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LETTER TO THE EDITOR

Sir,

In a recent paper, Wilson and Wilson* extend the classical paper by Stokes to the problem of an inertialess beam on multiple supports traversed by a mass at constant speed. Their method of solution is to solve the problem of the multiple-span beam subjected to a single concentrated force (authors' equation ⟨6⟩†) and substitute into the equation of motion of the mass ⟨equation 4⟩ obtaining a differential equation ⟨15⟩ for the trajectory.

However, their solution for the beam deflection ⟨equation 6⟩ includes the $N-1$ unknown support reactions which are themselves functions of the load position z and which have to be found at each stage of the integration process by solving a system of simultaneous equations. This method is mathematically correct, but it is very inefficient. The whole procedure can be forestalled by recognizing that the influence of the inactive spans can be described by a modification to the boundary conditions at each end of the active span. For example, when the load is between supports i and $i+1$, that portion of the continuous beam to the left of i has the effect of making the slope at i proportional to the local bending moment, i.e.

$$(\mathrm{d}y/\mathrm{d}z)_i = -k_i M_i \tag{1}$$

where k_i is constant for all load positions between z_i and z_{i+1} and can be found by solving the elementary beam problem of the section $0 < z < z_i$ subjected to an end moment M_i. In the same way, the inactive spans to the right of z_{i+1} are described by the boundary condition

$$(\mathrm{d}y/\mathrm{d}z)_{i+1} = k_{i+1} M_{i+1}. \tag{2}$$

Once the constants k_i, k_{i+1} have been found, we can find explicitly the local displacement due to a force Q on the active span, which in dimensionless form is given by

$$Y = F(Z)\,Q/m_0 g \tag{3}$$

where

$$F(Z) = 2Z^2(1-Z)^2\left(-1 + \frac{3(1-Z+Z^2) + K_i(1+Z)^2 + K_{i+1}(2-Z)^2}{(4(K_i+1)(K_{i+1}+1)-1)}\right) \tag{4}$$

$$K_i = 3k_i EI/l;\; K_{i+1} = 3k_{i+1} EI/l. \tag{5}$$

In these equations, the dimensionless variables are related to the length of the *active* span—here denoted by l—and the dimensionless coordinate Z is measured from the left hand support of the active span.

Notice that equation (4) replaces equation ⟨16⟩ in a form which shows explicitly the dependence of the dimensionless support reactions S_i on Z and is a polynomial of the sixth degree. The solution is then obtained from the differential equation ⟨15⟩, using equation (4), with initial conditions ⟨equation 17⟩.

In many cases, the maximum bending moment will be more important than the beam deflection. This must occur either under the load (M_0) or at one of the supports. The appropriate expressions are

$$M_i/M_{ss} = \frac{4QZ(1-Z)\{3(1-Z) + 2K_{i+1}(2-Z)\}}{m_0 g\{4(K_i+1)(K_{i+1}+1)-1\}} \tag{6}$$

$$M_{i+1}/M_{ss} = \frac{4QZ(1-Z)\{3Z + 2K_i(1+Z)\}}{m_0 g\{4(K_i+1)(K_{i+1}+1)-1\}} \tag{7}$$

$$M_0/M_{ss} = -4QZ(1-Z)/m_0 g + ZM_{i+1}/M_{ss} + (1-Z)M_i/M_{ss}, \tag{8}$$

where

$$Q/m_0 g = Y/8F(Z). \tag{9}$$

The approach described in this note has two major advantages. It speeds up the process of integration, by eliminating the need for solving the system of equations for S_i and it enables all possible beam configurations to be described in terms of only two dimensionless parameters K_i, K_{i+1}.

Furthermore, upper and lower bounds to these parameters can be established in terms of the length of the adjacent inactive span by imposing limiting conditions at the next support. For example, if the

* J. F. Wilson and D. M. Wilson, Responses of continuous inertialess beams to traversing mass—a generalisation of Stokes' problem. *Int. J. Mech. Sci.* **26**, 105–112 (1984).

† Throughout this note, references to the authors' equations will be made in the form ⟨equation 1⟩.

beam were built in at z_{i-1}, and the length of the $(i-1)$th span were fl, we should have $K_i = 0.75f$.

The other extreme value, obtained by assuming a simple support at z_{i-1} is $K_i = f$. All practical cases must lie between these extremes and hence we conclude that

$$0.75 < K_i/f < 1. \tag{10}$$

(Unless the active span is the first or the last span, in which case the appropriate value of K tends to infinity and equation (4) reduces to a simpler form.)

A numerical solution of equation ⟨15⟩ shows that a variation in K_i within such a restricted range only changes the maximum deflection by about 5%. This indicates that for most practical purposes attention can be restricted to the active span and the two adjacent spans. For more accurate calculations—as for example in an optimization routine—we conclude that convergence will be rapid because the effects of span dimensions are strongly localised.

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J. R. Barber

AUTHORS' REPLY

Sir,

(1) Professor Barber's method of solution, while computationally efficient, is nonetheless approximate.

(2) Our exact method of solving for the reaction forces at every time step is accomplished quite efficiently by our algorithm. Our program, including the numerical solution of the governing differential equation, took only 5 s of CPU time on an IBM 370/165 computer to calculate the following quantities for a six-span case: time histories at several selected span locations of both the deflection and bending moment; the complete trajectory followed by the moving mass; and the span slope and curvature at selected critical points. Thus, with the advent of high speed computation, there may be less need to make approximations of the type suggested by Professor Barber.

(3) The 5% deviation from our solutions of the approximate solution maxima for deflection is believable since the beam system is inertialess. Our intuition based on previous experience with solutions of multiple spans with static point loads serves us well. Of course, the small magnitude of the error incurred by Professor Barber's three-span approximation (using the active and two surrounding spans only) could not have been predicted *a priori*.

(4) It appears that, while the proposed approximations are appropriate for static and pseudostatic (inertialess) span systems, such approximations should *never* be applied to multiple-span dynamic analyses if span inertia is included. With span inertia, the deflection maxima build up considerably due to wave reinforcement as one successively compares dynamic span response deflections to a point load, proceeding from 3 through 6 spans (see [7] of our paper).

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