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ON PARTIAL CONTACT OF A THIN-WALLED CIRCULAR CYLINDER AND A RIGID HALF-SPACE

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Summary—A long circular cylinder with radius R and thickness t transmits a longitudinal eccentrically applied force to a rigid half-space. We study the influence of the eccentricity of the load and the ratio R/t on the stress state of the cylinder using the theory of thin shells in bending. The elastic displacements are presented in the form of a Fourier expansion, the coefficients of which are found by the variational formulation of the problem combined with the penalty approach for a numerical implementation of contact conditions. Results are applied to certain problems of buckling of a circular bar and compared with previous approximate solutions.

NOMENCLATURE

c	parameter of the thickness, defined by equation (10)
e	eccentricity of the force
E	Young's modulus
$f(\mathbf{u}, \mathbf{v}, \mathbf{w})$	functional of energy of elastic deformation
$F(u, v, w)$	dimensionless energy of elastic deformation
I	moment of inertia
k	stiffness of the artificial spring
L	length of the cylinder
$\mathbf{M}_x$	bending moment
$\mathbf{M}_{x\beta}$	twisting moment
$\mathbf{N}_x$	membrane axial force
$\mathbf{N}_{x\beta}$	tangential force
$N_x, N_{x\beta}, M_x$	dimensionless axial, tangential forces and bending moment, respectively
P	force
Q_{ef}, T_{ef}	dimensionless effective (Kirchhoff's) shearing and tangential forces
$\mathbf{Q}_x$	shearing force
$Q_x, M_{x\beta}$	dimensionless shearing force and twisting moment
α	end rotation
R	radius of the cylinder
s	dimensionless parameter defined by (41)
t	thickness
$\mathbf{u}, \mathbf{v}, \mathbf{w}$	components of elastic displacements of the cylinder
u, v, w	dimensionless displacements
u_{rb}	translational rigid body displacement
$\mathbf{x}$	axial coordinate
x	dimensionless axial coordinate
β	circumferential angular coordinate
θ	extent (polar angle) of the region of contact
ν	Poissons ratio

INTRODUCTION

Consider a circular semi-infinite cylinder of radius R and thickness t, one end of which is at rest on a half-space (see Fig. 1). The cylinder transmits an axial force P applied with some eccentricity e to the half-space. The strength of materials solution (bending plus compression) applies if $e \leqslant R/2$, but otherwise predicts tensile tractions, indicating that separation will occur. We treat the case $e \geqslant R/2$ for which the solution of elementary strength of materials is not applicable.

A solution of the problem described enables us to deal with a set of related problems. Suppose, for example, that two cylinders with nominally conforming ends are pressed together. Suppose also that as a result of imperfectness, the axes of the cylinders are not quite correctly aligned, or one of the cylinders has a contact surface which is not quite perpendicular to its axis. The following question present itself: how does the area of contact develop with the growth of the force, and what is the pressure distribution?

As another example, consider a cylinder placed between two rigid half-spaces (Fig. 2a). The

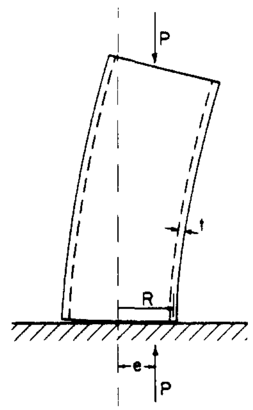

FIG. 1. Geometry of the problem.

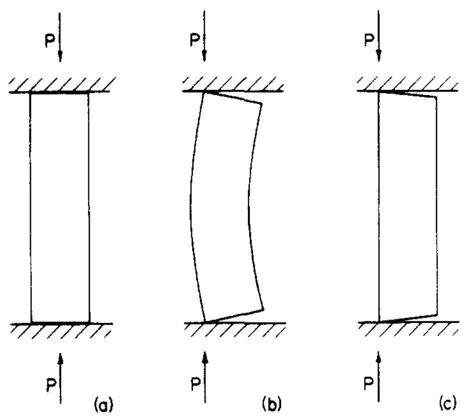

FIG. 2. Examples: (a) a cylinder placed between two rigid half-spaces; (b) one point scheme of buckling; (c) a cylinder with an imperfect face.

length L of the cylinder is sufficiently large for edge effects on the two ends to not interfere. The elementary solution predicts two extreme situations; (a) full contact between the surfaces if the force is smaller than some critical value; (b) one point contact (Fig. 2b) if the force is sufficiently large for instability to occur. The structure is sensitive to the imperfection shown in Fig. 2(c), and an accurate picture of the intermediate state is needed for an adequate evaluation of the critical load. The present paper provides results which can be used to deal with problems of that type.

We treat the cylinder using the theory of thin shells in bending. The elastic displacements are presented in the form of Fourier expansion, the unknown coefficients of which are found by the variational formulation of the problem, combined with the penalty approach for a numerical implementation of the contact conditions.

GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

If we introduce dimensionless quantities by

$$u = Et\mathbf{u}/P \tag{1}$$

$$v = Et\mathbf{v}/P \tag{2}$$

$$w = Et\mathbf{w}/P \tag{3}$$

$$x = \mathbf{x}/R \tag{4}$$

the dimensionless version of the governing equations [1] for a cylindrical semi-infinite shell in bending can be written.

$$u'' + (1-\nu)/2\ddot{u} + (1+\nu)/2\dot{v}' + w' + c^2[(1-\nu)/2\ddot{u} - w''' + (1-\nu)/2\ddot{w}'] = 0 \tag{5}$$

$$(1+\nu)/2u'^{\cdot} + v^{\cdot\cdot} + (1-\nu)/2v'' + w^{\cdot} + c^2[3(1-\nu)/2v'' - (3-\nu)/2w''^{\cdot}] = 0 \quad (6)$$

$$\nu u' + v^{\cdot} + w + c^2[(1-\nu)/2u'^{\cdot\cdot} - u''' - (3-\nu)/2v''^{\cdot} + w'''' + 2w''^{\cdot\cdot} + w^{\cdot\cdot\cdot\cdot} + 2w^{\cdot\cdot} + w] = 0. \quad (7)$$

In these equations **x**, β are the axial and angular coordinates, respectively, **u**, **v**, **w** are respectively axial, circumferential and radial displacements, ν is Poisson's ratio, and

$$(\ )' \equiv \partial/\partial x \quad (8)$$

$$(\ )^{\cdot} \equiv \partial/\partial \beta \quad (9)$$

$$c^2 = t^2/(12R^2). \quad (10)$$

The most interesting quantity—the dimensionless membrane force N_x is given by

$$N_x = \mathbf{N}_x R/P = (1-\nu^2)^{-1}[u' + \nu v^{\cdot} + \nu w - c^2 w'']. \quad (11)$$

The dimensionless bending moment, effective Kirchoff's shearing and tangential forces which are needed for formulation of the boundary conditions, are

$$M_x(x,\beta) = \mathbf{M}_x/P = c^2/(1-\nu^2)[w'' + \nu w^{\cdot\cdot} - u' - \nu v^{\cdot}] \quad (12)$$

$$Q_{ef} = Q_x + \partial M_{x\beta}/\partial\beta = R/P(\mathbf{Q}_x + 1/R\,\partial \mathbf{M}_{x\beta}/\partial\beta)$$
$$= c^2/(1-\nu^2)[w''' + (2-\nu)w'^{\cdot\cdot} - u'' - (3-\nu)/2v'^{\cdot} + (1-\nu)/2u^{\cdot\cdot}] \quad (13)$$

$$T_{ef} = N_{x\beta} - M_{x\beta} = R/P(\mathbf{N}_{x\beta} - \mathbf{M}_{x\beta}/R)$$
$$= 0.5/(1+\nu)[u^{\cdot} + (1+3c^2)v' - 3c^2 w'^{\cdot}]. \quad (14)$$

In these equations dimensional quantities are denoted by bold type.

The boundary conditions at $x = 0$ can be written as follows.

$$M_x(0,\beta) = 0 \quad \text{all } \beta \quad (15)$$

$$Q_{ef}(0,\beta) = 0 \quad \text{all } \beta \quad (16)$$

$$T_{ef}(0,\beta) = 0 \quad \text{all } \beta \quad (17)$$

$$u(0,\beta) = 0 \quad |\beta| \leqslant \theta \quad (18)$$

$$u(0,\beta) \geqslant 0 \quad |\beta| \geqslant \theta \quad (19)$$

$$N_x(0,\beta) = 0 \quad |\beta| \geqslant \theta \quad (20)$$

$$N_x(0,\beta) \leqslant 0 \quad |\beta| \leqslant \theta \quad (21)$$

$$\int_0^{2\pi} N_x(0,\beta)\,\mathrm{d}\beta = 1 \quad (22)$$

$$\int_0^{2\pi} N_x(0,\beta)\cos(\beta)\,\mathrm{d}\beta = M/PR = e/R, \quad (23)$$

where $M = Pe$ is the moment of the applied force about the diametral axis at the contact surface. The polar angle θ denotes the area of contact. Equation (18) and inequality (19) state that the surfaces conform in the area of contact, and there is no overlapping in the area out of contact. Equation (20) states that out of contact the edge is stress free; inequality (21) states that no tensile stresses are permitted in the region of contact. Finally, equations (22) and (23) are equations of equilibrium for the whole cylinder. This is a complete set of equations and inequalities that defines the solution to the problem, including the unknown area of contact in terms of the angle θ.

The solution depends on six parameters: ν, E, t, R, P, and M. We note, however, that all the dimensionless relationships (and hence the solution) depend on three non-dimensional parameters R/t, ν and e/R. Once the problem is solved, an average rotation and translation of the lower edge can be found in the form

$$u_{av} = g_0 + 0.5\,g_1\cos(\beta). \quad (24)$$

It is more convenient for computational purposes to invert the problem, namely to prescribe the angle of rotation $g_1/2R$ and find e/R from the solution.

SERIES REPRESENTATION

The displacements are sought in the form

$$u(x,\beta) = u_0(x) + u_1(x)\cos(\beta) + \sum_{m=2}^{\infty} A_m \cos(m\beta)\exp(s_m x) \quad (25)$$

$$v(x,\beta) = v_1(x)\sin(\beta) + \sum_{m=2}^{\infty} B_m \sin(m\beta)\exp(s_m x) \quad (26)$$

$$w(x,\beta) = w_0(x) + w_1(x)\cos(\beta) + \sum_{m=2}^{\infty} C_m \cos(m\beta)\exp(s_m x). \quad (27)$$

The terms with subscripts 0 and 1 have different (non-decaying) structure and correspond to rigid body displacements and the elementary solution of a pure compression and bending. After introducing equations (25) to (27) into (5) to (7) the factoring out $\cos(m\beta)\exp(s_m x)$ or $\sin(m\beta)\exp(s_m x)$, we obtain a system of three linear equations for each m ($m = 2, 3 \ldots$). Equating the determinant of this system to zero, we get the following equation

$$s^8 - 2(2m^2 - \nu)s^6 + [(1-\nu)/2 + 6m^2(m^2-1)]s^4 - 2m^2[2m^4 - (4-\nu)m^2 + (2-\nu)]s^2 + m^4(m^2-1)^2 = 0 \tag{28}$$

for finding complex roots $s_{jm} = r_{jm} + ip_{jm}$ ($j = 1$ to 8). Discarding roots with positive real parts which correspond to non-decaying solutions, we obtain the following expressions for the m-th term of the series ($m \geq 2$).

$$u_m(x,\beta) = \cos(m\beta)\{\exp(r_{1m}x)[A_{1m}\cos(p_{1m}x) + A_{2m}\sin(p_{1m}x)] + \exp(r_{2m}x)[A_{3m}\cos(p_{2m}x) + A_{4m}\sin(p_{2m}x)]\} \tag{29}$$

$$v_m(x,\beta) = \sin(m\beta)\{\exp(r_{1m}x)[B_{1m}\cos(p_{1m}x) + B_{2m}\sin(p_{1m}x)] + \exp(r_{2m}x)[B_{3m}\cos(p_{2m}x) + B_{4m}\sin(p_{2m}x)]\} \tag{30}$$

$$w_m(x,\beta) = \cos(m\beta)\{\exp(r_{1m}x)[C_{1m}\cos(p_{1m}x) + C_{2m}\sin(p_{1m}x)] + \exp(r_{2m}x)[C_{3m}\cos(p_{2m}x) + C_{4m}\sin(p_{2m}x)]\}. \tag{31}$$

Equations (25) to (31) can be found in [1], but they are repeated here in the interests of clarity.

Only four unknown coefficients (say, $C_{1m}, C_{2m}, C_{3m}, C_{4m}$) are independent. The others can be found in terms of C's from the system of three linear equations mentioned above. Moreover, since series (25) to (27) are orthogonal, the global equations (15) to (17) are uncoupled among m's. Therefore, these equations can be used to eliminate three of the four coefficients C_{jm}, leaving one unknown coefficient for each m. This can be done analytically, but the explicit expressions for the displacements are awkward and not especially useful. It is therefore more convenient to perform the operation numerically for any particular set of the three dimensionless parameters R/t, ν, e/R. In further considerations we assume that the procedure of elimination is already performed. The remaining series of coefficients has to be found from the contact boundary conditions (18) to (23). The unknown extent of the contact region θ introduces a non-linearity into this stage of the solution, which is treated by the variational formulation of the problem discussed in the following sections.

VARIATIONAL FORMULATION

If the cylinder were of finite length L, we could have stated the problem in terms of minimization of the functional [2, 3]:

$$f(\mathbf{u}, \mathbf{v}, \mathbf{w}) = V(\mathbf{u}, \mathbf{v}, \mathbf{w}) - P\mathbf{u}(L, \beta(P)) \tag{32}$$

on the kinematically admissible set of functions $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. In (32), $V(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is the energy of elastic deformation of the shell, $\mathbf{u}(L, \beta(P))$ is an axial displacement at the end $x = L$ and β corresponding to position of the force. The admissibility in particular requires that for any β

$$\mathbf{u}(0, \beta) \geq 0. \tag{33}$$

The functional (32) cannot be used directly in our case, because the energy of elastic deformation is infinite. This difficulty arises because of the non-decaying terms, corresponding to the uniform compression and bending of the cylinder which are hidden in (25) to (27) in terms with subscripts 0 and 1. It can be circumvented by finding these terms from other considerations and introducing them in (32) as known values. The form of (25) to (27) provides that these terms are orthogonal in the energy of elastic deformation with the others, and therefore, they have no influence on minimization of (32) and can be omitted. It seems at first sight that these terms should be present in (33), but their contribution can be absorbed in the rigid body translation in the axial direction and rotation. The most convenient way of satisfying the constraint (33) is to introduce the rigid body rotation of the lower end of the cylinder $g_1/(2R)$ as an input, and find e/R later when the problem is solved.

The functional (32) is more general than is required for our purposes. In particular, the variational statement described includes not only conditions (18) to (22), but also equations of equilibrium (5) to (7) and the natural boundary conditions (15) to (17) which have already been satisfied. After elimination of unnecessary terms, corresponding to (5) to (7) and (15) to (17), the functional of boundary conditions [4, 5] can be written

$$f(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \frac{1}{2}\int_0^{2\pi} \mathbf{u}(0,\beta)\mathbf{N}_x(0,\beta)R\,d\beta - P\mathbf{u}_{rb}, \tag{34}$$

where $\mathbf{u}_{rb}$ is the translational rigid body displacement of the cylinder. The dimensionless form of (34) is obtained by factoring out P^2/Et from (34), giving

$$F(u, v, w) = \frac{1}{2}\int_0^{2\pi} u(0,\beta)N_x(0,\beta)\,d\beta - u_{rb} \tag{35}$$

with the constraint

$$u(0, \beta) \geq 0, \tag{36}$$

where N_x is given by expression (11).

NUMERICAL IMPLEMENTATION

The problem of minimization of (35) with the constraint (36) can be replaced by the problem of unconstrained minimization, using the penalty approach [6–8]. We elected the exterior penalty method in which the functional (35) is replaced with

$$F(u, v, w) = \frac{1}{2}\int_0^{2\pi} u(0, \beta)N_x(0, \beta)\,d\beta - u_{rb} + \frac{1}{2}\int_0^{2\pi} ku(0, \beta)^2\,d\beta, \tag{37}$$

where k is equal to zero if $u \geqslant 0$, and is otherwise equal to a large number.

The form (37) has an obvious mechanical explanation. The last term in this functional can be viewed as a spring with a large stiffness k which is ineffective if there is separation of the surfaces, and effective if there is some overlapping. Thus, overlapping would be discouraged in the process of minimization.

In the software used (FORTRAN in MTS system) there are several subroutines available for the minimization of an unconstrained functional, based on [9–12]. It turned out, however, that for this problem a procedure in which (37) was minimized directly by giving small increments to all C's in turn, was more efficient. We adopted a strategy in which we started with a relatively rough increment. When after a few iterations this increment could not decrease the functional (37) any more, the increment was halved and the procedure was repeated several times until the minimum was reached with some prescribed accuracy.

Since the terms with $m = 0, 1$ were omitted from the elastic energy, we cannot obtain the corresponding derivatives needed for the stresses, directly from the series representation. However, they can be recovered from other considerations, thus the membrane force on the surface can be found using the artificial spring

$$N_x(0, \beta) = ku(0, \beta). \tag{38}$$

This is the most interesting stress in the problem. If necessary, the uniform compression and pure bending parts can be separated from it.

NUMERICAL RESULTS

The algorithm described was used to obtain results for R/t ranging from 5 to 50, e/R from 0.5 to 1 and $\nu = 0.3$. The relationship between the load eccentricity, e/R, and the extent of the area of contact is presented in Fig. 3.

An approximate solution to the problem can be obtained by 'unwrapping' the cylinder, to give the plane stress problem of periodic patch-like contact between an elastic body with a wavy surface and a rigid plane. This approximation is used for a related thermoelastic problem by Burton *et al.* [13]. It is of interest to know how the shell thickness ratio, R/t, affects the accuracy of this approximation—*i.e.* the extent to which shell bending influences the contact problem.

An elegant complex variable solution for the contact of a sinusoidally wavy surface and a plane was given by Westergaard [14]. The stress function, the contact pressure distribution and the extent of the periodic contact patches were all obtained in closed form. More recently, a series solution of the same problem was given by Dundurs *et al.* [15] who were apparently unaware of Westergaard's paper. Results from [14] are shown in Fig. 3, from which we see that the unwrapping approximation slightly overestimates the area of contact. The error increases with increasing R/t, as we should expect, since for thin shells, shell bending effects will be more important. The Hertzian approximation to the plane stress solution is also shown in Fig. 3 for comparison.

Shell bending has a more noticable effect on the flexibility of the shell as can be seen from Fig. 4 which shows the angle of rotation between the half-plane and the mean plane of the shell end surface, as a function of load eccentricity. In the following section we use these results to investigate the stability of certain contact problems for the circular cylinder.

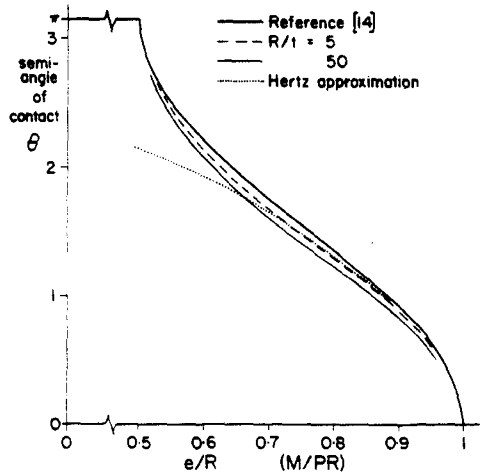

FIG. 3. Influence of the thickness parameter R/t on the relationship between the eccentricity of the load, e/R, and the extent of the area of contact.

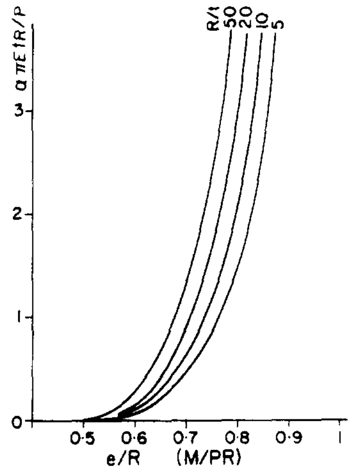

FIG. 4. Influence of the thickness parameter R/t on the relationship between the eccentricity of the load (e/R) and the rotation of the face $\alpha\pi EtR/P$.

EXAMPLES

We now return to the problem illustrated in Fig. 2. We first discuss the elementary solution and then show how this is modified when contact and shell bending are taken into account. Figure 2(b) shows the buckling-type deformation of a cylinder with initially conforming contact faces. This type of deflection is possible only if the force is greater than some critical value. The elementary theory ([16] p. 14) predicts that an equilibrium solution for this configuration gives an end rotation

$$\alpha = \frac{PRL}{2EI}\left|\frac{\tan(s)}{s}\right| \tag{39}$$

or

$$\alpha L/R = |2s\tan(s)|, \tag{40}$$

where

$$s = 0.5\sqrt{PL^2/EI} = 0.5L/R\sqrt{P/(E\pi Rt)}. \tag{41}$$

This solution has a branch in each of the ranges $0 < s < \pi/2$ and $\pi/2 < s < \pi$, which are shown as broken lines in Fig. 5. However, if the cylinder has initially conforming ends, only the upper branch is physically possible, since the other would involve interpenetration of the planes by the cylinder.

If there is an initial misalignment at the contact surfaces as shown in Fig. 2(c), the same equations (39) to (41) are applicable, but the cylinder will now follow the lower branch on loading, until the load is sufficiently large to produce

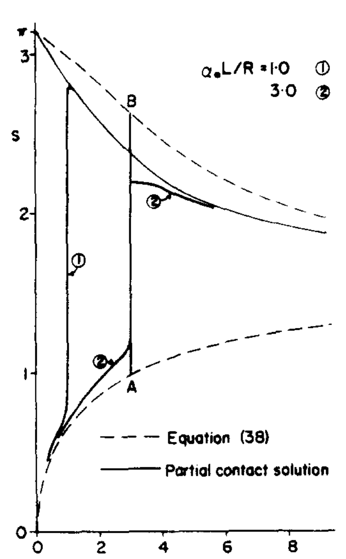

FIG. 5. Solution of the buckling problem of Fig. 2. The elementary strength of material solution is shown by broken lines, and the shell bending solution by solid lines.

a rotation at the ends which closes the gap—say at point A in Fig. 5. On further loading, the ends remain in complete contact and the loading curve follows the straight line AB in Fig. 5, until the upper branch is reached at B, when the system becomes unstable.

To obtain a more accurate solution we must take account of the fact that the force P is distributed over a finite area and hence the moment PR must be replaced by Pe. Equation (39) then becomes

$$\frac{2\alpha\pi EtR}{P} = \frac{e}{R}\frac{L}{R}\left|\frac{\tan s}{s}\right|. \tag{42}$$

The eccentricity ratio e/R can be eliminated numerically from (42), using the curves in Fig. 4 and we are left with a relation between the rotation α and the load which replaces (40) and the upper branch of which is shown as solid line in Fig. 5 for the parameters $L/R = 10$ and $R/t = 50$.

In the case of initial gap with the angle α_0, the governing equation (42) is unchanged, but e/R should be treated as a function of $|\alpha - \alpha_0|$ rather than of $|\alpha|$. Results for $\alpha_0 L/R = 1$ and $\alpha_0 L/R = 3$ are shown in Fig. 5. We note that the buckling loads are, respectively, 6 and 20% less than those given by the elementary theory.

CONCLUSION

It is easy to add further examples. In particular, the use of Figs 3 and 4 is not precluded for large displacements which might occur for a sufficiently long cylinder. The application of equations (5) to (7) is justified if the angle of relative rotation of the half-space and the face of the cylinder is small.

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