Non-Darcian effects on natural convection in porous media confined between horizontal cylinders

M. KAVIANY
Department of Mechanical Engineering and Applied Mechanics, University of Michigan, Ann Arbor, MI 48109, U.S.A.

Abstract—The inertia, boundary and velocity-square terms, normally not included in the flow analysis, are included in the study of natural convection between isothermal, concentric cylinders (inner cylinder heated) filled with saturated, porous media. The results show that all of these effects reduce the heat transfer rate with the boundary term being the most significant. It is shown that since at high Rayleigh numbers the flow adjacent to the confining walls becomes of boundary-layer type, with a very thin sublayer over which the velocity reaches its maximum value, then as long as the contribution of the velocity-square term is small, Darcy's model holds for very large Rayleigh and Prandtl numbers. A flow regime diagram showing the pseudo-conduction, Darcy and non-Darcy regimes, is given.

1. INTRODUCTION

Fluid flow in saturated, porous media with high permeabilities manifests some of the characteristics of the flow in the absence of a rigid matrix, i.e. the inertia and boundary effects not included in Darcy's model may become significant [1-4]. Also, at high velocities the rigid matrix resistance is no longer given by Darcy's law, i.e. the velocity-square term becomes significant [5]. The inertia or development term which may be significant in the leading edge region (external flows) [1-3] and in the entrance region (internal flows) [4] becomes less significant in the presence of a solid matrix because the lower the permeability the shorter is the development region [1, 3, 4]. This is due to the proportionality of the flow resistance to the local velocity which tends to make the velocity field uniform. In boundary-layer flows, the relative influence of the boundary term on the heat transfer rate is manifested in the ratio of the thermal to momentum boundary-layer thickness [1, 3, 4]. In general, for a given flow driving force (pressure gradient or buoyancy) the results have shown that each of the three effects, i.e. inertia, boundary and velocity-square, reduces the heat transfer rate.

In natural convection between isothermal, horizontal cylinders—as in other enclosure convections—as the buoyancy potential (Rayleigh number) increases, three regimes are observed. First is the regime of no significant convection, i.e. conduction regime, then the non-boundary-layer convection or the conduction-convection regime is observed. Finally the boundary-layer flow regime is reached. At still higher Rayleigh numbers the flow becomes unstable and various transitions have been observed. Convection between isothermal, horizontal cylinders has been studied in detail in the presence of a rigid matrix [10, 11]. It is found that when no rigid matrix is present, and for air, unsteady three-dimensional flows begin above Ra = 3 x 10^6 and 10^7 for R = 2 and 4, respectively [11], where R is the diameter ratio and Ra is the Rayleigh number. When the solid matrix is present and Darcy's law holds, it is found that for R = 2 and Rayle^{-2} greater than 134, unsteady three-dimensional convection will take place [5], where y^2 is proportional to the inverse of the permeability.

In this study, the non-Darcy effects on natural convection between two concentric, isothermal, horizontal cylinders (inner cylinder heated) filled with a fluid-saturated, porous medium, are studied by solving the two-dimensional conservation equations numerically. Some of the Ra and Rayle^{-2} values considered in this study are larger than the critical values given above for very small and very large permeabilities. However, since no definite statement can yet be made for moderate permeabilities the numerical results are assumed to be for stable flows. Figure 1 gives a schematic of the problem considered.

2. ANALYSIS

Assuming local thermal equilibrium between the solid and fluid, and also, a Boussinesq fluid, the conservation equations based on the available empirical data and the volume average principles [1] are

\[ \nabla \cdot u = 0 \]  
\[ (u \cdot \nabla)u = -\rho_0^{-1} \nabla p + \rho \omega^2 g + \nu \nabla^2 u \]  
\[ -K^{-1} \nabla \cdot \mu \nabla u = \nabla \cdot (\kappa \nabla T) \]  
\[ (u \cdot \nabla)T = \chi \nabla^2 T \]

where equation (2) allows for the Darcy resistance as
well as for the second-order resistance found for pore Reynolds numbers larger than unity [5]. However, these equations do not account for any dispersion or spreading of momentum and thermal energy due to inter-pore mixing in the presence of a velocity gradient. This effect which is only significant at relatively high velocities and for large gradients, is discussed in ref. [12]. For the present problem these equations can reasonably describe the phenomena.

The above equations have been non-dimensionalized using:

\[ r^* = \frac{r}{D_i}, \quad u^* = \frac{uD_i}{\alpha_c}, \quad T^* = \frac{T - T_o}{T_i - T_o}, \]

in the following all the quantities are dimensionless and the asterisks are dropped for convenience.

Introducing the streamfunction \( \psi = u \) and vorticity \( \omega = \nabla \times u \), we arrive at the following vorticity equation by taking the curl of equation (2)

\[ \frac{Pr}{r^2} \frac{\partial}{\partial r} \left( r \psi \frac{\partial \psi}{\partial r} \right) = \nabla^2 \omega - Ra (r^{-1} \cos \phi T_0 + \sin \phi T_1) - \gamma \psi + \Psi (r^{-1} \psi \omega, \phi), \]

\[ + r^{-2} (\nabla \psi, \phi). \]

We also have

\[ \nabla^2 \psi = -\omega \]

\[ r^{-1} (\psi T_1 - \psi T_0) = \nabla^2 T. \]

The emerging dimensionless parameters are:

\[ R = \frac{D_i}{D_o}, \quad Ra = \frac{g\beta(T_i - T_o)D_i^2}{\nu \alpha_c}, \]

\[ \gamma = \frac{\alpha_c D_i^2}{K}, \quad \Psi = \frac{Kc^2 D_i}{K^{1/2}}, \quad Pr = \frac{\nu}{\alpha_c}. \]

The physical significance of these parameters are:

(a) the relative buoyancy potential given by the Rayleigh number \( Ra \),

(b) the relative significance of the inertia term given by the Prandtl number \( Pr \),

(c) the first-order rigid matrix resistance given by the porous media shape parameter \( \gamma \), which is the ratio of the large length scale (taken to be the curvature instead of the gap spacing) to small length scale; and

(d) second-order rigid matrix resistance given by \( \Psi \).

The boundary conditions are (also shown in Fig. 1)

\[ r = \frac{R}{2} \]

\[ \psi = 0, \quad T = T_o, \quad \omega = -\psi \partial \phi / \partial \phi, \]

\[ r = 0 \]

\[ \psi = 0, \quad T = 0, \quad \omega = 0. \]

where \( R \) is the ratio of the outside to inside diameter.

### 3. Solution and Validation

Equations (4)–(6) subject to equations (7)–(9) are solved numerically using the finite-difference approximations (hybrid method [13]), a uniform grid.
The average normalized inner wall Nusselt number is defined as

\[ Nu = \frac{\ln R}{2\pi} \int_0^\pi T(r = 0.5) \, d\phi \] (10)

where \( 2/\ln R \) is the conduction Nusselt number.

### 4. RESULTS AND DISCUSSION

In special cases equation (4) simplifies to the following forms

(a) \( Ra \to \infty, Pr \to \infty, \Psi \to 0 \) (i.e. Darcy's law)

\[-Ray^{-2}(r^{-1} \cos \phi T_0 + \sin \phi T_0) - \omega = 0. \] (11)

(b) \( Ra \to \infty, Pr \to \infty \) (i.e. insignificant inertial force and shear stresses)

\[-Ray^{-2}(r^{-1} \cos \phi T_0 + \sin \phi T_0) - \omega + \Psi r^{-2}(\omega r^{-1} |\psi| \psi_r) + r^{-2}(\omega |\psi| \psi_r) = 0. \] (12)

(c) \( Pr \to \infty \) (i.e. insignificant inertial effects)

\[-\gamma^{-2}V^2 - Ray^{-2}(r^{-1} \cos \phi T_0 + \sin \phi T_0) - \omega + \Psi r^{-2}(\omega r^{-1} |\psi| \psi_r) + r^{-2}(\omega |\psi| \psi_r) = 0. \] (13)

Since we are interested in the net heat transfer rate between the cylinders, the results will be presented such that they demonstrate the effect of the inertia, boundary and velocity-square terms on the average inner wall Nusselt number, i.e. \( Nu = Nu(R, Pr, Ra, y, \Psi) \). All of these effects reduce the heat transfer rate, but their relative significance varies greatly. In this study two diameter ratios, namely \( R = 2 \) and 4, have been chosen. Table 1 gives examples of some porous media saturated with gases at near standard conditions [14]. The porous media are polyurethane foams and some of the values given are not encountered in practice and are given for appreciation of the order of magnitude for the

| Fluid, \( g\beta/\nu^2 \) (m\(^{-3}\) K\(^{-1}\)) | \( \Delta T \) (K) | \( D_0 \) (m) | \( \epsilon \) | \( K \) (m\(^2\)) | \( F^* \) | \( R \) | \( Pr^* \) | \( Ra \) | \( \gamma^2 \) | \( \Psi \) | \( Ray^{-2} \) | \( Nu, Nu(\Psi = 0) \) |
|----------------|----------|--------|-----|------|----|----|---------|--------|--------|------|--------|------|-----------|
| \( 1.6 \times 10^8 \) | \( 10^2 \) | 0.05 | 0.9 | 10\(^{-5} \) | 0.1 | 2 | 0.5 | 10\(^6 \) | 2 \times 10^2 | 1.3 | 3 \times 10^3 | 1.0 |
| \( 10^{-8} \) | 4 | \( 2 \times 10^3 \) | 41 | 5 | \( 1.0 \) |
| \( 10^{-5} \) | 2 | \( 2 \times 10^2 \) | 1.3 | 5 \times 10^3 | 0.90 |
| \( 10^{-8} \) | 6.4 | \( 2 \times 10^2 \) | 41 | 5 | 1.0 |
| \( 10^{-5} \) | 0.5 | \( 2 \times 10^2 \) | 200 | 5 | \( 0.94 \) |
| \( 10^{-8} \) | \( 2 \times 10^2 \) | 6.4 | 3 \times 10^3 | 0.81 |
| \( 10^{-5} \) | 4 | \( 2 \times 10^2 \) | 200 | 5 | 1.0 |
| \( 10^{-8} \) | \( 2 \times 10^2 \) | 6.4 | 3 \times 10^3 | 0.81 |

* For some polyurethane foams saturated with gases. In general \( k_0 \) and \( F \) change with \( K \). \( K = 10^{-5} \) m\(^2\) and \( F = 0.5 \) are given as examples of extreme cases and are not normally encountered.
various dimensionless parameters. This table will be further examined in the section on the effect of the velocity-square term on the heat transfer rate.

Since a relatively comprehensive presentation of the variations in $Nu$ with respect to each of the variables given above is rather lengthy, in the following some limited results are given with the objective of constructing flow regime diagrams where the regions of Darcy and non-Darcy convections can be identified.

4.1. Inertia and boundary terms

For a given relative buoyancy potential, as the permeability decreases the velocity decreases and the contribution of the inertia term to the overall force balance vanishes. A similar trend is expected for the shear-stress term. Figure 2 shows the variation in the average inner wall Nusselt number with the permeability for a given $R$, $Ra$, and $Pr$. The inertia effect is not significant at low permeabilities, while the boundary effect persists to relatively lower permeabilities and generally dominates the inertia effects.

As shown in ref. [6], for the Darcy regime, as the value of $Ray^{-2}$ increases the center of the cellular motion (one on each side) moves upward, i.e. above $\phi = \pi/2$. Of course, this behavior is also observed when no rigid matrix is present (e.g. [11]). Since the Darcy’s model does allow for velocity slip of the boundaries, it is expected that the inclusion of the boundary term would lower the center of the cell. Figure 3 shows the lines of constant temperature and streamfunction for $Ra = 10^6$ and $Ray^{-2} = 5 \times 10^3$. As expected, the boundary term causes the center of the circulation to move downward and also reduces the circulation (i.e. $\psi_{max}$). The results for no boundary

\[ Ray^{-2} = (g\beta v^{-1}a_e^{-1}e^{-1}\Delta T D)K. \]

Note that $Ray^{-2} = (g\beta v^{-1}a_e^{-1}e^{-1}\Delta T D)K$. The results are for $R = 2$, $Ra = 10^6$, $Pr = 0.5$ and $\gamma^2 = 2 \times 10^3$.

4.2. Prandtl number

The case of $Pr \to \infty$ corresponds to no inertia effects. As $Pr$ decreases, the effect of inertia increases. Figure 4 shows a further decrease in the Nusselt number with a decrease in the Prandtl number. Also when the conduction regime is approached, the results are independent of Prandtl number.

4.3. Rayleigh number

When both the buoyancy term and the first-order rigid matrix resistance term dominate over the other term are in agreement with the trend found in ref. [6] for $R = 2$. In ref. [6] the results are given for the values of $Ray^{-2}$ equal to 40 and 400 (Figs. 4 and 7 in [6]). Note that as shown in Fig. 3 and found in ref. [6], for large values of $Ray^{-2}$, the upward flow near the center of the cell is nearly vertical.

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4.4. Rayleigh number

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4.4. Velocity-square term

Since the velocity-square term contains the permeability, its magnitude must be changed in accordance with $y^2$. Table 1 gives a range for $\Psi$ characteristic of foams saturated with gases at near standard conditions [14]. The reduction in the Nusselt number, due to this term, is only significant when: (a) $Ray^{-2}$ is large (due to large $Ra$ and high $K$), which results in high velocities; and (b) $F$ is large. The maximum reduction in the Nusselt number, for the media considered, is about 20%. In general this term does not contribute significantly to the reduction of the Nusselt number.

4.5. Flow regimes

Since for low permeability media the inertia term is insignificant, the Prandtl number effect can be neglected. Also, the velocity-square term is significant only when the boundary term is significant. Therefore, the boundary between regimes of Darcy and non-Darcy flow can be determined by only specifying $Ra$ and $y^2$. Although the transition from Darcian to non-Darcian regime is not absolute, we have taken any difference less than a few percent in $Nu$ between the results based on the inclusion of all terms and exclusion of them, to indicate that the effects are negligible (Darcian convection). This is done in Fig. 7 for $R = 2$ and 4. As expected, pseudo-conduction regime persists up to higher $Ray^2$ for $R = 2$.

The results show that

(a) along a line of constant $K$ (or $\gamma^2$), as $\Delta T$ (or $Ra$) increases the conduction regime, Darcy regime and non-Darcy regime are encountered; and

(b) along a line of constant $Ray^{-2}$ (or $\Delta TK$), as $Ra$ (or $\Delta T$) increases the non-Darcy effect becomes less significant.

It is desirable to examine the experimental results of ref. [6] for $R = 2$ (where glass beads and water were used) which have been shown to be in good agreement with the predictions based on Darcy’s model.

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FIG. 5. The effect of inclusion of the inertia and boundary terms on the average inner wall Nusselt number for $R = 2$, $Pr = 0.5$ and various values of $Ra$. The numerical results of Caltagirone for no inertia and boundary effects are also given.

FIG. 6. The effect of inclusion of the inertia and boundary terms on the average inner wall Nusselt number for $R = 4$, $Pr = 0.5$ and various values of $Ra$. The numerical results of Caltagirone for no inertia and boundary effects are also given.

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Note that the Nusselt and Rayleigh numbers are based on $D_\circ$.

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FIG. 7. The regimes of significant inertia and boundary effects for $R = 2$ and 4.
Unfortunately, not all the necessary experimental conditions needed for evaluation of $Ra$ are given in ref. [6]. However, if we estimate the effective Prandtl number to be about 3.0 [15] and also assuming that data for $Ray^{-2} = 6$ in the experiments of ref. [6] is for $T_e - T_i = 10^\circ C$, then we have $3 \times 10^6 < Ra < 3 \times 10^8$. Now, examination of Fig. 7 shows that, as expected, the experimental results of ref. [6] are for high Rayleigh numbers and, therefore, are in Darcy's regime.

5. SUMMARY

Based on a semi-empirical momentum equation [1], which does not include the dispersion due to the inter-pore mixing in the presence of velocity gradients, the effects of inertia, boundary and velocity square terms on the heat transfer rate between two isothermal horizontal cylinders filled by a saturated porous medium is studied numerically. The results show that

(i) all of these effects reduce the heat transfer rate with the boundary effect being the most significant;

(ii) since at high values of $Ra$, the flow becomes nearly of boundary-layer type, then for small values of $F$ Darcy's law holds when $Ra$ and $Pr$ both become very large; and

(iii) for a given $R$, the boundary between the Darcy and non-Darcy regimes is given by $Ray^{-2}$ and $Ra$. The flow regime diagrams for $R = 2$ and 4 are determined.

REFERENCES


EFFETS NON DARCIENS SUR LA CONVECTION NATURELLE DANS LES MILIEUX POREUX CONFINES ENTRE DES CYLINDRES HORIZONTAUX

Résumé—Les termes d'inertie, de frontière et de carré de vitesse, normalement ignorés dans l'analyse de l'écoulement sont inclus dans l'étude de la convection naturelle entre des cylindres isothermes, concentriques (cylindre intérieur chaud), avec remplissage par un milieu poroux saturé. Les résultats montrent que tous ces effets réduisent le transfert de chaleur, le terme de frontière étant le plus important. On montre que puisque, aux grands nombres de Reynolds, l'écoulement adjacent aux parois devient du type une très fine sous-couche au-delà de laquelle la vitesse atteint son maximum, aussi longtemps que la contribution du terme de carré de vitesse est faible, le modèle de Darcy est valable pour de très grands nombres de Rayleigh et de Prandtl. On donne un diagramme montrant les régimes de pseudo-conduction, de Darcy et de non Darcy.

Nicht-Darcy-Effekte bei natürlicher Konvektion in Porösen Medien Zwischen zwei horizontalen Zylindern

ИССЛЕДОВАНИЕ ВЛИЯНИЯ ЭФФЕКТОВ, НЕ ПОДЧИНЯЮЩИХСЯ ЗАКОНУ ДАРСИ, НА ЕСТЕСТВЕННУЮ КОНВЕКЦИЮ В ЗАКЛЮЧЕННОЙ МЕЖДУ ДВУМЯ ГОРИЗОНТАЛЬНЫМИ ЦИЛИНДРАМИ ПОРИСТОЙ СРЕДЕ

Аннотация—Проведено исследование естественной конвекции между изотермическими концентрическими заполненными насыщенной пористой средой цилиндрами (внутренний цилиндр нагрет) с учетом влияния инерционных сил, граничных условий и процессов, описываемых квадратичными по скорости членами, что обычно не делалось ранее. В результате получено меньше значение плотности теплового потока, на которую больше всего влияют условия на границе. В силу того, что при больших значениях числа Рэлея поток вблизи ограничивающих стенок приобретает характер течения в пограничном слое, в котором возможна максимальная скорость в пределах очень тонкого подслоя, показано, что модель Дарси можно использовать при очень больших числах Рэлея и Прандтля только в том случае, пока влияние процессов, описываемых квадратичными по скорости членами, незначительно. Приведена диаграмма течений, иллюстрирующая наличие псевдожерема Дарси и режима, не подчиняющегося закону Дарси.