Comp & Maths with Appls Vol 11 Nos 1-3 pp 281-282 1985
Printed in the U S A
0097-4943/85 $3 00 + 00
Pergamon Press Ltd

SIMPLIFIED ANALYSIS OF A HYPERBOLIC SYSTEM

RICHARD BELLMAN
Departments of Mathematics, Electrical Engineering and Medicine,
University of Southern California, Los Angeles, CA 90007, U S A

and

SEMYON M MEERKOV
Department of Electrical Engineering and Computer Science, The University of Michigan, Ann Arbor, MI 48109-1105, U S A

Communicated by Matthew Witten

Abstract—The method of generating equation is used in order to reduce a weakly nonlinear hyperbolic system to the standard form, i e the form which admits an asymptotic treatment based on the averaging principle

Consider a weakly nonlinear system defined by the following canonical form

$$u_{tt} - a^2 u_{xx} = \beta u + \varepsilon \Phi(t, x, u, u_t, u_x) \tag{1}$$

along with the boundary conditions

$$\begin{aligned} u(t, 0) &= u(t, l) = 0, \\ u(0, x) &= f(x), \\ u_t(0, x) &= F(x) \end{aligned}$$

Φ, f and F are assumed to be continuous with respect to all variables, a, β, and ε constants, $0 < \varepsilon \ll 1$

The solution of the above problem is sought in the form

$$u(t, x) = \sum_{n=1}^{\infty} v_n(t) \sin \frac{\pi n}{l} x$$

To satisfy (1), functions $v_n(t)$, $n = 1, 2, \ldots$, have to obey the following set of ordinary differential equations

$$\ddot{v}_n + \omega_n^2 v_n = \varepsilon \Phi_n(t, v_1, v_2, \ldots, \dot{v}_1, \dot{v}_2, \ldots), \qquad n = 1, 2, \ldots \tag{2}$$

with the initial conditions

$$v_n(0) = f_n, \qquad \dot{v}(0) = F_n, \qquad n = 1, 2, \ldots,$$

where f_n and F_n are the Fourier coefficients of $f(x)$ and $F(x)$ with respect to $\sin \frac{\pi n}{l} x$, $n = 1, 2, \ldots$

A number of asymptotic techniques can be employed for analysis of the solutions of (2) (see, for instance, Bellman[1]) The purpose of this note is to demonstrate how the averaging principle can be applied to the analysis of (2) Namely, we show that the method of generating equation, developed by Bellman, Bentsman and Meerkov[2], reduces (2) to the standard form, i e to a form which admits an asymptotic treatment based on the averaging principle

Consider

$$\begin{aligned} \dot{v}_{1n} &= v_{2n} \\ \dot{v}_{2n} &= -\omega_n^2 v_{1n} \end{aligned} \tag{3}$$

According to the method of generating equation, the general solution of (3) gives the following substitution for (2)

$$\begin{aligned} v_{1n} &= y_{1n} \sin(\omega_n t + y_{2n}), \\ v_{2n} &= \omega_n y_{1n} \cos(\omega_n t + y_{2n}), \\ n &= 1, 2, \end{aligned} \tag{4}$$

The change of the dependent variables (4) has been known for a long time However, this and the analogous substitutions has been introduced somewhat ad hoc The method of generating equation derives all these and a number of new substitutions in a formal, regular manner (see Bellman *et al* [2] for more examples)

In terms of variables $y_{1n}, y_{2n}, n = 1, 2, \quad$, (2) can be rewritten as

$$\begin{aligned} \dot{y}_{1n} = \varepsilon \frac{\cos(\omega_n t + y_{2n})}{\omega_n} \Phi_n[t, y_{11} \sin(\omega_1 t + y_{21}), y_{12} \sin(\omega_2 t + y_{22}), \quad , \\ \omega y_{11} \sin(\omega_1 t + y_{21}), \omega y_{12} \sin(\omega_2 t + y_{22}), \quad], \end{aligned} \tag{5}$$

$$\begin{aligned} \dot{y}_{2n} = -\varepsilon \frac{\sin(\omega_n t + y_{2n})}{y_{1n}\omega_n} \Phi_n[t, y_{11} \sin(\omega_1 t + y_{21}), y_{12} \sin(\omega_2 t + y_{22}), \quad , \\ \omega y_{11} \sin(\omega_1 t + y_2), \omega y_{12} \sin(\omega_2 t + y_{22}), \quad], \quad n = 1, 2, \end{aligned}$$

Equations (5) are in the standard form Applying the averaging principle, from (5) we obtain

$$\begin{aligned} \dot{z}_{1n} &= \varepsilon \underset{t}{M} \left\{ \frac{\cos(\omega_n t + z_{2n})}{\omega_n} \Phi_n \right\} \\ \dot{z}_{2n} &= -\varepsilon \underset{t}{M} \left\{ \frac{\sin(\omega_n t + z_{2n})}{z_{1n}\omega_n} \Phi_n \right\}, \end{aligned} \tag{6}$$

where

$$\underset{t}{M} \{S(\ , t)\} = \bar{S}(\ ) = \lim_{T\to\infty} \frac{1}{T} \int_0^T S(\ , t) \, dt$$

and the existence of the required averages has been assumed

In some instances, the structure of (6) is simpler than that of (5) (For example, sometimes (6) admits solution in a consecutive manner whereas (5) does not) When this is the case, the above procedure might substantially simplify the analysis of weakly nonlinear hyperbolic systems An example of such treatment can be found in Benney and Niell[3]

REFERENCES

1 R Bellman, *Perturbation Techniques in Mathematics, Physics and Engineering* Holt, Rinehart & Winston, New York (1966)

2 R Bellman, J Bentsman and S M Meerkov, Nonlinear systems with fast parametric oscillations *J Math Anal Appl* **97**, 572–589 (1983)

3 D J Benney and A M Niell, Apparent resonances of weakly nonlinear standing waves *J Math & Phys* **41**, 254–263 (1962)