf{x}; R) = \dim R/I$ for every i, then using Remark 1.4, it is easy to check this must hold for every generating set of I.

The rigidity of the Koszul complex shows that nilrad(ann($H_i(I; R)$)) is the contained in nilrad(ann($H_{i+1}(I; R)$)). Let $x_1, \ldots, x_k$ be a maximal R-sequence in I. By Remark 1 the last nonvanishing Koszul homology is isomorphic to $((x_1, \ldots, x_k): I)/(x_1, \ldots, x_k)$. Hence the annihilator of this Koszul homology is nilrad $(\mathbf{x}: (\mathbf{x} : I))$. Thus, for arbitrary i (with $H_i(I; R) \neq 0$), $\dim R/I \geqslant H_i(I; R) \geqslant \dim R/(\mathbf{x}: (\mathbf{x} : I))$. To finish the proof of this remark it remains to show $\dim R/I = \dim R/(\mathbf{x}: (\mathbf{x} : I)))$. It is enough to show that if R is a Cohen–Macaulay local ring and I is an ideal of grade 0, then $\dim R/I = \dim R/(0: (0 : I))$. Let P be a prime containing I such that $\dim R/P = \dim R/I$. We claim $(0: (0 : I)) \subset P$. If not, then when we localize at P, $(0 : I)_P = (0 : I_P) = 0$. Since R is Cohen–Macaulay, $ht\, I = 0$ and so $ht\, P = 0$. Hence $\dim R_P = 0$ and so $(0 : I_P) \neq 0$ unless $I_P \not\subseteq P_P$. This contradiction proves our claim. In particular, we note that if $H_i(I; R)$ is a Cohen–Macaulay R-module, it is a maximal Cohen–Macaulay module for R/I.

Remark 1.6. Let R be a local Cohen-Macaulay and I an ideal. Let $I = (x_1, \ldots, x_n)$ and suppose $H_i(\mathbf{x}; R)$ are Cohen–Macaulay R-modules for all $i \geqslant 0$. Then if $(y_1, \ldots, y_m)$ is any other generating set for I, $H_i(\mathbf{y}; R)$ are

Cohen–Macaulay for all $i \geqslant 0$. This remark follows immediately from Remark 1.4.

There are two basic exact sequences which we will use throughout this paper. We separate these exact sequences in a lemma.

LEMMA 1.1. *Let I be an ideal of R generated by $x_1, \ldots, x_n$.*

(1) *If x_1 is a regular element of R, denote the homomorphism from R to $\bar{R} = R/Rx_1$ by "—". Then there is an exact sequence,*

$$0 \to H_i(I, R) \to H_i(\bar{I}, \bar{R}) \to H_{i-1}(I, R) \to 0$$

for $i \geqslant 1$. The middle homology is the Koszul homology of the elements $0, \bar{x}_2, \ldots, \bar{x}_n$.

(2) *Suppose $(0 : I) \cap I = (0)$. Let "—" denote the homomorphism from R to $R/(0 : I) = \bar{R}$. Then there is an exact sequence,*

$$0 \to K \to H_i(I; R) \to H_i(\bar{I}; \bar{R}) \to 0$$

for $i \geqslant 0$, where $K \simeq \bigoplus\ (0 : I)$.

Proof. First consider (1). Set $S = R[T_1, \ldots, T_n]$, where the T_i are algebraically independent over R. We may make R an S-module by mapping T_i to x_i as in Remark 1.2. Then with the notation of that remark,

$$\operatorname{Tor}_i^S(R_0, M) \simeq H_i(\mathbf{x}; M).$$

Set $x = x_1$, and consider the short exact sequence

$$0 \to R \xrightarrow{x} R \to \bar{R} \to 0$$

of S-modules. This gives a long exact sequence,

$$\to \operatorname{Tor}_i^S(R_0, R) \xrightarrow{x} \operatorname{Tor}_i^S(R_0, R) \to \operatorname{Tor}_i^S(R_0, \bar{R}) \to \operatorname{Tor}_{i-1}^S(R_0, R) \to$$

It is well known $\operatorname{ann}(\operatorname{Tor}_i^S(M, N))$ contains $\operatorname{ann} M + \operatorname{ann} N$. Consequently, $\operatorname{ann}(\operatorname{Tor}_i^S(R_0, R))$ contains $(T_1, \ldots, T_n) + (T_i - x_i)$. In particular, $x = x_1$ annihilates $\operatorname{Tor}_i^S(R_0, R)$, Thus the long exact sequence above breaks up into short exact sequences,

$$0 \to \operatorname{Tor}_i^S(R_0, R) \to \operatorname{Tor}_i^S(R_0, \bar{R}) \to \operatorname{Tor}_{i-1}^S(R_0, R) \to 0.$$

The discussion above shows $\operatorname{Tor}_i^S(R_0, R) \simeq H_i(\mathbf{x}; R)$, while $\operatorname{Tor}_i^S(R_0, \bar{R})$ is isomorphic to $H_i(0, \bar{x}_2, \ldots, \bar{x}_n; \bar{R})$. This shows (1).

To prove (2), consider the short exact sequence of S-modules,

$$0 \to (0 : I) \to R \to \bar{R} \to 0.$$

This gives rise to a long exact sequence,

$$\cdots \to \mathrm{Tor}_{i+1}^S(R_0, R) \to \mathrm{Tor}_{i+1}^S(R_0, \bar{R}) \to \mathrm{Tor}_i^S(R_0, (0:I)) \\ \to \mathrm{Tor}_i^S(R_0, R) \to \cdots.$$

We claim the map

$$\mathrm{Tor}_i^S(R_0, R) \to \mathrm{Tor}_i^S(R_0, \bar{R})$$

is surjective. This map is just the map from

$$H_i(I; R) \to H_i(\bar{I}; \bar{R})$$

given by restriction of a cycle in $K_i(I; R)$ to the same cycle in $K_i(\bar{I}; \bar{R})$. Suppose $\bar{\alpha}$ is a cycle in $K_i(\bar{I}; \bar{R})$ which represents a homology class in $H_i(\bar{I}; \bar{R})$. We wish to lift $\bar{\alpha}$ to a cycle in $K_i(I; R)$. Write

$$\bar{\alpha} = \sum \bar{r}_{(j)} e_{j_1} \wedge \cdots \wedge e_{j_i},$$

and let $r_{(j)}$, be any lifting of $\bar{r}_{(j)}$, to R. Set

$$\alpha = \sum r_{(j)} e_{j_1} \wedge \cdots \wedge e_{j_i}.$$

It clearly suffices to show $d\alpha = 0$, where d is the differential from $K_i(I; R)$ to $K_{i-1}(I; R)$. By assumption, $d\bar{\alpha} = 0$. Hence,

$$d\alpha \in (0:I)\, K_{i-1}(I; R).$$

In addition, $\mathrm{im}(d) \subseteq IK_{i-1}(I; R)$ by definition of the Koszul complex. Hence

$$d\alpha \in (0:I)\, K_{i-1}(I; R) \cap IK_{i-1}(I; R).$$

Since $K_{i-1}(I; R)$ is free and $(0:I) \cap I = (0)$, we see $d\alpha = 0$ as we required.

This shows the long exact sequence (3) breaks up into short exact sequences

$$0 \to \mathrm{Tor}_i^S(R_0, (0:I)) \to \mathrm{Tor}_i^S(R_0, R) \to \mathrm{Tor}_i^S(R_0, \bar{R}) \to 0$$

for $i \geqslant 0$. In terms of the Koszul complex these sequences become

$$0 \to H_i(I; (0:I)) \to H_i(I; R) \to H_i(\bar{I}; \bar{R}) \to 0.$$

Since I annihilates $(0:I)$, the Koszul homology of I on $(0:I)$ is just a direct sum of copies of $(0:I)$. This proves (2).

Corollary 1.1. *Let I be an ideal of a Cohen–Macaulay local ring R. Suppose $I=(x_1,\ldots,x_n)$ and $x=x_1$ is a regular element of R. Let "—" denote the homomorphism from R to $\bar{R}=R/Rx_1$. Then $H_i(I;R)$ are Cohen–Macaulay for all i if and only if $H_i(\bar{I};\bar{R})$ are Cohen–Macaulay for all i.*

Proof. Since this is independent of the generating set of I by Remark 1.6, we do not specify this set. By Lemma 1.1 there are exact sequences

$$0\to H_i(I;R)\to H_i(\bar{I};\bar{R})\to H_{i-1}(I;R)\to 0 \tag{4}$$

for $i\geqslant 1$. Set $d=\dim R/I$. If $H_i(I;R)$ are Cohen–Macaulay then Remark 1.5 shows $\dim H_i(I;R)=\operatorname{depth} H_i(I;R)=d$ for all i. It is immediate from the exact sequences (4) that $\operatorname{depth} H_i(\bar{I};\bar{R})=d$ for all $i\geqslant 1$. For $i=0$, $H_i(\bar{I},\bar{R})=\bar{R}/\bar{I}=R/I$ is Cohen–Macaulay.

Conversely suppose $\operatorname{depth} H_i(\bar{I};\bar{R})=\dim \bar{R}/\bar{I}=d$. Induct on i to show $\operatorname{depth} H_i(I;R)=d$. For $i=0$, $H_i(I;R)=R/I$ and by assumption this is Cohen–Macaulay. Suppose we have shown $H_{i-1}(I;R)$ is Cohen–Macaulay. It follows immediately from (4) that $H_i(I;R)$ also has depth at least d—and consequently has depth equal to d.

We next prove an important proposition dealing with the "link" of an ideal I. In [19], Peskine and Szpiro proved the following result:

Proposition. *Let R be a Gorenstein local ring and let I be an unmixed ideal such that*

(1) $(0:I)\cap I=(0)$

(2) *R/I is Cohen–Macaulay.*

Then $R/I(0:I)$ is also Cohen–Macaulay.

Peskine and Szpiro given an example in [19] to show the above proposition is false if one only assumes R is Cohen–Macaulay. The next proposition shows that if R is Cohen–Macaulay and I has Cohen–Macaulay Koszul homology then $R/(0:I)$ is Cohen–Macaulay.

Proposition 1.1. *Let R be a Cohen–Macaulay local ring and let I be an ideal of R. If $H_i(I;R)$ are Cohen–Macaulay for all i, then $R/(0:I)$ is Cohen–Macaulay.*

Proof. Of course this is vacuous unless $(0:I)\neq(0)$. If z is a nonzero divisor on $H_i(I;R)$ and R for all i, then Remark 1.3 shows $H_i(I;R)/zH_i(I;R)\simeq H_i(\bar{I},\bar{R})$, where "—" represents the homomorphism from R to $R/Rz=\bar{R}$. It follows that $H_i(\bar{I};\bar{R})$ are Cohen–Macaulay. We induct on $\dim R/I$ to prove $R/(0:I)$ is Cohen–Macaulay. If $\dim R/I>0$, then by induction if we choose z as above, $\bar{R}/(0:\bar{I})\simeq R/(z:I)$ is

Cohen–Macaulay. However, since z is not a zero divisor on $H_n(I; R)$, where $n =$ the number of generators of I, we see that $(z : I)/(z) = H_n(\bar{I}; \bar{R}) = H_n(I; R)/zH_n(I; R) = (0 : I)/z(0 : I)$. It follows that $(z : I) = ((0 : I), z)$. Since z is not a zero divisor on R, z is not a zero divisor on $R/(0 : I)$. As $R/(z : I) = R/((0 : I), z)$ is Cohen–Macaulay, we conclude that $R/(0 : I)$ is Cohen–Macaulay. We have reduced to $\dim R/I = 0$. However, if P is a minimal prime containing I, then $ht\, P = 0$. It follows that $(R/(0 : I))_P$ is Cohen–Macaulay.

Corollary 1.2. *Let R be a Cohen–Macaulay local ring, I and ideal of R. Suppose $(0 : I) \cap I = (0)$ and $ht((0 : I) + I) \geqslant 1$. Let "—" denote the homomorphism from R to $R/(0 : I)$. If $H_i(I; R)$ are Cohen–Macaulay for all i, then $H_i(\bar{I}; \bar{R})$ are Cohen–Macaulay.*

Proof. We may assume $(0 : I) \neq 0$. The last nonvanishing Koszul homology is $(0 : I)$, and hence this is Cohen–Macaulay. By Lemma 1.1, there are exact sequences

$$0 \to \bigoplus (0 : I) \to H_i(I; R) \to H_i(\bar{I}, \bar{R}) \to 0.$$

Since both $H_i(I; R)$ and $(0 : I)$ have depth $d = \dim R/I$, depth $H_i(\bar{I}; \bar{R}) \geqslant d - 1$. However $\dim H_i(\bar{I}; \bar{R}) = \dim R/((0 : I) + I)$ since by Proposition 1.1, $R/(0 : \mathrm{I})$ is Cohen–Macaulay, and we may apply Remark 1.5. As $ht((0 : I) + I) \geqslant 1$, $\dim H_i(\bar{I}; \bar{R}) \leqslant d - 1$. This show depth $H_i(\bar{I}; \bar{R}) = \dim H_i(\bar{I}; \bar{R}) = d - 1$.

We remark that the condition $(0 : I) \cap I = (0)$ forces $ht((0 : I) + I) \geqslant 1$ if R_P is Gorenstein for all prime ideals P such that $ht\, P = 0$. For if $(0 : I) + I \subseteq P$ for some prime ideal of height 0, then this implies that $I_P \neq 0$. Since R_P is 0-dimensional, $(0 : I)_P = (0 : I_P) \neq 0$ also. However $(0 : I_P) \cap I_P = 0$ by assumption. Since R_P is Gorenstein, (0) is irreducible and this is a contradiction.

2

In this section we discuss the exactness of the complex $\mathscr{M}$ and the ideals I which have Cohen–Macaulay Koszul homology. We will find these properties are intimately connected with d-sequences. We first recall their definition and elementary properties.

Definition [11]. A sequence of elements $x_1, \ldots, x_n$ in a commutative ring R is said to be a d-sequence if $x_i \notin Rx_1 + \cdots Rx_{i-1} + Rx_{i+1} + \cdots + Rx_n$ and $((x_0, \ldots, x_i) : x_{i+1}x_k) = (x_0, \ldots, x_i : x_k)$ for all $0 \leqslant i \leqslant n - 1$ and all $k \geqslant i + 1$, where we set $x_0 = 0$. An ideal J is said to be a *related* *ideal* to the d-sequence $x_1,\ldots,x_n$ if J is of the form $J=((x_0,\ldots,x_i):I)$ or $J=((x_0,\ldots,x_i):I)+I$ for some $0\leqslant i\leqslant n$, where $I=(x_1,\ldots,x_n)$.

In the papers [11, 12, and 13] many properties of d-sequences were developed; we separate those properties we shall use in a series of remarks. Throughout these remarks we fix a local ring R and a d-sequence $x_1,\ldots,x_n$ in R. Let $I=(x_1,\ldots,x_n)$, and set $x_0=0$.

Remark 2.1. If $0\leqslant i\leqslant n-1$ then $((x_0,\ldots,x_i):x_{i+1})=((x_0,\ldots,x_i):I)$. This follows easily from the definition.

Remark 2.2. If $0\leqslant i\leqslant n-1$, then the images of $x_{i+1},\ldots,x_n$ in $R/((x_1,\ldots,x_i):I)$ form a d-sequence. The images of $x_{i+1},\ldots,x_n$ in $R/(x_1,\ldots,x_i)$ also form a d-sequence.

Remark 2.3. If $0\leqslant i\leqslant n-1$, then $(x_0,\ldots,x_i:I)\cap I=(x_0,\ldots,x_i)$. In particular, $(0:I)\cap I=(0)$.

Remark 2.4. Any d-sequence is analytically independent; in particular, $n\leqslant \dim R$.

Remark 2.5. If $1\leqslant j\leqslant n-1$, then

$$(x_1,\ldots,x_j)\cap I^d=(x_1,\ldots,x_j)\,I^{d-1}.$$

Remark 2.6. If grade $I\geqslant k$, then $x_1,\ldots,x_k$ is an R-sequence.

Remark 2.7. If P is a prime ideal containing I, then there exists a subset $x_{i_1},\ldots,x_{i_k}$ of $(x_1,\ldots,x_n)$ such that $I_P=(x_{i_1},\ldots,x_{i_k})_P$ and $x_{i_1}/1,\ldots,x_{i_k}/1$ form a d-sequence in R_P.

Remark 2.8. The symmetric algebra $\mathrm{Sym}(I/I^2)$ is isomorphic to the graded algebra $\mathrm{gr}_I(R)=R/I\oplus I/I^2\oplus\cdots$, and $\mathrm{Sym}(I)\simeq R[It]$.

We are now ready to consider the exactness of $\mathscr{M}$. We will call a d-sequence $x_1,\ldots,x_n$ in a Cohen–Macaulay local ring R a *Cohen–Macaulay d-sequence* (or simply a C–M d-sequence) if R/J is Cohen–Macaulary for every related ideal J of $(x_1,\ldots,x_n)$.

THEOREM 2.1. *Let R be a Cohen–Macaulay local ring and let $x_1,\ldots,x_n$ be a Cohen–Macaulay d-sequence. Then depth $H_i(\mathbf{x};R)\geqslant i$ for all $i\geqslant 0$.*

Remark. The condition on the depth of the Koszul homology is *not* independent of the generating set. In fact since $H_i(\mathbf{x},0;R)=H_i(\mathbf{x};R)\oplus H_{i-1}(\mathbf{x};R)$ by Remark 1.4, it is clear that depth $H_i(\mathbf{x},0;R)\geqslant i-1$ is all one can say in this case. Of course in general we start with a minimal generating set of I to calculate Koszul homology; however after localization this minimal generating set may no longer be minimal.

Proof. We induct on n. Let $I=(x_1,\ldots,x_n)$.

Case 1 (grade $I>0$). We show in this case we may always reduce to a smaller n or $k=\text{grade}\, I=ht\, I$. We are assuming $k\geqslant 1$. By Remark 2.6, $x_1,..., x_k$ form an R-sequence. Let "—" denote the map from R to R/Rx_1. By Lemma 1.1.1, there are exact sequences,

$$0\to H_i(I; R)\to H_i(\bar{I}; \bar{R})\to H_{i-1}(I; R)\to 0,$$

where the middle homology is taken with respect to 0, $\bar{x}_2,..., \bar{x}_n$. By Remark 1.4, the middle homology is isomorphic to

$$H_i(\bar{x}_2,..., \bar{x}_n; \bar{R})\oplus H_{i-1}(\bar{x}_2,..., \bar{x}_n; \bar{R}).$$

By induction on n, we may assume the depth of this module to be at least $i-1$. We now induct on j to show depth $H_{n-j}(I; R)\geqslant n-j$. The first non-zero homology is when $j=k$, and in this case, $H_{n-k}(I; R)\simeq ((x_1,..., x_k):I)/(x_1,..., x_k)$. By assumption, $R/((x_1,..., x_k):I)$ is Cohen–Macaulay, clearly of depth $R/I=\dim R-k$. It follows from the exact sequence,

$$0\to((x_1,..., x_k):I)/(x_1,..., x_k)\to R/(x_1,..., x_k)\to R/((x_1,..., x_k):I)\to 0$$

that depth $H_{n-k}(I; R)\geqslant \text{depth}\, R-k$. It remains to show $n-k\leqslant$ depth $R-k$. However, R is Cohen–Macaulay so that depth $R=\dim R$. In addition by Remark 2.4, $n\leqslant \dim R$. Thus our assertion is verified for $j=k$.

Suppose $j>k$. Consider the exact sequence

$$0\to H_{n-j+1}(I; R)\to H_{n-j+1}(\bar{I}; \bar{R})\to H_{n-j}(I; R)\to 0.$$

By our induction on n, depth $H_{n-j+1}(\bar{I}; \bar{R})\geqslant n-j$. By our induction on j, depth $H_{n-j+1}(I; R)\geqslant n-j+1$. It now follows depth $H_{n-j}(I; R)\geqslant n-j$.

Case 2 (grade $I=0$). Let "—" denote the homomorphism from R to $R/(0:I)$. By Remark 2.3, $(0:I)\cap I=(0)$. By Lemma 1.1.2 there are exact sequences,

$$0\to \bigoplus (0:I)\to H_i(I; R)\to H_i(\bar{I}; \bar{R})\to 0.$$

Since we are assuming $R/(0:I)$ is Cohen–Macaulay (since $(0:I)$ is a related ideal), it follows that $(0:I)$ is also Cohen–Macaulay. As $(0:I)=(0:x_1)=(0:x_1^2)$, grade $\bar{I}\geqslant 1$, and we may apply Case 1 and induction on n to conclude depth $H_i(\bar{I}, \bar{R})\geqslant i$. Since depth$(0:I)=\text{depth}\, R/I=\text{depth}\, R$, it easily follows from the exact sequence above that depth $H_i(I; R)\geqslant i$.

THEOREM 2.2. *Let R be a Cohen–Macaulay local ring, and let $x_1,..., x_n$ be a Cohen-Macaulay d-sequence. Set $I=(x_1,..., x_n)$. Then the complex $\mathcal{M}$ associated to I is acyclic.*

Proof. We show each graded piece of $\mathcal{M}$ is acyclic. Fix a $t > 0$; then the tth graded piece of $\mathcal{M}$ is

$$\begin{aligned}&\to H_n(I; R) \otimes \mathrm{Sym}_{t-n}(F/IF) \to \cdots \\ &\to H_{n-1}(I; R) \otimes \mathrm{Sym}_{t-n+1}(F/IF) \to \cdots \to \mathrm{Sym}_t(F/IF) \to 0, \qquad (1)\end{aligned}$$

where $F \simeq R^n$. Simis and Vasconcelos show the homology of $\mathcal{M}$ is independent of the generating set of I. Localize R at a prime ideal P, minimal with respect to the property that $H_i(\mathcal{M}')_P \neq 0$ for some $i \geqslant 1$. Using Remark 2.7, we see we may reduce to the case where the homology $H_i(\mathcal{M}')$ has finite length. Let $I = (x_1, ..., x_k)$ (notice k could be less than n), where $x_1, ..., x_k$ is a Cohen–Macaulay d-sequence. Since $H_i(\mathcal{M}')$ is independent of the generating set, it follows that $H_i(\mathcal{M}') = 0$ for $i > k$ (since $H_i(\mathbf{x}; R) = 0$ in this case). The complex $\mathcal{M}'$ is now

$$0 \to H_k(\mathbf{x}; R) \to H_{k-1}(\mathbf{x}; R) \otimes \mathrm{Sym}_1(F/IF) \to \cdots \to \mathrm{Sym}_k(F/IF) \to 0$$

if $k \geqslant t$. If $k > t$, the complex $\mathcal{M}'$ becomes

$$0 \to H_t(\mathbf{x}; R) \to H_{t-1}(x; R) \otimes \mathrm{Sym}_1(F/IF) \to \cdots \to \mathrm{Sym}_t(F/IF) \to 0.$$

By Theorem 2.1, depth $H_i(\mathbf{x}; R) \geqslant i$. Therefore an application of the acyclicity lemma below proves Theorem 2.2.

LEMMA [18]. *Let R be local ring and $C_.$ a complex of finitely generated R-modules,*

$$0 \to C_n \to \cdots \to C_0.$$

Suppose that depth $C_i \geqslant i$ *and the length of $H_i(C_0)$ is a finite for $i \geqslant 1$. Then $C_.$ is exact.*

We would like to apply Theorem 2.2 to a variety of situations. In particular we note the following two propositions.

PROPOSITION 2.2 [11]. *Let $X = (x_{ij})$ be a generic n by $n+1$ matrix over the integers $\mathbb{Z}$. Put $R = \mathbb{Z}[x_{ij}]_{(x_{ij})}$ and set $I = I_n(X) =$ the ideal generated by the maximal minors of X. Then I is generated by a Cohen–Macaulay d-sequence; in fact, if d_i is the determinant of the matrix determined by deleting the* ith *column of X, then $d_1, ..., d_{n+1}$ is a Cohen–Macaulay d-sequence.*

Remark. It is not hard to compute $(d_1, ..., d_k : I)$ using the paper of Eagon and Hochster [7]. We refer the reader to [10] for specifics. However we will need to know one fact: $ht((d_1, ..., d_k) : I) + I) = k + 1$.

PROPOSITION 2.3. *Let $X=(x_{ij})$ be a generic $2n+1$ by $2n+1$ alternating matrix with zeroes down the diagonal. (Thus $x_{ij}=-x_{ji}$ and $x_{ii}=0$ by convention.) Set $R=\mathbb{Z}[x_{ij}]_{(x_{ij})}$ and let I be the ideal of R generated by the maximal Pfaffians of X. Label these $p_1,..., p_{2n+1}$. Then $p_1,..., p_{2n+1}$ is a Cohen–Macaulay d-sequence.*

Remark. We again remark that

$$ht(((p_1,..., p_k):I)+I)=k+1.$$

We would like to apply these results to nongeneric cases. To do this we first prove two lemmas.

LEMMA 2.6. *Let R be a local ring and I an ideal satisfying $v(I_P)\leqslant ht\, P$ for all prime ideals P which contain I. Then there is a minimal generating set $a_1,..., a_n$ of I such that $(a_1,..., a_k)_P=I_P$ if P is a prime ideal containing I and $ht\, P\leqslant k$.*

Proof. This is a standard exercise using the prime avoidance lemma. It follows that $ht(((a_1,..., a_k):I)+I)\geqslant k+1$. For if P is a prime and $P\supset((a_1,..., a_k):I)+I$, then $(a_1,..., a_k)_P\neq I_P$ while $I\subset P$. This shows $ht\, P>k$.

LEMMA 2.7. *Let R be a Cohen–Macaulay local ring and let $x_1,..., x_n$ be a Cohen–Macaulay d-sequence. Set $I=(x_1,..., x_n)$. Suppose y is a nonzero divisor of R which is not a zero divisor on $R/((x_0,..., x_j):I)+I)$ for all j, $0\leqslant j\leqslant n-1$. Then y is not a zero divisor on R/J for any related ideal J. Further, $((x_0,..., x_j, y):I)=(((x_0,..., x_j):I), y)$. Hence if "—" denotes the homomorphism from R to R/Ry, $\bar{x}_1,..., \bar{x}_n$ form a Cohen–Macaulay d-sequence in $\bar{R}$ and if $\bar{J}$ is a related ideal of $\bar{x}_1,..., \bar{x}_n$, then $\bar{J}=(J, y)/(y)$ for some related ideal J of $x_1,..., x_n$.*

Proof. By induction on $n-k$, we claim y is not a zero divisor on $R/((x_1,..., x_k):I)$. For $k=n-1$, i.e., $n-k=1$, we have assumed that y is not a zero divisor on $R/((x_1,..., x_{n-1}):I)+(x_n))$. Since x_n is not a zero divisor on $((x_1,..., x_{n-1}):I)$, $\{x_n, y\}$ form a $R/((x_1,..., x_{n-1}):I)$-sequence. As R is local, it follows that y is regular on $R/((x_1,..., x_{n-1}):I)$. If $k<n-1$, then it follows from our Remark 2.3 that

$$(((x_1,..., x_k):I), x_{k+1})=((x_1,..., x_{k+1}):I)\cap(((x_1,..., x_k):I)+I).$$

We know y is regular on $R/(((x_1,..., x_k):I)+I)$, while by induction y is regular on $R/((x_1,..., x_{k+1}):I)$. It follows y is regular on $(((x_1,..., x_k):I), x_{k+1})$. Again since x_{k+1} is regular on $R/((x_1,..., x_k):I)$, we obtain y is regular on $R/((x_1,..., x_k):I)$. This proves y is regular on R/J for every related ideal J of R.

Clearly, $A_1 = ((y, x_0, \dots, x_k) : I)$ contains $A_2 = (y, ((x_0, \dots, x_k) : I))$. To prove equality it is enough to show $(A_2)_P = (A_1)_P$ for every prime ideal P in $\mathrm{Ass}(R/A_2)$. Since $R/((x_0, \dots, x_k) : I)$ is Cohen–Macaulay and y is not a zero divisor on this ring, there are no embedded primes in $\mathrm{Ass}(R/A_2)$. We claim if $P \in \mathrm{Ass}(R/A_2)$, then $P \not\supseteq I$. For if $P \supseteq I$, then $P \supseteq (x_{k+1}, y)$ which we have shown to be a $R/((x_0, \dots, x_k) : I)$-sequence. This contradicts $P \in \mathrm{Ass}(R/A_2)$. Therefore, $(A_2)_P = (y, ((x_0, \dots, x_k)_P : I_P)) = (y_1, x_0, \dots, x_k)_P = (A_1)_P$. Hence, $A_1 = A_2$. The last statement of Lemma 2.7 is an immediate consequence of the first two statements.

THEOREM 2.3. *Let R be a Cohen–Macaulay local ring and suppose I is an ideal such that $v(I_P) \leqslant ht\, P$ for all prime ideals P containing I and such that either*

(1) *$ht\, I = 2 = pd\, R/I$, or*

(2) *$ht\, I = 3 = pd\, R/I$, and both R and R/I are Gorenstein rings.*

Then I is generated by a Cohen–Macaulay d-sequence and in particular the complex $\mathcal{M}$ associated to I is exact.

Proof. We will prove (1); (2) follows similarly. By the Hilbert–Burch theorem [4], we know that any minimal generating set of I can be realized as the n by n minors of an n by $n+1$ matrix A. Choose generators $b_1, \dots, b_{n+1}$ of I satisfying the conditions of Lemma 2.6, and choose the matrix $A = (a_{ij})$ such that b_i is the minor determined by deleting the ith column of A.

Choose $n(n+1)$-generic indeterminates x_{ij}, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n+1$, and let $S = R[x_{ij}]_{(x_{ij}, m)}$, and $J = I_n(X) =$ the ideal generated by the maximal minors of the matrix $X = (x_{ij})$. Consider the homomorphism f from S to R by sending x_{ij} to a_{ij}. This has kernel equal to the ideal $N = (x_{ij} - a_{ij})$, and $f(J) = I$. Let Δ_i be the minor of X determined through deleting the ith column of X. Then $f(\Delta_i) = b_i$. We claim that the $x_{ij} - a_{ij}$ form a regular-sequence on the ring S/B, where B is any ideal of the form $B = ((\Delta_1, \dots, \Delta_i) : J) + J$. By Proposition 2.2, S/B is Cohen–Macaulay so to show the $x_{ij} - a_{ij}$ are a regular S/B-sequence it suffices to prove $ht(N+B)/B \geqslant n(n+1)$. Since R (and thus S) is Cohen–Macaulay, $ht(N+B)/B = ht(N+B) - ht\, B$. By the remark after Proposition 2.2, $ht\, B = i+1$. On the other hand $ht(N+B) = ht(N+B)/N + ht\, N = n(n+1) + ht\, f(B)$. We claim $ht\, f(B) = ht((b_1, \dots, b_i) : I) + I)$. In fact we will show nilrad $f(\mathrm{B})$ contains $((b_1, \dots, b_i) : I) + I$. For this, it is clearly enough to prove nilrad$(f((\Delta_1, \dots, \Delta_i) : J)))$ contains $(b_1, \dots, b_i : I)$.

Let $d \in ((b_1, \dots, b_i) : I)$. Then

$$dJ \subseteq (b_1, \dots, b_i, x_{ij} - a_{ij}) = (\Delta_1, \dots, \Delta_i, x_{ij} - a_{ij}).$$

Since the $(x_{ij}-a_{ij})$ form an S/J sequence, (by the structure theorem), we see

$$dJ \subseteq (\Delta_1, \ldots, \Delta_i) + JN.$$

Therefore modulo $(\Delta_1, \ldots, \Delta_i)$,

$$dJ \subseteq JN.$$

From the well-known "determinant argument" there exists a *monic* polynomial,

$$g(Z) = Z^n + c_1 Z^{n-1} + \cdots + c_n$$

such that $c_k \in N^k$ and such that $g(d)$ annihilates J-modulo $(\Delta_1, \ldots, \Delta_i)$. Hence

$$g(d) \in ((\Delta_1, \ldots, \Delta_i) : J).$$

In particular,

$$d^n \in (((\Delta_1, \ldots, \Delta_i) : J), x_{ij} - a_{ij})$$

which shows $d^n \in f(B)$.

It follows from Lemma 2.6 that $ht((b_1, \ldots, b_i) : I) + I) \geqslant i+1$. Since nilrad$(f(B))$ contains this ideal, $ht\, f(B) \geqslant i+1$. We conclude,

$$\begin{aligned} ht(N+B)/B &= ht(N+B)/N + ht\, N - ht\, B \\ &\geqslant n(n+1) + (i+1) - (i+1) = n(n+1). \end{aligned}$$

This shows $x_{ij} - a_{ij}$ are a regular sequence on $S/((\Delta_1, \ldots, \Delta_i) : J) + J)$. From Lemma 2.7 we may now conclude

$$\begin{aligned} N + B &= ((x_{ij} - a_{ij}), (\Delta_1, \ldots, \Delta_i : J) + J)) \\ &= ((x_{ij} - a_{ij}, b_1, \ldots, b_i) : I) + I \\ &= ((b_1, \ldots, b_i) : I) + I + (x_{ij} - a_{ij}). \end{aligned}$$

This shows $f(B) = ((b_1, \ldots, b_i) : I) + I$. By Lemma 2.7, $b_1, \ldots, b_{n+1}$ is a Cohen–Macaulay d-sequence. An application of Theorem 2.1 proves $\mathcal{M}$ is exact.

The proof of (2) is similar, using the structure theorem for ideals of this type due to Buchsbaum and Eisenbud [3].

We next wish to discuss the Cohen–Macaulayness of the Koszul homology. To do this we first show, THEOREM 2.4. *Let R be a Cohen–Macaulay local ring and I an ideal whose Koszul homology is Cohen–Macaulay and such that $v(I_p) \leqslant ht\, P$ for all prime ideals P containing I. Then*

(1) *I is generated by a Cohen–Macaulay d-sequence,*

(2) *The complex $\mathcal{M}$ associated to I is exact and $H_0(\mathcal{M}) \simeq \mathrm{gr}_I(R) = R/I \oplus I/I^2 \oplus \cdots$, and*

(3) depth $I^k/I^{k+1} \geqslant$ depth $R/I - k$ *for all* $k \geqslant 0$.

Remark. By Theorem 2.2, condition (1) implies $\mathcal{M}$ is exact. Since $H_0(\mathcal{M}) = \mathrm{Sym}(I/I^2)$, the rest of (2) follows from Remark 2.8; $\mathrm{Sym}(I/I^2) \simeq \mathrm{gr}_I(R)$ whenever I is generated by a d-sequence. We remark that the condition "$H_i(I; R)$ is Cohen–Macaulay" can be replaced by the condition,

(a) $H_i(I; R)$ satisfies Serre's condition S_i (compare this with Theorem 4.2, [21].)

Proof. We first prove (1). It is well known that if I is an ideal in a commutative ring, there are reduced ideals $G_k(I)$ with the property that a prime ideal P contains $G_k(I)$ if and only if $v(I_P) \geqslant k+1$ [15]. Let $B(I)$ be the set of all prime ideals of R which are minimal over some $G_k(I)$. This is a finite set. We now induct on $n = v(I)$ to prove (1). There are two cases.

Case 1 (grade $I > 0$). Choose an x in I which is a nonzero divisor and such that x is a minimal generator of I_Q for all prime ideals Q in $B(I)$, and a minimal generator of I. This is possible since $B(I)$ is a finite set. Let "—" denote the homomorphism from R to $\bar{R} = R/Rx$. By Corollary 1.1, $H_i(\bar{I}; \bar{R})$ are Cohen–Macaulay. We also claim $v(\bar{I}_{\bar{Q}}) \leqslant ht\, \bar{Q}$ for all prime ideals $\bar{Q} \supseteq \bar{I}$. Let $\bar{Q}$ be such a prime ideal and let Q be the lifting of $\bar{Q}$ to R. Then $ht\, Q = ht\, \bar{Q} + 1$. If $Q \in B(I)$, then

$$v(\bar{I}_{\bar{Q}}) = v(I_Q) - 1 \leqslant ht\, Q - 1 = ht\, \bar{Q}.$$

If Q is not in $B(I)$, set $k = v(I_Q)$, and choose a $P \in B(I)$ such that $Q \supset P \supseteq G_{k-1}(I)$. Then

$$v(\bar{I}_{\bar{Q}}) \leqslant v(I_Q) = k = v(I_P) \leqslant ht\, \bar{Q},$$

the last inequality holding since $ht\, P < ht\, Q$.

By induction on $v(I)$ we may assume that $\bar{I}$ is generated by a d-sequence $\bar{x}_2, \ldots, \bar{x}_n$. Set $x_1 = x$. Since x_1 is a nonzero divisor, we may lift each $\bar{x}_i$ to an x_i in R which is a nonzero divisor. Then $x_1, x_2, \ldots, x_n$ clearly form a d-sequence, and $R/(x_1, \ldots, x_i : x_{i+1}) = \bar{R}/(\bar{x}_2, \ldots, \bar{x}_i : \bar{x}_{i+1})$ so that if $\bar{x}_2, \ldots, \bar{x}_n$ is a Cohen–Macaulay d-sequence, so is $x_1, \ldots, x_n$.

Case 2 (grade $I=0$). Choose an x in I such that x is a minimal generator of I, a minimal generator of I_Q for all $Q \in B(I)$ and not an element of any minimal prime ideal Q of height 0 which does not contain I.

We claim the assumptions of (1) still hold for the ring $\bar{R}=R/((0:I), x)$. First we show $(0:I)=(0:x)=(0:x^2)$. Clearly $(0:I) \subset (0:x)$. Let

$$(0)=\bigcap_{i=1}^{N} q_i$$

be a primary decomposition of (0). Let $q_1, \ldots, q_k$ be those q_i such that $I \not\subseteq \text{nilrad}(q_i)$. Then $(0:I)=\bigcap_{i=1}^{k} q_i$. To see this, let $y \in (0:I)$. Then $yI=(0)$. Since $I \not\subseteq \text{nilrad}(q_i)$ for $i=1, \ldots, k$ this shows $y \in \bigcap_{i=1}^{k} q_i$. Thus $(0:I) \subseteq \bigcap_{i=1}^{k} q_i$. If $y \in \bigcap_{i=1}^{k} q_i$, then $yI=0$. To show this it is enough to show

$$I=\bigcap_{i=k+1}^{n} q_i.$$

However if nilrad $q_i=P_i$, then the primary component of I in P_i is just $R \cap I_{P_i}$. Hence it suffices to observe $I_{P_i}=0$. But, $v(I_{P_i}) \leqslant ht\, P_i$ by assumption, which shows $I_{P_i}=0$. We have established that

$$(0:I)=\bigcap_{i=1}^{k} q_i.$$

Since $x \notin \text{nilrad}(q_i)$ for $i=1, \ldots, k$, we see $(0:x) \subseteq \bigcap_{i=1}^{k} q_i$. As $(0:I) \subseteq (0:x)$, this shows

$$(0:x)=(0:I).$$

Furthermore, since x is not in nilrad(q_i), x is not a zero-divisor on $R/\bigcap_{i=1}^{k} q_i=R/(0:I)$. Thus $(0:x)=(0:x^2)$.

By Proposition 1.1, $R/(0:I)$ is Cohen–Macaulay and by Corollary 1.2, $H_i(I, R/(0:I))$ are Cohen–Macaulay. (Note that the above calculations show $ht((0:I)+I) \geqslant 1$.) Since x is not a zero divisor on $R/(0:I)$ by above, Lemma 1.2 now shows $H_i(\bar{I}; \bar{R})$ are Cohen–Macaulay. We need to show if $\bar{q}$ is a prime ideal of $\bar{R}$ containing $\bar{I}$, then $v(\bar{I}_q) \leqslant ht\, \bar{q}$. Since $ht((0:I)+I)=1$, if q lifts $\bar{q}$, we see that $ht\, \bar{q}=ht\, q-1$. Suppose $q \in B(I)$. Then if $I_q \neq 0$,

$$v(\bar{I}_{\bar{q}}) \leqslant v(I_q)-1 \leqslant ht\, q-1=ht\, \bar{q}.$$

If $q \notin B(I)$ set $k=v(I_q)$ and choose a $p \in B(I)$ such that $q \supset p \supseteq G_{k-1}(I)$. Then

$$v(\bar{I}_{\bar{q}}) \leqslant v(I_q)=k=v(I_p) \leqslant ht\, p \leqslant ht\, \bar{q}.$$

Since $\bar{q} \supseteq ((0:I), x)$, we see that $I_q \neq 0$, and this shows the conditions of (1) hold for the ring $\bar{R}$ and the ideal $\bar{I}$. By induction we conclude that $\bar{I}$ is generated by a Cohen–Macaulay d-sequence $\bar{x}_2, \ldots, \bar{x}_n$. To show I is generated by a Cohen–Macaulay d-sequence we need the following lemma.

LEMMA 2.1. *Let R be a Cohen–Macaulay local ring and I an ideal such that $(0:I) \cap I = (0)$. If the image of I in $R/(0:I)$ is generated by a Cohen–Macaulay d-sequence of grade at least one, then I is generated by a Cohen–Macaulay d-sequence.*

Proof. Choose $\bar{x}_1, \ldots, \bar{x}_n$ which generate I modulo $(0:I)$ and form a Cohen–Macaulay d-sequence. Lift $\bar{x}_i$ to any x_i. I claim $(0:x_1) = (0:I)$ and $((x_1, \ldots, x_i) : x_{i+1}) \equiv (\bar{x}_1, \ldots, \bar{x}_i : \bar{x}_{i+1})$ modulo $(0:I)$. The first claim holds since $\bar{x}_1$ is not a zero divisor on $R/(0:I)$. If $\bar{r}\bar{x}_{i+1} \in (\bar{x}_1, \ldots, \bar{x}_i)$ then

$$rx_{i+1} \in (0:I) + (x_1, \ldots, x_i).$$

Since $(0:I) \cap I = (0)$ this shows that $r \in ((x_1, \ldots, x_i) : x_{i+1})$. This proves $x_1, \ldots, x_n$ is a d-sequence.

Now return to the theorem. If $\bar{x}_2, \ldots, \bar{x}_n$ generate a d-sequence in $\bar{R}$ generating $\bar{I}$, the proof of Case 1 shows that we may lift $\bar{x}_2, \ldots, \bar{x}_n$ to elements $x'_2, \ldots, x'_n$ in $R/(0:I)$, where x', $x'_2, \ldots, x'_n$ form a d-sequence in $R/(0:I)$, where x' = the image of x in $R/(0:I)$. Lemma 2.1 shows that any lifting of $\bar{x}_i$ to x_i in R makes $x, x_2, \ldots, x_n$ a d-sequence which generate I. Further, if J is a related ideal of $x_1, x_2, \ldots, x_n$ and $J \neq (0:I)$, then R/J is isomorphic to $\bar{R}/\bar{J}_1$, where $\bar{J}_1$ is related ideal of $\bar{x}_1, \ldots, \bar{x}_n$. Since $\bar{x}_1, \ldots, \bar{x}_n$ are a Cohen–Macaulay d-sequence, $\bar{R}/\bar{J}_1$ is Cohen–Macaulay for every related ideal J not equal to $(0:I)$. By Proposition 1.1, however, $R/(0:I)$ is Cohen–Macaulay. This proves (1).

Our opening remark shows that (2) is also satisfied. It remains to prove (3). Since $\mathcal{M}$ is exact and $\mathrm{Sym}(I/I^2) \simeq \mathrm{gr}_I(R)$, we have a resolution of I^k/I^{k+1},

$$0 \to H_k(I; R) \to H_{k-1}(I; R) \otimes \mathrm{Sym}_1(F/IF) \to \cdots \to \mathrm{Sym}_k(F/IF) \to I^k/I^{k+1} \to 0,$$

where $F = R^n$, $n = v(I)$.

We are assuming depth $H_i(I; R)$ = depth R/I. The conclusion (3) now follows from

LEMMA 2.2. *Let R be a local ring and C an exact sequence of finitely generated R-modules,*

$$0 \to C_n \to \cdots \to C_0.$$

If depth $C_i \geqslant$ depth R *for every i, then* depth $H_0(C) \geqslant$ depth $(R) - n$.

Proof. This is an easy exercise.

We now wish to prove the converse of Theorem 2.3.

THEOREM 2.5. *Let R be a Cohen–Macaulay local ring, and I an ideal generated by a Cohen–Macaulay d-sequence $x_1,..., x_n$. If*

$$\text{depth } I^k/I^{k+1} \geq \text{depth } R/I - k$$

for all $k \geq 0$, then $H_i(I; R)$ are Cohen–Macaulay for each $i \geq 0$.

To prove this, we first recall the definition of a relative regular sequence due to Fiorentini [8].

DEFINITION 2.1. Let R be a commutative ring, M an R-module and N a submodule of M. A sequence $a_1,..., a_n$ of elements of R is said to be a relative M-regular sequence with respect to N, if for all i, $a_{i+1}x \in \sum_{j=1}^{i} a_j N$ and $x \in N$ imply $x \in \sum_{j=1}^{i} a_j M$, $0 \leq i < n$.

THEOREM E [8]. *Let N, M, $a_1,..., a_n$ be as in Definition 2.1, and let $f: K(a_1,..., a_n; N) \to K(a_1,..., a_n; M)$ be the induced map on the Koszul complexes. If $a_1,..., a_n$ are a relative M-regular sequence with respect to N, then $H_p(f) = 0$ for all $p \geq 1$.*

LEMMA 2.3. *Let $I = (a_1,..., a_n)$ be generated by the d-sequence $a_1,..., a_n$. Then $a_1,..., a_n$ are a relative I^{d-1} sequence with respect to I^d for all $d \geq 1$. ($I^0 = R$ by convention).*

Proof. Suppose $x \in I^d$ and $a_j x \in (a_1,..., a_{j-1}) I^d$. We must show $x \in (a_1,..., a_{j-1}) I^{d-1}$. Clearly,

$$x \in (a_1,..., a_{j-1} : a_j) \cap I^d.$$

Since $(a_1,..., a_{j-1} : a_j) \cap I = (a_1,..., a_{j-1})$ by Remark 2.3,

$$x \in (a_1,..., a_{j-1}) \cap I^d.$$

Thus, by Remark 2.5,

$$x \in (a_1,..., a_{j-1}) I^{d-1}.$$

Following common usage, let us call a sequence of elements $x_1,..., x_n$ j-syzygetic if for all i and $1 \leq k \leq j$, the induced map from

$$H_i(I; I^k) \to H_i(I; I^{k-1})$$

is zero, where $I = (x_1,..., x_n)$. By Lemma 2.3 and Theorem E, any d-sequence is j-syzygetic for all j.

LEMMA 2.4. *Let $I=(x_1,\dots,x_n)$ be an ideal of R. Suppose z is an element of R which is not a zero divisor on R, R/I, $I/I^2,\dots,I^{d-1}/I^d$. Let "—" denote the homomorphism from R to $\bar{R}=R/Rz$. If $x_1,\dots,x_n$ are* dth *syzygetic, then $\bar{x}_1,\dots,\bar{x}_n$ are* $(d-1)$st *syzygetic.*

Proof. Consider the exact sequences

$$0\to I^k\to I^{k-1}\to I^{k-1}/I^k\to 0.$$

These induce long exact sequences on Koszul homology,

$$\cdots\to H_j(I,I^k)\to H_j(I,I^{k-1})\to H_j(I;I^{k-1}/I^k)\to H_{j-1}(I;I^k)\to\cdots.$$

For $k\leqslant d$, the maps $H_j(I;I^k)\to H_j(I;I^{k-1})$ are zero, and so we get short exact sequences, for $k\leqslant d$,

$$0\to H_j(I;I^{k-1})\to H_j(I;I^{k-1}/I^k)\to H_{j-1}(I;I^k)\to 0.$$

As I^{k-1}/I^k is annihilated by I, $H_j(I;I^{k-1}/I^k)\simeq\bigoplus I^{k-1}/I^k$. Hence, we obtain exact sequences

$$0\to H_j(I;I^{k-1})\to\bigoplus I^{k-1}/I^k\to H_{j-1}(I;I^k)\to 0. \tag{2}$$

Since z is not a zero divisor on I^k/I^{k+1} for $j\leqslant d-1$, (2) shows z is not a zero divisor on $H_j(I;I^k)$ for $k\leqslant d-1$, i.e., $0=\operatorname{Tor}_1(R/Rz,H_j(I;I^j))$. Hence if we tensor (2) with R/Rz for $k\leqslant d-1$, we obtain short exact sequences,

$$0\to\frac{H_j(I;I^{k-1})}{zH_j(I;I^{k-1})}\to\bigoplus I^{k-1}/(I^k+zI^{k-1})\to\frac{H_{j-1}(I;I^k)}{zH_{j-1}(I;I^k)}\to 0. \tag{3}$$

From the exact sequences

$$0\to I^j\xrightarrow{z}I^j\to\bar{I}^j\to 0$$

and

$$0\to I^j/I^{j+1}\xrightarrow{z}I^j/I^{j+1}\to\bar{I}^j/\bar{I}^{j+1}\to 0$$

and because z is a nonzero divisor on $H_j(I;I^{k-1})$ for all j, $j\leqslant d-1$, it follows that

$$\frac{H_j(I;I^{k-1})}{zH_j(I;I^{k-1})}\simeq H_j(\bar{I};\bar{I}^{k-1})$$

and

$$I^{k-1}/(I^k,zI^{k-1})\simeq\bar{I}^{k-1}/\bar{I}^k.$$

It easily follows the exact sequence (3) is just part of the long exact sequence on Koszul homology associated to

$$0 \to \bar{I}^k \to \bar{I}^{k-1} \to \bar{I}^{k-1}/\bar{I}^k \to 0.$$

Since image $(H_j(\bar{I}; \bar{I}^k) \to H_j(\bar{I}; \bar{I}^{k-1})) = \ker(H_j(\bar{I}; \bar{I}^{k-1}) \to \bigoplus \bar{I}^{k-1}/\bar{I}^k)$, it follows from (3) that the map

$$H_j(\bar{I}; \bar{I}^k) \to H_j(\bar{I}; \bar{I}^{k-1})$$

is zero for $k \leqslant d-1$. Hence $\bar{x}_1, \ldots, \bar{x}_n$ is $(d-1)$th syzygetic.

To finish the proof of Theorem 2.4, it is enough to prove the following claim. Let $I = (x_1, \ldots, x_n)$ be an ideal of R. Suppose depth $R/I \geqslant d$, depth $I/I^2 \geqslant d-1, \ldots,$ depth $I^{d-1}/I^d \geqslant 1$. If $(x_1, \ldots, x_n)$ are d-syzygetic, then depth $H_i(I; R) \geqslant d$ for all i.

Proof. We induct on d. If $d=1$, we must show depth $H_i(I; R) \geqslant 1$. However, since $(x_1, \ldots, x_n)$ is 1-syzygetic there are exact sequences

$$0 \to H_j(I; R) \to H_j(I; R/I) \to H_{j-1}(I; I) \to 0$$

for $j \geqslant 1$. Hence for $j \geqslant 1$, $H_j(I; R)$ imbeds in $H_j(I; R/I) \simeq \bigoplus R/I$. For $j=0$, $H_j(I; R) = R/I$. Thus since depth $R/I \geqslant 1$, depth $H_j(I; R) \geqslant 1$.

Suppose $d > 1$. Since depth $I^j/I^{j+1} \geqslant 1$ for $1 \leqslant j \leqslant d-1$, we may choose an element z in R which is not a zero divisor on any of these modules. Let "—" denote the homomorphism from R to $\bar{R} = R/Rz$. Lemma 2.4 shows that $\bar{x}_1, \ldots, \bar{x}_n$ are $(d-1)$-syzygetic. In addition as in the proof of Lemma 2.4 there are exact sequences,

$$0 \to H_j(I; I^{k-1}) \to \bigoplus I^{k-1}/I^k \to H_{j-1}(I; I^k) \to 0$$

which show z is not a zero divisor on $H_j(I; I^k)$ for $0 \leqslant k \leqslant d-1$ and

$$\frac{H_j(I; I^k)}{zH_j(I; I^k)} \simeq H_j(\bar{I}; \bar{I}^k), \qquad \bar{I}^k/\bar{I}^{k+1} \simeq I^k/I^{k+1} \otimes R/Rz.$$

By induction depth $H_j(\bar{I}; \bar{R}) \geqslant d-1$. Hence depth $H_j(I; R) \geqslant d$. The following result was first discovered by Avramov and Herzog [1] and independently by Buchsbaum and Eisenbud.

COROLLARY 2.1. *Let R be a Cohen–Macaulay local ring and let I be an ideal of R such that $pd\ R/I = ht\ I = 2$. Then $H_i(I; R)$ are Cohen–Macaulay.*

Proof. The theorem of Hilbert–Burch shows I is generated by the n by n minors of an n by $(n+1)$ matrix $A = (a_{ij})$. Put $S = R[x_{ij}]_{(x_{ij}, m)}$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n+1$, and let J be the ideal generated by the n by n minors of $X = (x_{ij})$.

We claim $H_i(J; S)$ are Cohen–Macaulay. Since J is generated by a Cohen–Macaulay d-sequence, it is enough to show by Theorem 2.4 that

$$\operatorname{depth} J^k/J^{k+1} \geqslant \operatorname{depth} R/J - k.$$

This follows from the work of Weyman [24]. The ideal J has a resolution

$$F = 0 \to S^n \xrightarrow{X} S^{n+1} \to J \to 0.$$

Weyman constructs complexes $\operatorname{Sym}_i(F)$ which in this case become a resolution of $\operatorname{Sym}_i(J)$. (See Theorem 1.b, p. 338 [24].) By Remark 2.8, $\operatorname{Sym}_i(J) = J^i$. The length of $\operatorname{Sym}_i(F_0)$ is $\min(n, i)$ (p. 336 [24]). It follows that

$$\begin{aligned}\operatorname{depth} I^{k-1}/I^k &= \operatorname{depth} R - \min(n, k) - 1\\ &= \operatorname{depth} R/I - \min(n, k) + 1.\end{aligned}$$

In particular,

$$\operatorname{depth} I^{k-1}/I^k \geqslant \operatorname{depth} R/I - k + 1$$

for all k. Hence $H_i(J; S)$ are Cohen–Macaulay.

Now define a map f from S onto R by sending x_{ij} to a_{ij}. Then $f(J) = I$ and $f(H_j(J; S)) = H_j(J; S)/(x_{ij} - a_{ij})\, H_j(J; S) = H_j(I; R)$. Since the $x_{ij} - a_{ij}$ are a regular sequence on $H_j(J; S)$, and $H_j(J; S)$ are Cohen–Macaulay, we conclude $H_j(I; R)$ are Cohen–Macaulay $S/(x_{ij} - a_{ij}) = R$-modules.

Finally we discuss some other examples where the complex $\mathcal{M}$ will be exact. As Theorems 2.2 and 2.3 show, such examples arise from Cohen–Macaulay d-sequences.

Ideals close to a complete intersection were studied by Simis and Vasconcelos in [21]. The following was proved in [12].

Proposition 2.4. *Suppose R is a Cohen–Macaulay local ring and I is an ideal such that either*

(1) *$v(I) = ht\, I + 1$ and I_P is a complete intersection for all prime ideals P minimal over I, or*

(2) *$v(I) = ht\, I + 2$, R is Gorenstein and R/I is Cohen–Macaulay and $v(I_P) \leqslant ht\, P$ for all prime ideals P containing I.*

Then I is generated by a d-sequence. In case (2), I is generated by a Cohen–Macaulay d-sequence. In case (1), I is generated by a Cohen–Macaulay d-sequence if and only if R/I is Cohen–Macaulay. In particular if R/I is Cohen–Macaulay, $\mathcal{M}$ is exact.

EXAMPLE 2.1. Let $X=(x_{ij})$ be an r by s matrix $(r \leqslant s)$ of indeterminates over a field K and let I be the ideal in $K[x_{ij}]_{(x_{ij})}$ generated by the t by t minors of X. Set $R=K[x_{ij}]_{(x_{ij})}/I$. Then the images of any row or column of X in R form a Cohen–Macaulay d-sequence [11] and hence $\mathscr{M}$ is exact for these ideals.

EXAMPLE 2.2. Let R be a local ring and $x_1,\ldots,x_n$ a Cohen–Macaulay d-sequence. By $\bar{x}_1,\ldots,\bar{x}_n$ denote the images of $x_1,\ldots,x_n$ in the first graded piece of $\mathrm{Sym}(I)$, where $I=(x_1,\ldots,x_n)$. (It was shown in [12] and [23] that $\mathrm{Sym}(I) \simeq R[It]$ when I is generated by a d-sequence.) Then in [13] it was shown that $\bar{x}_1,\ldots,\bar{x}_n$ are also a Cohen–Macaulay d-sequence; in particular the complex $\mathscr{M}$ associated to $\bar{x}_1,\ldots,\bar{x}_n$ is exact. The same statement also holds when we replace $\mathrm{Sym}(I)$ by $\mathrm{Sym}(I/I^2)$.

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