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ON $\mathbf{K}^{\Delta}$

Sung-Jin CHUNG

Department of Industrial Engineering, Seoul National University, South Korea

Katta G. MURTY*

Department of Industrial & Operations Engineering, The University of Michigan, Ann Arbor, MI 48109, USA

Soo Y. CHANG

Department of Industrial & Operations Engineering, The University of Michigan, Ann Arbor, MI 48109, USA

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Let $\mathbf{K}$ be an unbounded convex polyhedral subset of $\mathbf{R}^n$ represented by a system of linear constraints, and let $\mathbf{K}^{\Delta}$ be the convex hull of the set of extreme points of $\mathbf{K}$. We show that the combinatorial-facial structure of $\mathbf{K}$ does not uniquely determine the combinatorial-facial structure of $\mathbf{K}^{\Delta}$. We prove that the problem of checking whether two given extreme points of $\mathbf{K}$ are nonadjacent on $\mathbf{K}^{\Delta}$, is NP-complete in the strong sense. We show that the problem of deriving a linear constraint representation of $\mathbf{K}^{\Delta}$, leads to the question of checking whether the dimension of $\mathbf{K}^{\Delta}$ is the same as that of $\mathbf{K}$, and we prove that resolving this question is hard because it needs the solution of some NP-complete problems. Finally we provide a formula for the dimension of $\mathbf{K}^{\Delta}$, under a nondegeneracy assumption.

Keywords. Unbounded convex polyhedron, convex hull of extreme points, facial structure, NP-complete and NP-hard problems, dimension, Hamiltonian chains, maximum capacity cuts.

1. Introduction

Let $\mathbf{K}\subset\mathbf{R}^n$ be the set of feasible solutions of a finite system of linear constraints in $x\in\mathbf{R}^n$ with integer data. Assume that $\mathbf{K}\neq\emptyset$ and that $\mathbf{K}$ is unbounded. Let $\mathbf{K}^{\Delta}$ be the convex hull of extreme points of $\mathbf{K}$. The study of the structure of $\mathbf{K}^{\Delta}$ is a very important problem in mathematical programming. Some of the known important results associated with $\mathbf{K}^{\Delta}$ are listed below.

(1) A fundamental theorem in the theory of convex polyhedra, the *resolution theorem* (see [4, 7]), with important algorithmic consequences, states that if $\mathbf{K}$ is the set of feasible solutions of

$$\begin{aligned} &Ax=b,\\ &x\geq 0, \end{aligned} \tag{1}$$

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then $\mathbf{K}$ can be expressed as $\mathbf{K}^{\Delta}+\mathbf{K}^{<}=\{x+y\colon\ x\in\mathbf{K}^{\Delta},\ y\in\mathbf{K}^{<}\}$, where $\mathbf{K}^{<}=\{y\colon Ay=0,\ y\geq 0\}$.

(2) Let $z(x)=cx$, where c is an integer row vector in $\mathbf{R}^n$, be a linear function which is unbounded below on $\mathbf{K}$. The problem of minimizing $z(x)$ on $\mathbf{K}^{\Delta}$, more specifically, that of finding an extreme point of $\mathbf{K}$ that minimizes $z(x)$ over the set of extreme points of $\mathbf{K}$, is an NP-hard problem, see [3]. Several NP-hard problems are special cases of it, in a direct way. We mention some of these problems below.

(a) *The shortest hamiltonian chain problem*

Let $G_1=(\mathcal{N},\mathcal{A})$ be a complete directed network with $\mathcal{N}=\{1,\dots,n\}$, $\mathcal{A}=\{(i,j)\colon i,\ j=1 \text{ to } n,\ i\neq j\}$. Let $d=(d_{ij})>0$ denote the given positive integer vector of arc lengths in $\mathcal{A}$.

The nodes 1, n are the specified origin and destination nodes in G_1. A hamiltonian chain from 1 to n in G_1 is a simple chain from 1 to n in G_1 that passes through each of the other nodes in G_1. The problem is to determine the shortest hamiltonian chain from 1 to n in G_1 with d as the vector of arc lengths. Define the incidence vector of a simple chain $\mathcal{C}$ from 1 to n in G_1 to be the vector $x=(x_{ij})$, where

$$x_{ij}=1,\quad \text{if } (i,j) \text{ is on } \mathcal{C},$$
$$=0,\quad \text{otherwise.}$$

Then the vector x satisfies

$$\sum_{\substack{j=1\\ j\neq i}}^{n} x_{ij}-\sum_{\substack{j=1\\ j\neq i}}^{n} x_{ji}=\begin{cases}0, & \text{if } i\neq 1,\ n,\\ 1, & \text{if } i=1,\\ -1, & \text{if } i=n,\end{cases} \tag{2}$$

$$x_{ij}\geq 0,\quad \text{for all } i,\ j.$$

It can be shown that every basic feasible solution (BFS) of (2) (an extreme point of the set of feasible solutions of (2)) is the incidence vector of a simple chain from 1 to n in G_1, and vice versa.

Let $\alpha=2(1+\text{maximum }\{d_{ij}\colon (i,j)\in\mathcal{A}\})$. For each $(i,j)\in\mathcal{A}$, define $d'_{ij}=\alpha-d_{ij}$. From the definition of α, we have $d'_{ij}>0$ for all $(i,j)\in\mathcal{A}$ and for any $i,\ j,\ k\in\mathcal{N}$, $d'_{ik}+d'_{kj}>d'_{ij}$. So, the vector $d'=(d'_{ij})$ satisfies the triangle inequality. The objective function $l(x)=\sum(-d'_{ij}x_{ij}\colon\ i,\ j=1 \text{ to } n,\ i\neq j)$, can be verified to be unbounded below on the set of feasible solutions of (2). But the problem of finding the BFS of (2) that minimizes $l(x)$ among all the BFSs of (2) (a special case of minimizing $z(x)$ on $\mathbf{K}^{\Delta}$) is equivalent to the shortest hamiltonian chain problem, since $d'=(d'_{ij})>0$, and these distances satisfy the triangle inequality. See [6].

(b) *The problem of finding a maximum capacity cut separating the source and sink nodes*

Let $G=(\mathcal{N},\mathcal{A})$ be a directed single commodity flow network in which $\mathcal{N}$ is the set of nodes, and $\mathcal{A}$ is the set of arcs. Suppose nodes 1, n are the specified source and sink nodes in G. Let $k=(k_{ij}:(i,j)\in\mathcal{A})$ be a positive integer arc capacity vector associated with the arcs in $\mathcal{A}$. A cut in G separating 1 and n is a set of arcs $\{(i,j): i\in\mathbf{X}, j\in\bar{\mathbf{X}} \text{ and } (i,j)\in\mathcal{A}\}$ where $(\mathbf{X},\bar{\mathbf{X}})$ is a partition of the node set $\mathcal{N}$ with $1\in\mathbf{X}$, $n\in\bar{\mathbf{X}}$, and this cut itself is denoted by the symbol $(\mathbf{X},\bar{\mathbf{X}})$. The capacity of this cut $(\mathbf{X},\bar{\mathbf{X}})$ is defined to be $\sum(k_{ij}:(i,j)\in(\mathbf{X},\bar{\mathbf{X}}))$.

The well-known problem of finding a minimum capacity cut separating 1 and n can be solved efficiently by using any efficient algorithm for finding a maximum value flow from 1 to n in G and then using the max-flow min-cut theorem. On the other hand, the problem of finding a maximum capacity cut is an NP-hard problem. Define the vectors of variables $\Pi=(\Pi_i: i\in\mathcal{N})$, $\gamma=(\gamma_{ij}:(i,j)\in\mathcal{A})$. Define the constraint system

$$
\begin{aligned}
&-\Pi_1+\Pi_n=1,\\
&\Pi_i-\Pi_j+\gamma_{ij}\geq 0, \quad \text{for all } (i,j)\in\mathcal{A},\\
&\Pi_1=0,\\
&\gamma_{ij}\geq 0, \quad \text{for all } (i,j)\in\mathcal{A}.
\end{aligned}
\tag{3}
$$

If $(\mathbf{X},\bar{\mathbf{X}})$ is any cut separating 1 and n in G, define the corresponding (Π,γ) by

$$
\begin{aligned}
\Pi_i&=\begin{cases}0, & \text{if } i\in\mathbf{X},\\ 1, & \text{if } i\in\bar{\mathbf{X}},\end{cases}\\
\gamma_{ij}&=\begin{cases}1, & \text{if } (i,j)\in(\mathbf{X},\bar{\mathbf{X}}),\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}
\tag{4}
$$

It can be verified that if $(\mathbf{X},\bar{\mathbf{X}})$ is any cut separating 1 and n in G, then the corresponding (Π,γ) defined by (4) is a BFS of (3); and conversely, if (Π,γ) is any BFS of (3), define $\mathbf{X}=\{i:\Pi_i=0\}$, $\bar{\mathbf{X}}=\{i:\Pi_i=1\}$, then $(\mathbf{X},\bar{\mathbf{X}})$ is a cut separating 1 and n in G. Thus every cut separating 1 and n in G corresponds to a BFS of (3) and vice versa. The objective function $v(\Pi,\gamma)=\sum(k_{ij}\gamma_{ij}:(i,j)\in\mathcal{A})$ is clearly unbounded above on the set of feasible solutions of (3), but the problem of finding a maximum capacity cut separating 1 and n in G is equivalent to the problem of finding a BFS that maximizes $v(\Pi,\gamma)$ among all BFSs of (3), this again is a special case of the problem of minimizing $z(x)$ on $\mathbf{K}^{\Delta}$.

(c) *The separation problem*

Consider the original sets $\mathbf{K}$, $\mathbf{K}^{\Delta}$ again. Suppose we are given a point $x\in\mathbf{K}$ and are required to determine whether

either $x\in\mathbf{K}^{\Delta}$

or else determine a hyperplane in $\mathbf{R}^n$ separating x and $\mathbf{K}^{\Delta}$.

No efficient algorithm for this problem is known. If an efficient algorithm for this problem can be developed, by combining it with the ellipsoid method, we can generate an efficient algorithm for minimizing $z(x)$ on $\mathbf{K}^{\Delta}$, see [5].

2. Results on the combinatorial-facial structure of $\mathbf{K}^{\Delta}$

The combinatorial structure of $\mathbf{K}$ is determined by the incidence relationships of its faces. Here we investigate whether the combinatorial structure of $\mathbf{K}^{\Delta}$ can be deduced purely from the combinatorial structure of $\mathbf{K}$. While the combinatorial structure of $\mathbf{K}$ has an effect on the combinatorial structure of $\mathbf{K}^{\Delta}$, it turns out that it does not determine it completely.

Theorem 1. *Let x^1, x^2 be two extreme points of $\mathbf{K}$. x^1, x^2 are adjacent on $\mathbf{K}^{\Delta}$ if either* (i) *x^1, x^2 are adjacent on $\mathbf{K}$,*
or (ii) *x^1, x^2 are the two extreme points incident to the two unbounded edges in an unbounded two-dimensional face of $\mathbf{K}$.*

Proof. By definition, two extreme points of a convex polyhedron are not adjacent on it iff their midpoint can be expressed as a convex combination of two distinct points in the polyhedron, neither of which is on the line segment joining the two extreme points, see [7]. Since $\mathbf{K}^{\Delta} \subset \mathbf{K}$, this definition directly implies that if x^1, x^2 are adjacent extreme points of $\mathbf{K}$, then x^1, x^2 are also adjacent on $\mathbf{K}^{\Delta}$.

Now, suppose that x^1, x^2 are extreme points which are not adjacent on $\mathbf{K}$, but they are both extreme points on a two-dimensional face, $\mathbf{F}$, of $\mathbf{K}$, incident to the two unbounded edges on the face. See Fig. 1.

The line segment joining x^1 and x^2 partitions this face $\mathbf{F}$ into two regions $\mathbf{F}^{\Delta}$ and $\mathbf{P}$. Let $\bar{x} = \frac{1}{2}(x^1 + x^2)$. Since $\mathbf{F}$ is a two-dimensional face of $\mathbf{K}$, if $\bar{x} = \alpha x^3 + (1-\alpha)x^4$, where $0<\alpha<1$; and x^3, x^4 are points in $\mathbf{K}$ not contained on the line segment joining x^1, x^2; one of the points among x^3, x^4, say x^3, is in $\mathbf{F}^{\Delta}$, and the other point x^4 must be in $\mathbf{P}$. So $x^4 \notin \mathbf{K}^{\Delta}$, and these facts imply that $\bar{x}$ cannot be expressed as a convex combination of two points in $\mathbf{K}^{\Delta}$ both of which are not contained on the line segment joining x^1, x^2. So, in this case, x^1, x^2 are also adjacent on $\mathbf{K}^{\Delta}$. □

The converse of Theorem 1 may not be true, as the following example illustrates. See Figs. 2 and 3.

Note that $\mathbf{K}_1$, $\mathbf{K}_2$ have the same combinatorial structure; and yet the combinatorial structure of $\mathbf{K}_1^{\Delta}$ and $\mathbf{K}_2^{\Delta}$ is different. This shows that the combinatorial structure of $\mathbf{K}^{\Delta}$ not only depends on the combinatorial structure of $\mathbf{K}$, but on the actual data in the linear constraints defining $\mathbf{K}$.

The points x^1, x^2 in Fig. 3 are not contained together on any two-dimensional face of $\mathbf{K}_2$, and yet they are adjacent on $\mathbf{K}_2^{\Delta}$, providing a counterexample to the converse of Theorem 1. We now provide some additional results on this problem.

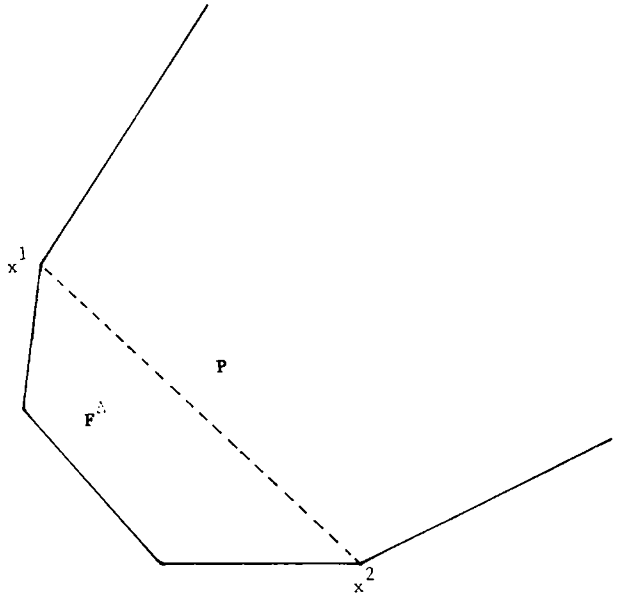

Fig. 1.

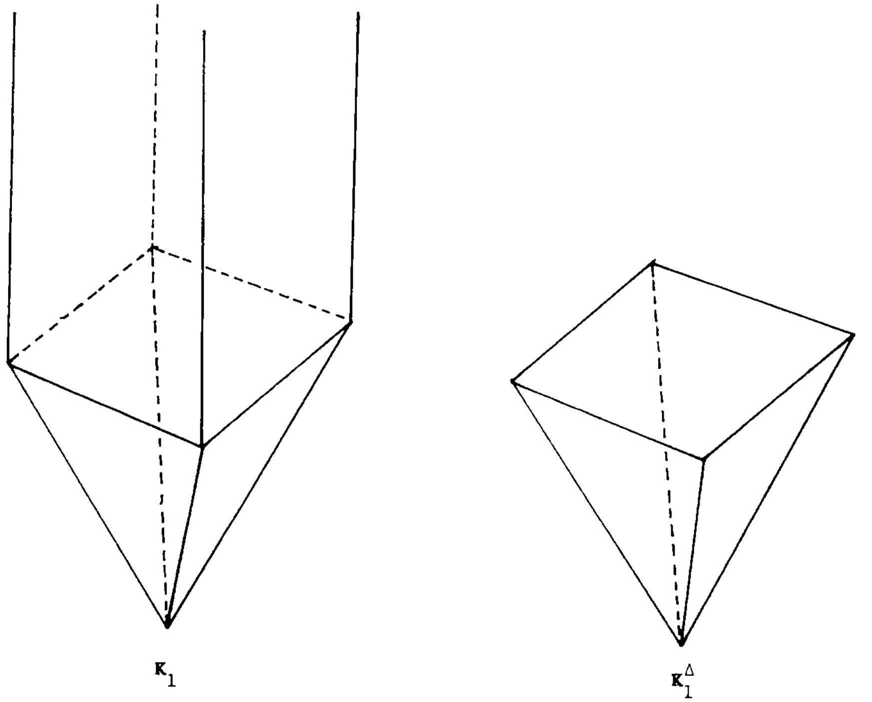

Fig. 2.

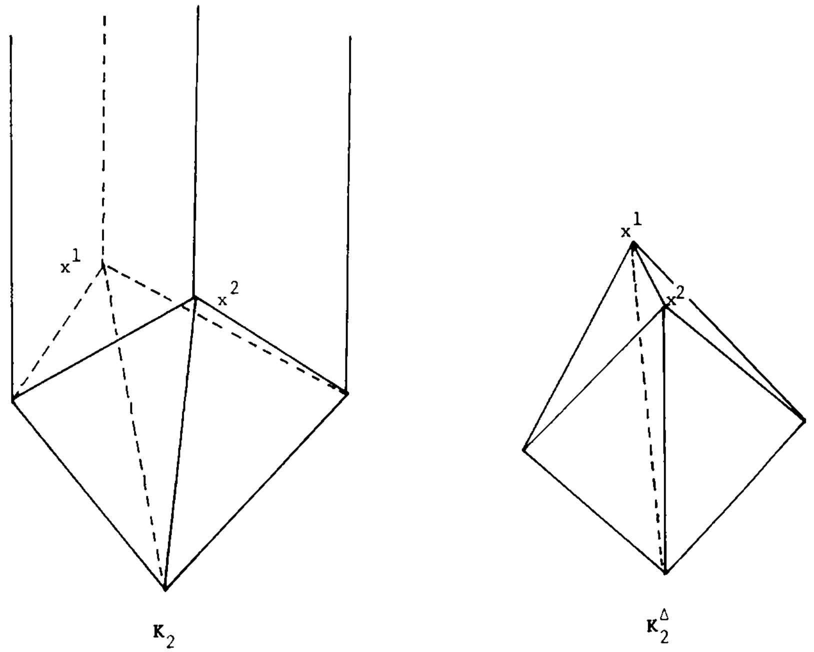

Fig. 3.

Let Γ denote the set of feasible solutions of (2), and let Γ^{Δ} denote the convex hull of all extreme points of Γ. So, Γ^{Δ} is the convex hull of the incidence vectors of simple chains from 1 to n in $G_1=(\mathcal{N},\mathcal{A})$, where $\mathcal{N}=\{1,\dots,n\}$, $\mathcal{A}=\{(i,j)\colon i, j=1$ to n, $i\neq j\}$. Let the symbol τ denote the incidence vector of a hamiltonian chain in G_1 from 1 to n, or that chain itself. For any chain $\mathcal{C}$ in G_1, let the symbol $|\mathcal{C}|$ denote the linear function which is the number of arcs in $\mathcal{C}$. Then τ is any simple chain from 1 to n in G_1 satisfying $|\tau|=n-1$, and vice versa. Let $\mathbf{T}$ denote the convex hull of all the incidence vectors of hamiltonian chains from 1 to n in G_1. So $\mathbf{T}\subset\Gamma^{\Delta}$.

Lemma 1. τ^1, τ^2, *the incidence vectors of two distinct hamiltonian chains from* 1 *to* n *in* G_1, *are nonadjacent on* $\mathbf{T}$ *iff they are nonadjacent on* Γ^{Δ}.

Proof. Since $\mathbf{T}\subset\Gamma^{\Delta}$, if τ^1, τ^2 are nonadjacent on $\mathbf{T}$, they are nonadjacent on Γ^{Δ}, by the definition of adjacency (see [7, 8]).

Suppose τ^1, τ^2 are adjacent on $\mathbf{T}$ but not on Γ^{Δ}. We will now show that this leads to a contradiction. By definition of nonadjacency (see [7, 8]), since Γ^{Δ} is a convex polytope, this implies that there exist simple chains from 1 to n in G_1 with incidence vectors $x^1,\dots,x^r$ and real numbers α_1, α_2, $\beta_1,\dots,\beta_r$, such that

$$\alpha_1 \tau^1 + \alpha_2 \tau^2 = \sum_{t=1}^{r} \beta_t x^t,$$

$$\alpha_1 + \alpha_2 = \sum_{t=1}^{r} \beta_t = 1, \tag{5}$$

$$\alpha_1, \alpha_2, \beta_t > 0, \quad \text{for } t = 1 \text{ to } r.$$

Since $|\tau^1| = |\tau^2| = n-1$, we have $\alpha_1|\tau^1| + \alpha_2|\tau^2| = n-1$, so from the above $\sum_{t=1}^{r} \beta_t |x^t| = (n-1)$, too. Since each x^t is the incidence vector of a simple chain in G_1, we have $|x^t| \le n-1$ for all $t=1$ to r. So, $\sum_{t=1}^{r} \beta_t |x^t| = n-1$ implies that $|x^t| = n-1$ for all $t=1$ to r, that is, each x^t is the incidence vector of a hamiltonian chain from 1 to n in G_1. From (5), this is a contradiction to the hypothesis that τ^1 and τ^2 are adjacent on $\mathbf{T}$. This proves the lemma. □

Theorem 2. *The problem of checking whether two given extreme points of* $\mathbf{K}$, *are nonadjacent on* $\mathbf{K}^{\Delta}$, *is NP-complete in the strong sense.*

Proof. The fact that the problem of checking whether two given extreme points of $\mathbf{K}$ are nonadjacent on $\mathbf{K}^{\Delta}$, is in the class NP of problems, follows from the definition of nonadjacency, see [7, 8].

Now consider the special case of this problem on Γ. By Lemma 1, the problem of checking whether the incidence vectors of two hamiltonian chains from 1 to n in G_1 are nonadjacent on Γ^{Δ} is equivalent to checking whether they are nonadjacent on $\mathbf{T}$. However, from the results in [9], the problem of checking whether the incidence vectors of two hamiltonian chains are nonadjacent on $\mathbf{T}$ is strongly NP-complete. This clearly implies that the problem of checking whether two extreme points of Γ are nonadjacent on Γ^{Δ} is strongly NP-complete. This proves the theorem. □

Corollary 1. *Let* $G_2 = (\mathcal{N}, \mathcal{A})$ *be a directed network. Let* 1, n *be an origin, destination pair of nodes in* G_2. *Let* $\mathbf{S}$ *denote the convex hull of the arc-incidence vectors of all simple chains from* 1 *to* n *in* G. *Then the problem of checking whether the arc-incidence vectors* x^1, x^2 *of two distinct simple chains from* 1 *to* n *in* G, *are nonadjacent on* $\mathbf{S}$, *is NP-complete in the strong sense.*

Proof. This follows from the arguments used in the proofs of Lemma 1 and Theorem 2. □

Recently, there has been a lot of interest in studying linear programming through combinatorial abstraction (see [1]). These examples show that it may be difficult to derive any results about $\mathbf{K}^{\Delta}$, or carry out algorithms for optimization over $\mathbf{K}^{\Delta}$, using a combinatorial abstraction of $\mathbf{K}$.

Finally, the problem of designing a reasonable scheme for generating a system of linear constraints, which when combined with those in the constraint system defining $\mathbf{K}$ leads to $\mathbf{K}^{\Delta}$, using the data in the constraint system defining $\mathbf{K}$, remains an open problem. We investigate one question related to this open problem, in the next section.

3. On the dimension of $\mathbf{K}^{\Delta}$

Here we study the problem of determining the dimension of $\mathbf{K}^{\Delta}$, using the data in the constraint system specifying $\mathbf{K}$. If the constraint system specifying $\mathbf{K}$ consists of equality constraints only, $\mathbf{K}$ is an affine space and has no extreme points, in this case $\mathbf{K}^{\Delta}=\emptyset$, and the problem is trivial. So we assume that the constraint system specifying $\mathbf{K}$ consists of at least one inequality constraint. If there are any equality constraints in the system, using them some variables can be eliminated, thus reducing the system. So, for the sake of this study, we can, without any loss of generality, assume that the system specifying $\mathbf{K}$ consists of inequality constraints only. Suppose that $\mathbf{K}$ is the set of feasible solutions of

$$Dx \geq d \tag{6}$$

where D, d are given integer matrices of orders $m \times n$ and $m \times 1$ respectively. As before, we assume that $\mathbf{K} \neq \emptyset$, and that $\mathbf{K}$ is unbounded.

We use the symbols $D_{i.}$, $D_{.j}$, to denote the ith row vector, the jth column vector of the matrix D respectively.

A constraint in (6) which holds as an equation at every feasible solution for (6) is called a *binding inequality constraint* in the system (6). It is well known that the dimension of $\mathbf{K}$ is $<n$ iff there exists at least one binding inequality constraint in (6), that is, iff there exists an i such that $D_{i.}x=d_i$ for all $x \in \mathbf{K}$. For any i, $1 \leq i \leq m$, whether the ith constraint in (6) is a binding inequality constraint or not, can be determined by solving the linear program

$$\begin{aligned} &\text{maximize } D_{i.}x \\ &\text{subject to } D_{t.}x \geq d_t, \quad t=1 \text{ to } m,\ t \neq i. \end{aligned} \tag{7}$$

Since we assumed that $\mathbf{K} \neq \emptyset$, the optimum objective value in this linear program is $\geq d_i$. If the optimum objective value in (7) is $>d_i$, the ith constraint is not a binding inequality constraint in (6); if it is $=d_i$, then the ith constraint is a binding inequality constraint in (6). Thus to check whether the dimension of $\mathbf{K}$ is n or $<n$, requires the solution of at most m linear programming problems. In fact, the dimension of $\mathbf{K}$ is $n-$rank of $\{D_{i.}: i$ such that the ith constraint in (6) is a binding inequality constraint$\}$. Thus computing the dimension of $\mathbf{K}$ defined by (6), requires the solution of at most m linear programming problems.

If there is a binding inequality constraint in (6), that constraint can be treated as an equality constraint, and a variable eliminated using it, and the process can be repeated. We assume that such reduction steps have been carried out as far as possible.

So, we assume that for each $i = 1$ to m, there exists an $x^i \in \mathbf{K}$ satisfying $D_{i.}x > d_i$. Hence, the dimension of $\mathbf{K}$ is n.

It is well known (see, for example, [7]) that $\mathbf{K}$ has an extreme point iff the set of column vectors of D is linearly independent. If $\mathbf{K}$ has no extreme points, $\mathbf{K}^{\Delta} = \emptyset$, and our problem becomes trivial. So, we assume that this condition holds, that is that $\mathbf{K}$ has at least one extreme point.

Also, it is well known that $\mathbf{K}$ is bounded iff the system

$$D\xi \geq 0 \tag{8}$$

has $\xi = 0$ as its unique solution. See [7]. If $\mathbf{K}$ is bounded, $\mathbf{K}^{\Delta} = \mathbf{K}$, in this case the dimension of $\mathbf{K}^{\Delta}$ = dimension of $\mathbf{K}$. Our problem becomes interesting if $\mathbf{K}$ is unbounded, that is, when (8) has at least one nonzero solution. In this case $\mathbf{K}^{\Delta}$ is a proper subset of $\mathbf{K}$, it is the set of feasible solutions of a system of constraints consisting of (6) and some additional constraints. The number of these additional constraints needed to represent $\mathbf{K}^{\Delta}$ could be very large, but so far there is no systematic method known for generating them in a reasonable manner. Suppose $\mathbf{K}^{\Delta}$ is the set of feasible solutions of (6) and the additional constraints (9).

$$Q_{u.}x \geq q_u, \quad u = 1 \text{ to } V \tag{9}$$

In the system (6), (9), defining $\mathbf{K}^{\Delta}$, there is a binding inequality constraint (this is an inequality constraint among (6), (9), which holds as an equation at every $x \in \mathbf{K}^{\Delta}$) iff the dimension of $\mathbf{K}^{\Delta}$ is $< n$ = dimension of $\mathbf{K}$. Thus the system of additional constraints needed to define $\mathbf{K}^{\Delta}$ consists of inequalities none of which are binding iff the dimension of $\mathbf{K}^{\Delta}$ is n = dimension of $\mathbf{K}$. In fact, if dimension of $\mathbf{K}^{\Delta}$ is r, then the system of additional constraints over those in (6), needed to define $\mathbf{K}^{\Delta}$, consists of a set of $n - r$ linearly independent equality constraints and a system of inequality constraints none of which are binding. Thus the determination of the dimension of $\mathbf{K}^{\Delta}$ is an essential first step in determining the structure of the additional constraints needed to define $\mathbf{K}^{\Delta}$. Under a nondegeneracy assumption, we derive a formula for the dimension of $\mathbf{K}^{\Delta}$, which only needs the data in (6), and the solution to some problems on the convex polyhedron $\mathbf{K}$, however, these problems are NP-complete.

The results

Lemma 2. *Let* $\mathbf{K}$, *the set of feasible solutions of* (6), *be unbounded, and suppose* i *is such that* $D_{i.}x$ *is unbounded above over* $\mathbf{K}$. *The problem of checking whether there exists of extreme point of* $\mathbf{K}$ *satisfying* $D_{i.}x > d_i$, *is NP-complete.*

Proof. Clearly, this problem is in NP. Let $\mathbf{F}$ be the set of feasible solutions of

$$D_{t.}x \geq d_t, \quad t \neq i$$

Then by hypothesis, $\mathbf{F}$ is an unbounded convex polyhedron and $D_{i.}x$ is unbounded above on it. Since the problem of finding the extreme point of $\mathbf{F}$ that maximizes $D_{i.}x$ is NP-hard, the problem of checking whether there exists an extreme point of $\mathbf{F}$ satisfying $D_{i.}x>d_i$ is NP-complete. By the results of [7, 8], each extreme point of $\mathbf{K}$ has to belong to one of the following types.

(a) Extreme points of $\mathbf{F}$ satisfying $D_{i.}x>d_i$.

(b) Extreme points of $\mathbf{F}$ satisfying $D_{i.}x=d_i$.

(c) Points of intersection of edges of $\mathbf{F}$ (bounded or unbounded) which do not totally lie in the hyperplane $\{x: D_{i.}x=d_i\}$, with that hyperplane.

So the only extreme points of $\mathbf{K}$ which satisfy $D_{i.}x>d_i$, are those of type (a) above, that is, those extreme points of $\mathbf{F}$ satisfying $D_{i.}x>d_i$. But from the argument made above, the problem of checking whether there exists an extreme point of $\mathbf{F}$ satisfying $D_{i.}x>d_i$ is NP-complete. So the problem of checking whether there exists an extreme point of $\mathbf{K}$ satisfying $D_{i.}x>d_i$, is NP-complete. $\square$

Since we assumed that the dimension of $\mathbf{K}$, defined by (6), is n, there exists an $\bar{x}\in\mathbf{K}$ satisfying $D\bar{x}>d$, or equivalently, for each $i=1$ to m there exists an $x^i\in\mathbf{K}$ satisfying $D_{i.}x>d_i$. We have the following result.

Theorem 3. *Let* $\mathbf{K}$, *the set of feasible solutions of* (6), *be of dimension n. Also, assume that system* (6) *is nondegenerate* (*under this assumption,* $\mathbf{K}$ *is a regular convex polyhedron, that is, each extreme point of* $\mathbf{K}$ *is incident to exactly n edges of* $\mathbf{K}$, *these may be bounded or unbounded*). *Then the dimension of* $\mathbf{K}^{\Delta}$ *is also n iff for each* $i=1$ *to m there exists an extreme point* $\hat{x}^i$ *of* $\mathbf{K}$ *satisfying* $D_{i.}x>d_i$.

Proof. Since $\mathbf{K}^{\Delta}\subset\mathbf{K}$, every point in $\mathbf{K}^{\Delta}$ satisfies (6), this implies that if the dimension of $\mathbf{K}^{\Delta}$ is n, there must exist a point $\bar{x}\in\mathbf{K}^{\Delta}$ satisfying $D\bar{x}>d$. Since $\bar{x}\in\mathbf{K}^{\Delta}$, it is a convex combination of extreme points of $\mathbf{K}$, so $D\bar{x}>d$ holds iff for each $i=1$ to m, there exists an extreme point $\hat{x}^i$ of $\mathbf{K}$ satisfying $D_{i.}x>d_i$.

Conversely, suppose for each $i=1$ to m there exists an extreme point $\hat{x}^i$ of $\mathbf{K}$ satisfying $D_{i.}x>d_i$. It is well known that between every pair of extreme points of $\mathbf{K}$, there exists an edge path of $\mathbf{K}$ joining them, consisting of only bounded edges of $\mathbf{K}$. See [7]. Using this and the hypothesis, we prove below that the dimension of $\mathbf{K}^{\Delta}$ is n.

Introducing the vector of slack variables $s=(s_1,\dots,s_m)^{\mathrm{T}}$, the system (6) can be expressed as

$$Dx-I_m s=d, \qquad s\geq 0$$

where I_m is the unit matrix of order m. In this, the equality constraints can be used to eliminate the unrestricted variables $x_1,\dots,x_n$. Suppose this leads to a system

$$Es=p, \qquad s\geq 0 \tag{10}$$

where E, p are matrices of orders $r\times m$ and $r\times 1$ respectively, where $r=m-n$, and

E has rank r. Every extreme point of $\mathbf{K}$ corresponds to a basic feasible solution (BFS) of (10). Let $s^0=(s_1^0,\dots,s_m^0)^{\mathrm{T}}$ be a BFS of (10) corresponding to a basic vector $(s_1,\dots,s_r)$ for (10). By our assumption, the system (10) is nondegenerate. In s^0, the nonbasic variables $s_{r+1},\dots,s_m$ are all zero. By the hypothesis, for each $t=r+1$ to m, there exists a BFS of (10) in which the variable $s_t>0$. And each BFS of (10) is connected to s^0 by an edge path as mentioned above. So, among the nonbasic variables $s_{r+1},\dots,s_m$, at least some of them must enter the basic vector $(s_1,\dots,s_r)$ of (10) leading to adjacent BFSs of s^0, and not to unbounded edges. Suppose these are the nonbasic variables s_{r+j}, $j=1$ to q. Let $s^j=(s_1^j,\dots,s_m^j)^{\mathrm{T}}$ be the BFS obtained when the nonbasic variable s_{r+j} is entered into the basic vector $(s_1,\dots,s_m)$, for $j=1$ to q. So

$$\begin{aligned} s_i^j &= 0, \quad \text{for } i=r+1 \text{ to } m,\ i\neq r+j,\\ &>0, \quad \text{for } i=r+j. \end{aligned} \tag{11}$$

In each of the BFSs s^0, s^j, $j=1$ to q, all the variables $s_{r+q+1},\dots,s_m$ are zero, and by the hypothesis there are BFSs of (10) in which these variables are >0. Let $\mathbf{H}$ be the face of the set of feasible solutions of (10) obtained by setting $s_{r+q+1}=\cdots=s_m=0$. By the edge path connectedness property, there must exist an extreme point in $\mathbf{H}$ which has an adjacent extreme point, s^{r+q+1} say, not in $\mathbf{H}$, in which exactly one of the variables among $s_{r+q+1},\dots,s_m$ is >0, and the others are zero. Suppose $s_{r+q+1}^{r+q+1}>0$. Now consider the face of the set of feasible solutions of (10) obtained by setting $s_{r+q+2}=\cdots=s_m=0$, and repeat the same argument. Eventually we get BFSs $s^j=(s_1^j,\dots,s_m^j)^{\mathrm{T}}$ of (10), $j=1$ to $m-r$, satisfying the property that

$$\begin{aligned} s_i^j &= 0, \quad \text{for } i=r+j+1 \text{ to } m,\\ &>0, \quad \text{for } i=r+j. \end{aligned} \tag{12}$$

By (12) we conclude that the rank of the set $\{s^j-s^0: j=1 \text{ to } m-r\}$ is $m-r$ which implies that the dimension of the convex hull of BFSs of (10) is $m-r=n$, and hence the dimension of the convex hull of extreme points of $\mathbf{K}$ is n. □

Theorem 4. *Let* $\mathbf{K}$ *be the set of feasible solutions of* (6), *and assume that* $\mathbf{K}$ *has at least one extreme point, and that the system* (6) *is nondegenerate. Let* $\mathbf{J}=\{i$: *there exists no extreme point of* $\mathbf{K}$ *satisfying* $D_{i.}x>d_i\}$. *Then the dimension of* $\mathbf{K}^{\Delta}=n-$ *rank of* $\{D_{i.}: i\in\mathbf{J}\}$.

Proof. By the definition of the set $\mathbf{J}$, all the extreme points of $\mathbf{K}$ satisfy

$$\begin{aligned} D_{i.}x &\geq d_i, \quad i\notin\mathbf{J},\\ D_{i.}x &= d_i, \quad i\in\mathbf{J}. \end{aligned} \tag{13}$$

The result follows by applying Theorem 3 to the reduced system obtained by eliminating variables using the equality constraints in (13). ❑

By Theorem 3, to check whether the dimension of $\mathbf{K}^{\Delta}$ is n, we must check whether there exists an extreme point of $\mathbf{K}$ satisfying $D_{i.}x>d_i$, for each $i=1$ to m. However, by Lemma 2, for any i, the problem of checking whether there exists an extreme point of $\mathbf{K}$ satisfying $D_{i.}x>d_i$, is NP-complete. This suggests that the problem of checking whether the dimension of $\mathbf{K}^{\Delta}$ is n, or computing the dimension of $\mathbf{K}^{\Delta}$, may be hard problems.

Corollary 2. *If* $\mathbf{K}$ *is the set of feasible solutions of* (6), *and if the system* (6) *is nondegenerate*; *any equality constraints in the constraint system defining* $\mathbf{K}^{\Delta}$, *are a subset of the constraints in* (6) *treated as equations.*

Proof. Follows from Theorems 3 and 4. ❑

Eventhough it is hard to find the equality constraints satisfied by all the points in $\mathbf{K}^{\Delta}$, Corollary 2 provides a nice characterization of them, by showing that they must be a subset of the constraints in (6) treated as equations, when the system (6) is nondegenerate. We were quite hopeful that the results in Theorems 3, 4, and Corollary 2, would also hold even when the constraint system (6) is degenerate. But a simple three dimensional counterexample turned up. This polyhedron $\mathbf{K}$ of dimension 3 is given in Fig. 4. It has three extreme points x^0, x^1, x^2; only two bounded

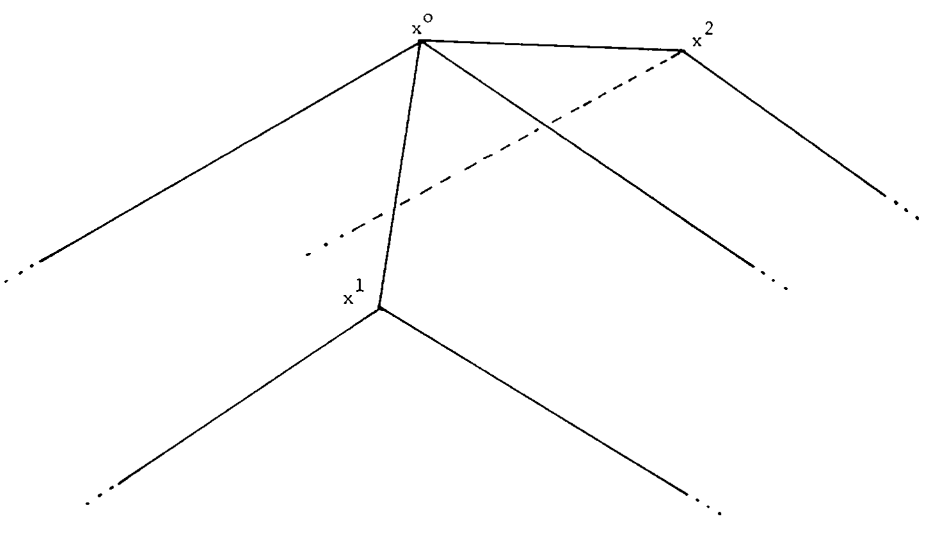

Fig. 4. Polyhedron in $\mathbf{R}^3$ showing a violation of Theorem 3 when the system of constraints defining it is degenerate. Each unbounded edge has dots at the end.

edges $[x^0, x^1]$ and $[x^0, x^2]$, 6 unbounded edges and 6 two-dimensional facets. Since x^0 has 4 edges incident at it, $\mathbf{K}$ is not regular, and the constraint system defining this polyhedron is degenerate. $[x^1, x^2]$ is not an edge of $\mathbf{K}$. All the conditions of Theorem 3 are satisfied, but the dimension of $\mathbf{K}^{\Delta}$ is only 2, since it is the convex hull of x^0, x^1, x^2.

In case when $\mathbf{K}$ is the set of feasible solutions of (6), where (6) is a degenerate system, even the problem of characterizing the equality constraints satisfied by all the points of $\mathbf{K}^{\Delta}$ is unresolved.

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