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THE CURL OF GRAPHS AND NETWORKS

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Abstract—We investigate a vector calculus for graphs and networks, thereby initiating a corresponding classification and structure theory. In particular, any network $G(V, E, \sigma, c)$ may be decomposed as

$$G = G_1 \oplus G_2 \oplus G_3,$$

where G_1 is divergence-free and curl-free, G_2 is divergence-free but not curl-free, and G_3 is curl-free but not divergence-free. A number of questions, implications, and future directions are discussed in this semi-expository study.

1. INTRODUCTION

Recently graph theory has been applied to fluid dynamics in some interesting ways: see Gustafson and Hartman [1] for its use in resolving questions about finite element subspaces for the Navier-Stokes equations. It is an intriguing idea to try to go the other way: what can fluid dynamics say to graph theory? Our purpose is to consider some of the new theorems, structures, and questions so induced.

For simplicity we restrict our attention to planar graphs, and regard them as two-dimensional. From the conceptual viewpoint, this restriction corresponds in the usual setting of vector calculus to looking only at the rotation:

$$(\text{curl } \mathbf{v}) \cdot \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1(x, y) & v_2(x, y) & 0 \end{vmatrix} \cdot \mathbf{k} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}.$$

As will be seen, even in this limited setting, the possible notions of the curl of graphs and networks are several. However, the considerations will apply as well to three-dimensional structures where the fluid dynamical analogues would have even larger play.

An *orientation* σ of a graph $G(V, E)$ results in a directed graph $G(V, E, \sigma)$, by assigning a unique direction $\sigma(e)$ to each edge $e \in E$. More analysis is possible from a network $G(V, E, \sigma, c)$, where the capacity function c assigns a numerical vlaue $c(e)$ to each $e \in$

E. Thus V denotes the set of vertices; E the set of edges; $\sigma = \pm 1, 0$; $c > 0$. We may also write, with c any real number, $G(V, E, c)$ for the same effect. We follow the graph theory notation and terminology in [2].

In Sec. 2 we will consider the specific question of the curl of graph-theoretic entities. Some motivating examples are presented. For lack of a more direct approach, we proceed analogously to [1], utilizing flow vector fields on the graph and decomposing the graph in terms of them. This allows a large number of decompositions from which, in principle, one may take his pick of curl representations according to his needs. On the other hand, one might prefer a unique curl resulting from a more geometrical formulation, as we indicate in our analyses.

Because [1] may not be accessible to the reader, we include in Sec. 2 some explanation of how the approach of [1] works. For further details, especially as to its application to determining the dimension and bases of the French finite element schemes, we refer to [1].

In Sec. 3 we give the network decomposition theorem stated in the abstract. From the descriptive development of Sec. 2, and by use of the results of [1], the proof is immediate.

In Sec. 4 we comment on a number of the interesting questions and possibilities raised by this initial investigation. Among these are a general vector calculus of graphs, vorticity and invariance properties desired in the resultant structure theory, a historical prospective, and a possible link to the Atiyah-Singer index theory.

2. THE CURL OF A GRAPH

We begin with a very specific question: given a directed graph $G(V, E, \sigma)$, what is its "curl"? One may prefer to ask, what is its "divergence"? Immediately one sees the possibility of a general decomposition of a graph into three parts: (1) its potential component; (2) its solenoidal component; (3) its irrotational component. This we will prove in the next section.

Our approach is as follows. Given an oriented graph $G(V, E, \sigma)$, associate with it a flow vector field $U(V, E, \sigma)$. For simplicity let the digraph be planar: then the flow field will have two components $\mathbf{u} = (u_1, u_2)$. In general, the flow field will have at least as many components as the minimum dimension in which the graph exists. The flow vector field $U(V, E, \sigma)$ is defined on the edges $e \in E$, and for simplicity one may initially visualize it as defined to be constant on each edge. Notice that in so doing we are led to, essentially, representing $G(V, E, \sigma)$ in terms of networks $G(V, E, c)$ but with c as a vector field. Moreover, as the following example shows, we are further led to networks $G(V, E, c_0, c_1)$ where c_0 is a vector field on the vertices and c_1 a vector field on the edges.

At this point let us describe the setting and methods of [1]. A viscous incompressible fluid motion in a vessel Ω is modelled continuously by its momentum equation

$$u_t - \gamma \Delta u + (u \cdot \nabla) u = -\nabla p + f,$$

with the incompressibility constraint

$$\nabla \cdot u = 0,$$

and appropriate initial and boundary conditions. In the finite element schemes treated in [1], the domain Ω is first triangulated, then "incompressible" approximation spaces $APX1$, $APX2, \ldots$, of increasing order, are set up on the triangulated Ω, and finally a discretized momentum equation is followed in time through the chosen incompressibility subspace.

The only $APXi$ we shall consider here are $APX1$, a finite difference approximation,

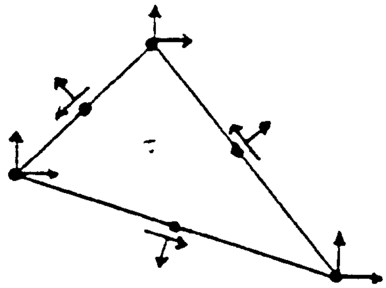


Fig. 1. An element example.

and $APX2$, a quadratic element approximation. Moreover, we have no need for the momentum equation.

There is an easy way to understand the analysis of [1], by first looking at the $APX2$ fit on the single element τ shown in Fig. 1. The problem is to specify a vector field $\phi = (\phi_1, \phi_2)$ in which ϕ_1 and ϕ_2 are restricted to be polynomials of degree at most 2. The means of carrying out this specification is depicted in Fig. 2. Recall that any vector field **u** permits a decomposition (Helmholtz Theorem)

$$\mathbf{u} = \mathbf{u}^1 \oplus \mathbf{u}^2 \oplus \mathbf{u}^3$$

in which $\mathbf{u}^1$ is both curl-free and divergence-free, $\mathbf{u}^2$ is divergence-free but not curl-free, and $\mathbf{u}^3$ is curl-free but not divergence-free.

How is this carried out analytically? Let any two vertices of the triangle τ be specified by A_i and A_j and the corresponding midedge by A_{ij}. In barycentric coordinates,

$$\phi(x) = \sum_{i=1}^{3} (2\lambda_i^2 - \lambda_i)\phi(A_i) + 4 \sum_{i<j} \lambda_i\lambda_j\phi(A_{ij}).$$

The incompressibility constraint is to be satisfied in a weak sense according to

$$\int_\tau \nabla \cdot \phi = \int_{\partial\tau} \phi \cdot n = 0.$$

The modelling of [1] for $APX2$ is now carried out as follows: First we specify ϕ at the vertices. This corresponds to the "potential" component as depicted in Fig. 2. Along any

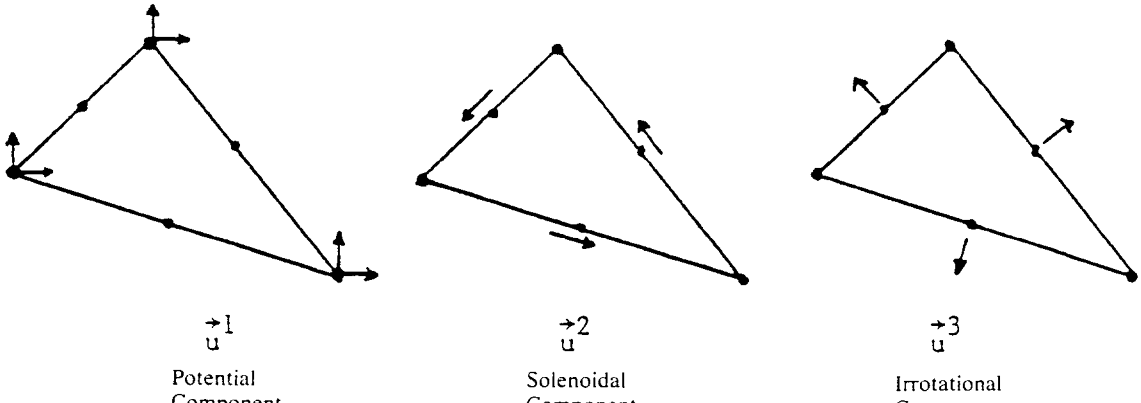


Fig. 2. A flow space decomposition.

edge one has

$$\int_{A_i}^{A_j} \phi \, dl = \frac{|A_i - A_j|}{6} \{\phi(A_i) + \phi(A_j) + 4\phi(A_{ij})\}$$

and we agree to guarantee incompressibility by requiring this quantity to be zero on each edge. Thus ϕ at A_{ij} is completely determined and so is its potential component $\mathbf{u}^1$.

We next define the tangent vectors $\mu_{ij} = (A_j - A_i)/ | A_j - A_i |$ and the normal vectors $\gamma_{ij} = (\mu_{ij}^2, - \mu_{ij}^1)$. To specify the solenoidal component $\mathbf{u}_2$ of ϕ, we specify a value of zero at the vertices and an arbitrary value $c_{ij}\mu_{ij}$ at the midedges. Then along any edge one has incompressibility in the sense:

$$\int_{A_i}^{A_j} \phi \cdot \gamma_{ij} \, dl = \frac{2}{3} c_{ij} \, | A_i - A_j | \, \mu_{ij} \cdot \gamma_{ij} = 0.$$

Finally, the irrotational component $\mathbf{u}^3$ of ϕ is determined as zero at the vertices, as an arbitrary $d_{ij}\gamma_{ij}$ at the midedges, and must satisfy:

$$\int_\tau \nabla \cdot \phi = \pm \frac{2}{3} [\, | A_1 - A_2 | \, d_{12} + | A_2 - A_3 | \, d_{23} + | A_3 - A_1 | \, d_{31}] = 0.$$

It follows that the dimension of the subspace of a quadratic which fits on an arbitrary admissibly triangulated domain Ω is twice (because it is a vector field of two components) the number of interior vertices (interior only because in the setting of [1], ϕ is specified to be zero at boundary vertices to meet a Dirichlet boundary condition) from its $\mathbf{u}^1$ components, plus the number of interior midedges from its $\mathbf{u}^2$ component, plus whatever comes in finishing the fit with the $\mathbf{u}^3$ component. The scheme of *APX*2 also requires the continuity of ϕ across element boundaries. At this point, we refer to [1] and simply assert the result obtained by graph theory that the $\mathbf{u}^3$ component still allows further degrees of freedom equal to the number of interior vertices plus the number of interior "holes" in Ω.

Let us summarize this description in graph-theoretic terms.

EXAMPLE 1. Consider the graph $G = K_3$ having three vertices and edges as in Fig. 1. Let us, analogous to the finite element scheme *APX*2 analyzed in [1], use a flow space $U(V, E, c_0, c_1)$ of quadratic functions $\mathbf{u} = (u_1, u_2)$ defined on the element τ. The flow space function $\mathbf{u}$ may be specified by its values at the three vertices, its three normal components at midedges, and its three tangential components at midedges, as indicated in Fig. 1 and Fig. 2. This gives the vertex and edge assignments of c_0 and c_1, respectively.

PROPOSITION 1. In the quadratic flow space, Example 1 decomposes as in Fig. 2.

Outline of proof. The irrotational component can have sources but no vortices; the solenoidal component can have vortices but no sources; the potential component can have neither. From the analysis of [1], the decomposition follows, the details as described above.

Thus, for Example 1, with $G = \tau(V, E, c_0, c_1)$, curl G is represented by the $\mathbf{u}^2$ component, curl G being zero on the other two components.

Let us consider another example, somewhat closer to networks themselves. This is analogous to the finite difference formulation *APX*1 in [1]. Virtual nodes are added to the original grid to permit differencing at each original grid point.

EXAMPLE 2. Consider the network shown in Fig. 3.

This graph $G(V, E, \sigma)$ came from a grid of 8 vertices and has been completed (under a left to right, bottom to top convention) by the addition of 7 virtual vertices. The flow space $U(V, E, \sigma)$ of functions is taken to be the assignment of a real value (representing capacity) to each of the edges, horizontal or vertical, as shown in Fig. 3. Imposing a local solenoidal condition corresponds to a net flow of zero at each vertex, from which one obtains two basis elements for $U(V, E, \sigma)$ corresponding to flows through the large and small cycles in the graph. This gives us the combined potential (which is solenoidal and irrotational) and solenoidal component $\mathbf{u}^1 + \mathbf{u}^2$ of a flow space element $\mathbf{u}$. The irrotational component $\mathbf{u}^3$ of $\mathbf{u}$ is then taken to be $\mathbf{u} - (\mathbf{u}^1 + \mathbf{u}^2)$. Curl G is represented by the $\mathbf{u}^2$ component of the field.

One may find useful the following interpretation of divergence-freeness of a network such as that in Fig. 3.

PROPOSITION 2. For the $APX1$ flow space $U(V, E, \sigma)$, where S is the vertex-edge incidence matrix of the graph G,

$$\text{div } G = 0 \text{ if and only if } Su = 0.$$

Outline of proof. Recall that $s_{ij} = +1, -1$, or 0 if the jth edge is directed away from, into, or neither, respectively, for the ith vertex. This is the same set-up, although incidence matrices were not used there, as used in the analysis of $APX1$ in [1], and the result follows.

The incidence matrix for Example 2 is shown in Fig. 4. The edges and vertices of the directed graph of Fig. 3 have been numbered from left to right, starting at the lowest level. The incidence matrix may be seen to be an equivalent formulation of the Eq. 3.1.1 of [1].

It would be interesting to have a matrix formulation for curl-freeness of a graph, similar to that for divergence-freeness given in the above proposition.

From Example 1 we concluded that the curl of G should have the geometrical meaning of the middle diagram in Fig. 2. From Example 2, where the geometrical meaning of the curl was less clear, we were led to cycles and incidence matrices to analyze div $G = 0$. Whatever graph context remains in Example 2 would appear to contain that needed to ascertain curl G there. What is left over? From Example 1 it would appear that all that remains in Example 2 is the fact that there are two cycles. The number of cycles is given by the "inner dual" of the graph, [3], defined as follows. For a plane graph G (a particular

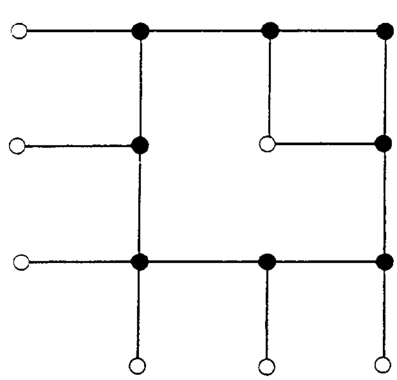


Fig. 3. A network example.

Edges

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Vertices
	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2
	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	3
	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	4
	-1	0	0	-1	1	0	1	0	0	0	0	0	0	0	0	0	5
	0	-1	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	6
	0	0	-1	0	0	-1	0	1	0	0	0	0	0	0	0	0	7
	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	8
	0	0	0	0	0	0	-1	0	-1	0	1	0	0	0	0	0	9
	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	10
	0	0	0	0	0	0	0	-1	0	-1	0	0	1	0	0	0	11
	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	12
	0	0	0	0	0	0	0	0	0	0	-1	0	0	-1	1	0	13
	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	-1	1	14
	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	-1	15

Fig. 4. Corresponding incidence matrix.

plane drawing of a planar graph), the inner dual G^* of G has the inner faces (the faces with finite area) of G as its points, and two points of G^* are adjacent whenever their cycles (faces) in G have a common edge.

Let us therefore consider two more examples, but now without regard to the flow space approach put forth above.

EXAMPLE 3. In graph G, independent of directions and flow spaces, there are 4 face cycles available. Let us take this as a coarse measure of curl in the graph. Its dual counts them but has no divergence. This may be taken as an instance of the relation div curl $G = 0$ which intimates (far from conclusively) that cycles, as a coarse measure of curl, are not inconsistent with a possible vector calculus of graphs.

EXAMPLE 4. Example 4 contrasts two graphs with the same number of faces (4) and a third with a "hole" in it. See Fig. 6. Intuitively, one feels that (b) should have more "intrinsic curl" than (a), and (c) more than twice that of (b).

The inner dual G^* of (a) is a path (P_4) which has no cycles, that of (b) is a quadrilateral cycle with four vertices (C_4), and (c) has as its inner dual the octagon cycle with eight vertices (C_8). There is one more (exterior) dual point for cases (a) and (b), and two more for case (c).

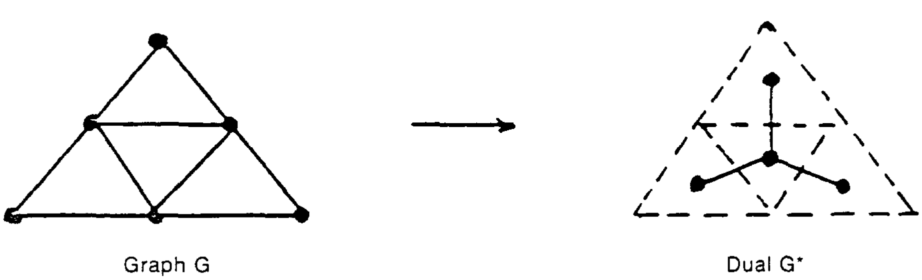
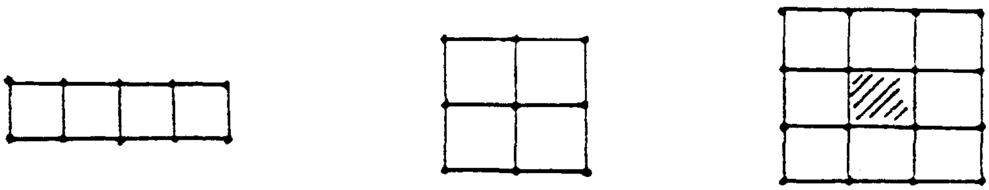


Fig. 5. Duality.



(a) (b) (c)

Fig. 6. Homology.

The first graph (a) is an outerplanar graph, sometimes called a "ladder" in the literature. Its co-cycles have cardinalities 2–5 in the sense that for any integer n between 2 and 5 inclusive, there is a set of n edges constituting a minimal disconnecting set for the graph. For (b) the cardinalities are 2–4 and for (c) they are 2–6.

From these two further examples we conclude that the dual of a graph can be associated with our notions of what the intrinsic curl of a graph or network should be, and that full (including exterior dual components) are needed. An argument for the inclusion of considerations of co-cycle cardinalities and cut sets is less clear.

The approach we have taken in the present study, to be elaborated in the next section, distinguishes cases (a), (b), and (c). For example, consider the quadratic $APX2$ scheme used above in Example 1 and Proposition 1 and found in [1] to have dimension

$$\begin{aligned} d &= \dim V_1 + \dim V_2 + \dim V_3 \\ &= 2\,(\text{number of interior vertices}) + (\text{number of interior midedges}) \\ &\quad + (\text{number of interior vertices} + \text{number of interior holes}). \end{aligned}$$

This formula was found for the case of zero boundary conditions and for a certain class of triangulated domains. Let us remain within that context, so to complete the triangulations of (a), (b), and (c), draw a diagonal from the upper left corner to the lower right corner of each square. We then arrive at the delineation (a) $d = 0 + 7 + 0 = 7$, (b) $d = 2 + 8 + 1 = 11$, and (c) $d = 0 + 17 + 1 = 18$.

3. A GRAPH DECOMPOSITION

Turning then to the approach of Sec. 2, we now obtain the network decomposition theorem.

THEOREM 1 (NETWORK DECOMPOSITION THEOREM). A directed graph $G = G(V, E, \sigma)$ or a network $G(V, E, c_0, c_1)$ may be decomposed as

$$G = G_1 \oplus G_2 \oplus G_3$$

where: G_1 is divergence-free and curl-free,
G_2 is divergence-free but not curl-free,
G_3 is curl-free but not divergence-free,

and then

$$\text{curl } G = \text{curl } G_2$$

Outline of proof. Establish any vector field over the graph domain and decompose it. See [1] for examples. For each vector field, the theorem holds.

We would like to insert a clarification here. All of the schemes *APXi* treated in [1] were divergence-free vector subspaces. This was because the application was to incompressible flow. They also satisfied a Dirichlet boundary condition (vanishing) on the boundary of the domain (triangulation) being treated. But any vector field, no matter whether divergence-free or not, and regardless of boundary condition, possesses the Helmholtz decomposition into potential, solenoidal, and irrotational parts. If the vector field is already divergence-free, so much the better, since the irrotational part is already absent, and to represent curl G one needs only to eliminate the potential component.

Clearly Theorem 1 allows too many solutions: each user with his vector field obtains a curl G, but they may not agree. Theorem 1 may be regarded as an example of an existence proof needing more information for a uniqueness proof.

As we saw in the previous section, it leads to some geometrical and representation insights, e.g., re $U(G_2)$ as the curl representation component. Moreover, the representation approach we have taken allows more freedom in distinguishing curls of different graphs.

To illustrate the latter point, consider the examples of the previous section. In Example 1, a quadratic flow space was imposed, but should a cubic flow space be deemed advisable, one could take into account further graph geometry, e.g., further subdivision of edges, departure from the rectangular orientation, in arriving at a curl G. More to the point, perhaps, after Examples 2 and 3 we suggested that a coarse measure of curl G is the number of cycles. But even the piecewise constant flow space of Example 2 contains more information than that, and delineates, for example, the case of a large cycle containing a small cycle, as in Example 2, from other graphical configurations containing two cycles. Moreover, as shown by Example 4, it is desirable to distinguish the relative orientations and juxtapositions of the cycles in the graph.

Let us consider one more example to further illustrate this point.

EXAMPLE 5. Consider the triangulated domain of Fig. 7 in which a center simplex is missing. We employ, for simplicity, a piecewise quadratic vector flow field over the (undirected) graph, such as *APX*2, previously considered. Under the requirement of a zero boundary condition, one finds that the dimension of this divergence-free subspace is 37. The $\mathbf{u}^3$ component contributed dimension equal to the number of interior vertices plus number of interior holes, $3 + 1 = 4$. This corresponds to the three small cycles and one large one in the inner dual of the graph of Fig. 7. If one chooses only the $\mathbf{u}^2$ component to represent curl G, one loses this contribution of "net curl zero."

The representation by vector field thus allows curl G to distinguish between types of cyclicity. As pointed out at the end of the last section, it connects to topological inter-

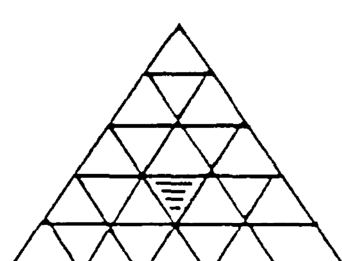


Fig. 7. Undirected triangulation graph.

pretations of curl G involving cohomology (Example 3) and homology (Example 4) of the graph or network. Moreover, the choice between lower and higher order vector fields with prescribed conformity (continuity) between their local representations allows alternatives in curl G.

4. FURTHER CONSIDERATIONS

4.1. *Vector calculus of graphs*

We have considered a very specific question, that of the conceptual investigation of the curl of a graph-theoretic structure. To do so, we were naturally led to the question of the decomposition of the structure into its curl-free and non-curl-free parts.

Evidently, one could go further, to attempt to develop a full graph structure theory properly encompassing all of the important notions of vector analysis and of the exterior differential calculus.

The view that we have taken is that this would be done more advantageously after, rather than before, further analysis of a specific question such as that of curl G.

4.2. *Curl and vorticity*

In fluid dynamics, the curl of the velocity field v is the important quantity, vorticity w. For graph structures, curl G should have vorticity properties such as:

(i) curl G is itself solenoidal in the graph theoretic sense of the latter, if one recalls Example 3, and
(ii) curl G should measure the limit of the "circulation" of the graph, in a graph theoretic sense, shrinking to the point at which one is measuring curl G.

4.3. *Representations and invariants*

From the various flow spaces allowed (see [1] and references therein), from desired invariance properties, and perhaps by using representations Ug for the relevant groups in the plane, one may be able to select preferred representations of the graphs and determine irreducibles for them. This would correspond to determining curl G by exponentiation.

For example, div v, curl v, and grad ϕ are invariant under the rigid motions, e.g., rotations and translations, in the plane. The same properties are desired for div G, curl G, and grad G, along with perhaps additional properties of finite subgroups related to the geometry and group structure of G itself.

4.4. *Graph theoretic comparisons*

A digraph $G(V, E, \sigma)$ is *eulerian* if and only if it is connected and, for each vertex, the outdegree and indegree are equal, and it may be thus interpreted as divergence-free; recall Example 2. It may be partitioned into cycles. In taking the latter as a measure of curl G, one is looking at a global curl G. To obtain local curl G, one may again resort to flow space representations, which, as noted earlier, correspond to putting local network quantities onto G. One could, in fact, allow the capacities c_0 and c_1 to be complex or group valued, if curl interpretations so justified.

4.5. *Simplical topology*

Interpretations of curl G in terms of homology and cohomology classes came into our analysis. While we would not be opposed to a "final outcome" involving them, we hold for the moment to our view expressed in Sec. 4.1 above. That is, at this point we wish to continue to assert the question of curl G from the physical and intuitive sides and to further expose the question, without anticipation of an algebraic solution which could foreclose other useful viewpoints and interpretations.

4.6. *A historical perspective*

The relationships between graph and network theory and applications are historically pervasive, all the way from the pioneering early work by Kirchhoff [4] and Maxwell [5] on electrical circuits to Kron's attempt [6] to model the Schrödinger Equation in terms of equivalent electrical networks. The solutions of the Dirichlet Problem and other differential equations by means of nets and circuits go back at least as far as Wiener and Weyl in the 1920s and already involved notions such as contravariant and covariant vectors, exterior forms, and the like, in increasing generality. But we are not aware of specific attention given to the vector properties of a graph structure *per se*, or to the use of fluid dynamic continuous or finite element methods to arrive at analogies sufficient to enable the determination of those vector properties of a graph.

Nonetheless, we have asked ourselves, for historical perspective, which past treatments to our knowledge bear comparison, and one which comes to mind is that of Eckmann [7]. There, the following generalization of a discrete version of the Dirichlet Problem was considered: given a function u on the vertices of a subgraph R of a graph K, determine u in K–R such that, on each vertex therein, u is the average of its values on all adjacent vertices. It was shown in [7] that there exists a unique solution to this "Dirichlet Problem" if and only if the subgraph R has at least one vertex in each component of K. Stated another way, the condition on R corresponds to the necessity of knowing the boundary data on the entire boundary in the classical Dirichlet Problem, and the condition on u in K–R corresponds to the classical mean value characterization of harmonic functions.

The proof in [7] depends on a unique representation of a chain with real coefficients as the sum of a harmonic part (a chain which is both a cycle and a co-cycle), a boundary, and a co-boundary. One could argue a conceptual parallel between that representation and the unique decompositions of vector fields used in the approach of the present article. On the other hand, certainly in [7], there was no attempt to define or discern vector properties, e.g., curl, of the graph or network K itself from, for example, functions defined on K. In [1], where we were concerned with the modelling of div $u = 0$, and in the following papers [8, 9] where, among others, we calculate Δu (the Laplacian of u) by the graph theoretic methods of [1] and apply it to Dirichlet and Stokes flow problems, we do not concern ourselves with the intrinsic vector properties of the graph, grid, or network itself. In other words, the perspective of the present paper is that of a vector calculus *of* (not *on*) graphs.

4.7. *An analogy*

To conclude, we wish to note an analogy with the celebrated Atiyah-Singer Index Theorem. In the Atiyah-Singer theory [10], one calculates the index of an elliptic operator on a manifold in terms of the Chern classes. In our theory one calculates the curl of a graph in terms of, in our interpretation of Sec. 4.5, certain topological invariants. In their theory, the elliptic partial differential operators are of second order and the index problem, by use of adjoints, is a null space problem. In our theory, the partial differential operator is a first order one and the representation approach of Sec. 3 enables a decomposition to null spaces for the curl.

The parallel with [10] is heightened when one recalls that Atiyah and Singer were really studying the first order Dirac operator. The distinction of the present study from [10] remains as stated in the previous section, the perspective here being that of a vector calculus of graphs (or, if one prefers, manifolds), not on them. This does not rule out a theory embracing to advantage both perspectives within a single structure.

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