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On Rings and Modules with DICC

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INTRODUCTION

Let us recall from [C, Definition 1] that a module is said to be DICC if any chain of submodules indexed by the integers $\mathbb{Z}$ stabilizes either to the right or to the left (or to both sides). A ring is DICC if it is so as a module.

In Section 1 we present some further results on DICC rings to clarify the study initiated in [C], where, after establishing a series of properties, we proved in Theorem 2 that a non-Noetherian ring S with no prime ideal both minimal and maximal (in brief, min/max ideal) is DICC if and only if: S reduced is Noetherian, $\mathfrak{n}$, the nilradical of S, is nilpotent and has *DCC*, and for all $x \in S - \mathfrak{n}$, $\mathfrak{n}/Sx \cap \mathfrak{n}$ has finite length. Such a result allowed us to establish with Theorem 3 that a DICC ring is a direct product of a ring such as described in Theorem 2 and an Artinian ring. Also, Condition 1 of Proposition 1.1 suggests the definition of almost-divisible module which provides a way of constructing DICC rings and a connection with Matlis' duality. This is done in Theorems 1.3 and 1.6. Some of these results are needed in the study of DICC modules initiated and extensively carried out in Section 2. For example, to prove Theorem 2.8 we need the corresponding result for rings first. Also in proving Theorem 2.4 we are illuminated by a similar result established in [C, Theorem 2] for rings, i.e., an R-module M is DICC iff M contains a submodule N such that (α) M/N is Noetherian; (β) N has DCC; (γ) for all $x \in M - N$, $N/Rx \cap N$ has finite length.

All rings considered in this paper are commutative with unit; $\mathfrak{n}$ will denote the nilradical of the ring. If M is an R-module, then $R \oplus M$ denotes the idealization of M in R. For a local (Noetherian) ring $(R, \mathfrak{m})$, $\hat{R}$ will denote the $\mathfrak{m}$-adic completion of R. All modules will be unitary. If M is an R-module, R local Noetherian, $M^{\vee}$ will denote its dual in the sense of

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Matlis. Also, the symbol: $\nmid$ means "does not divide"; $\subset$ means strict inclusion and $\subseteq$ allows equality. Moreover, when we quote a result in the same section we omit the section number. All the other notation is otherwise standard or specified.

1. DICC Rings

This section is a continuation of [C], where DICC rings were first introduced and investigated. As a matter of fact, the next proposition is strictly related to [C, Theorem 2].

PROPOSITION 1.1. *Let R be a non-Noetherian, non-reduced ring such that the nilradical $\mathfrak{n}$ of R has* DCC *and the reduced ring $\bar{R} = R/\mathfrak{n}$ is Noetherian. Then the following conditions are equivalent for any element $x \in R - \mathfrak{n}$:*

(i) $\mathfrak{n}/x\mathfrak{n}$ *has finite length;*

(ii) $\mathfrak{n}/Rx \cap \mathfrak{n}$ *has finite length.*

Proof. Let $x \in R - \mathfrak{n}$. The ideal $\mathfrak{j} = \mathfrak{n}:_R x = \{r \in R \mid rx \in \mathfrak{n}\} \supset \mathfrak{n}$ and $x\mathfrak{j} = Rx \cap \mathfrak{n} \supset x\mathfrak{n}$, hence (i) $\Rightarrow$ (ii). Let us now assume that (ii) holds and let us consider the short exact sequence

$$0 \to x\mathfrak{j}/x\mathfrak{n} \to \mathfrak{n}/x\mathfrak{n} \to \mathfrak{n}/x\mathfrak{j} \to 0.$$

Then $\mathfrak{n}/x\mathfrak{n}$ has finite length $\Leftrightarrow$ $x\mathfrak{j}/x\mathfrak{n}$ has finite length. Therefore let us prove that $x\mathfrak{j}/x\mathfrak{n}$ has finite length. For this proof let us observe that $\mathfrak{j} \supset \mathfrak{n}$ implies $\mathfrak{j}/\mathfrak{n}$ is finitely generated as an ideal of the Noetherian ring $R/\mathfrak{n}$. Let $\bar{j}_1, ..., \bar{j}_k$ be a set of generators for $\mathfrak{j}/\mathfrak{n}$. Then $\mathfrak{j} = \mathfrak{n} + Rj_1 + \cdots + Rj_k$, and therefore $x\mathfrak{j}/x\mathfrak{n} = (x\mathfrak{n} + Rxj_1 + \cdots + Rxj_k)/x\mathfrak{n}$ is finetely generated as R-module, which means it has finite length since it is Artinian.

Since in [C, Theorem 2] it was established that (ii) holds in a non-Noetherian DICC ring with no min/max ideals, this proposition will allow us to construct a class of DICC rings, as the next theorem shows, but first let us make a definition.

Let R be a ring and let Q be an R-module.

DEFINITION 1.2. We say that Q is *almost-divisible* if Q/aQ has finite length for all $a \in R - \mathfrak{n}$.

For example, if R is a non-Noetherian DICC ring, the nilradical $\mathfrak{n}$ of R is almost-divisible by Proposition 1.

We shall sometimes abbreviate almost-divisible by a.d.

Let us also remark that an almost-divisible module Q of infinite length must have $\operatorname{Ann}_R Q \subseteq \mathfrak{n}$.

THEOREM 1.3. (A construction of DICC rings.) *Let R be a reduced Noetherian ring and let Q be a DCC R-module of infinite length. Then the ring $R \oplus Q$ is a DICC ring if and only if Q is almost-divisible.*

Proof. It is straightforward by using [C, Theorem 2] and Proposition 1 once we observe that the nilradical $\mathfrak{n}$ of $R \oplus Q$ is $O \oplus Q$ and that for $x \notin \mathfrak{n}$, i.e., $x = (a, q)$, $a \neq 0$,

$$\frac{\mathfrak{n}}{x\mathfrak{n}} = \frac{O \oplus Q}{O \oplus aQ} \cong \frac{Q}{aQ}.$$

Another interesting result is

PROPOSITION 1.4. *Let R be DICC, non-Noetherian and with no* min/max *ideals. Then R has only one minimal prime ideal.*

Proof. The nilradical is almost-divisible. Let us assume that there exist $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Min} \operatorname{Spec}(R)$. Pick $x_1 \in \mathfrak{p}_1 - \mathfrak{p}_2$ and $x_2 \in \mathfrak{p}_2 - \mathfrak{p}_1$ so that $x_i \notin \mathfrak{n}$, $i = 1, 2$, and $x_1 x_2 \in \mathfrak{n} = \mathfrak{p}_1 \cap \mathfrak{p}_2$. We have that $\mathfrak{n}/x_1\mathfrak{n}$ has finite length and that $\mathfrak{n}/x_2\mathfrak{n}$ has finite length, hence $\mathfrak{n}/(x_1 x_2)\,\mathfrak{n}$ has finite length, which implies that $\mathfrak{n}/(x_1 x_2)^t\,\mathfrak{n}$ has finite length for all t. This means that $\mathfrak{n}$ has finite length since $x_1 x_2^t = 0$ for some t, a contradiction.

As a corollary then we get

COROLLARY 1.5. *If a reduced Noetherian ring has two or more minimal, not maximal, prime ideals, then almost-divisibility implies finite length.*

The next result relates almost-divisibility and Matlis' duality.

THEOREM 1.6. *Let $(R, \mathfrak{m})$ be a local Noetherian ring, not necessarily complete, not necessarily a domain, and let Q be a DCC non-Noetherian R-module. Then Q is almost-divisible if and only if*

(1) $\exists!$ *minimal prime ideal* $\mathfrak{p}_0 \subset R$;

(2) *If $F = \bigcup_{t \geqslant 1} \operatorname{Ann}_{Q^\vee} \mathfrak{m}^t$, then $Q^\vee/F$ is torsion-free with respect to $R - \mathfrak{p}_0$.*

For the proof we need a

LEMMA 1.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with more than one minimal prime ideal and let Q be a DCC R-module. Then Q almost-divisible $\Leftrightarrow$ Q has finite length $\Leftrightarrow$ Q is Noetherian.*

Proof. Of course, Q Noetherian $\Leftrightarrow$ Q has finite length $\Rightarrow$ Q almost-divisible. Let us now assume that Q is almost divisible. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$, $k \geqslant 2$, be the minimal prime ideals of R. Pick elements $x_i \in \mathfrak{p}_i - \bigcup_{j \neq i} \mathfrak{p}_j$, $i = 1, \ldots, k$.

Then Q/x_1Q and Q/x_2Q of finite length imply that Q/x_1x_2Q has finite length, hence by induction $Q/x_1 \cdots x_kQ$ has finite length and also Q/x^tQ, where $x = x_1 \cdots x_k$, has finite length. But $x \in \mathfrak{n}$, i.e., $x^t = 0$ for large t, implies that Q has finite length.

Proof of the theorem. $(\Rightarrow)$ 1. Assume not and let $\mathfrak{p}_1, ..., \mathfrak{p}_h$, $h \geqslant 2$, be the minimal prime ideals of R. Then Q should be Noetherian by the Lemma, a contradiction. 2. Let $E = E_R(k)$ be an injective envelope of $K = R/\mathfrak{m}$. Then $\hat{R} \cong \mathrm{Hom}_R(E, E)$ is Noetherian. For any $x \in R - \mathfrak{p}_0$ ($\mathfrak{n} = \mathfrak{p}_0$ by part (1)), $Q \xrightarrow{.x} Q \to C_x \to 0$ with $l(C_x) < \infty$ implies $Q^\vee \xleftarrow{.x} Q^\vee \leftarrow C_x^\vee \leftarrow 0$ with $l(C_x^\vee) < \infty$ and equal to $l(C_x)$, hence $C_x^\vee = \mathrm{Ann}_{Q^\vee} x \subset F$. This implies that $x \nmid 0$ on $Q^\vee/F$, for otherwise $x\bar{q}^\vee = \bar{0}$ and $\bar{q}^\vee \neq \bar{0}$ yields $\overline{xq}^\vee = \bar{0}$, i.e., $xq^\vee \in F$. Pick any $y \notin \mathfrak{p}_0$ and such that $yF = 0$. Then $y(xq^\vee) = (yx)\,q^\vee = 0$ implies $q^\vee \in \mathrm{Ann}_{Q^\vee} xy \subseteq F$, hence $\bar{q}^\vee = \bar{0}$, which is a contradiction. $(\Leftarrow)$ We shall prove it by way of a contradiction. Let us assume that Q is not almost divisible, i.e., there exists an element $x \notin \mathfrak{p}_0$ such that in the exact sequence $Q \xrightarrow{.x} Q \to C_x \to 0$, $l(C_x) \nless \infty$. Then in $Q^\vee \xleftarrow{.x} Q^\vee \leftarrow C_x^\vee \leftarrow 0$ $l(C_x^\vee = \mathrm{Ann}_{Q^\vee} x) \nless \infty$, which means $\mathrm{Ann}_{Q^\vee} x \not\subset F$. By (2) $x \nmid 0$ on $Q^\vee/F$ since $x \notin \mathfrak{p}_0$ and therefore $xq^\vee = 0$ with $q^\vee \neq 0$ gives $\overline{xq}^\vee = x\bar{q}^\vee = \bar{0}$, which in turn implies $\bar{q}^\vee = \bar{0}$, i.e., $q^\vee \in F$. Hence $\mathrm{Ann}_{Q^\vee} x \subset F$, a contradiction.

2. DICC Modules

In this section we shall concentrate on DICC modules.

LEMMA 2.1. *Let R be a ring and let M be a DICC R-module. Then*

(1) $M \neq 0 \Leftrightarrow \mathrm{Ass}(M) \neq \varnothing$.

(2) $\mathrm{Ass}(M)$ *is finite.*

Proof. (1) It suffices to prove $(\Rightarrow)$. For this pick an element $x \in M - \{0\}$. Then $R/\mathrm{Ann}\, x \cong Rx \subseteq M$ has DICC and is a ring hence it has associated prime ideals by [C, Proposition 2]. The assertion then follows since $\mathrm{Ass}(Rx) \subseteq \mathrm{Ass}(M)$. (2) Assume not. Then $R/\mathfrak{p} \cong Rx \subset M$ for infinitely many primes $\mathfrak{p}$ imlies $\bigoplus_{\mathfrak{p} \in \mathrm{Ass}(M)} R/\mathfrak{p} \subseteq M$, which is a contradiction by [C, Proposition 1].

With the next lemma we shall investigate the relation between $\mathrm{Ass}(M)$ and $\mathrm{Max\,Spec}(R)$, M being an R-module. First let us fix some notation. $E(M)$ will denote the injective hull of M and $S(M)$ the socle of M, i.e., the sum of all simple submodules of M. Thus $S(M)$ is a direct sum of simple modules. Also, let us recall from [V] the following

DEFINITION. An R-module M is said to be *finitely embedded* if $E(M) = E(S_1) \oplus \cdots \oplus E(S_k)$, where each S_i is a simple R-module.

We shall sometimes abbreviate finitely embedded by f.e.

LEMMA 2.2. *Let R be a ring and let M be a DICC module over R. Set $\mathfrak{M} = \operatorname{Ass}(M) \cap \operatorname{Max\,Spec}(R)$. Then the following statements hold:*

(1) *$\mathfrak{M} = \varnothing$ implies M Noetherian;*

(2) *$\mathfrak{M} = \operatorname{Ass}(M)$ and M not Noetherian imply that M is Artinian.*

Proof. (1) Assume not and let $0 \neq M_0 \subset M_1 \subset \cdots \subseteq M$ be a strictly increasing chain of submodules of M. Then Rx must have DCC for all $x \in M_0 - \{0\}$, and therefore by [Bal, Theorem 1] $\operatorname{Ass}(Rx) \subseteq \operatorname{Ass}(M)$ consists of maximal ideals, a contradiction. (2) Note that M is not finitely generated. If not, $R/\operatorname{Ann} x_i \cong Rx_i$ would be Noetherian, in fact Artinian, and so would be $M = \sum_{i=1}^{t} Rx_i$, where $x_1, \ldots, x_t$ is a set of generators of M. This implies that Rx has finite length for all $x \in M$. Also, M is an essential extension of $S(M)$ and this is finitely generated, hence M is f.e. by [V, Lemma 1]. For all submodules M_0 of M, M/M_0 is either Noetherian, and in fact Artinian, or is DICC, not Noetherian and $\operatorname{Ass}(M/M_0) \subseteq \mathfrak{M}$, hence f.e. Thus M is Artinian by [V, Proposition 2*].

Let all notation in Lemma 2 be fixed. Then

COROLLARY 2.3. *Let R be a ring and let M be a DICC R-module such that Rx has finite length for all $x \in M$. Then M has DCC.*

Proof. We can assume without loss of generality that M is not finitely generated since the case of M being finitely generated is straightforward. Pick any $\mathfrak{p} \in \operatorname{Ass}(M)$ which is not empty by Lemma 1. Then $R/\mathfrak{p} \cong Rx$ for some $x \in M - \{0\}$ is a domain which has a finite length, hence an Artinian domain, which means a field. Therefore $\operatorname{Ass}(M) \subseteq \operatorname{Max\,Spec}(R)$ and the conclusion follows from Lemma 2.

Next we prove a structure theorem for DICC modules.

THEOREM 2.4. *Let R be a ring and let M be an R-module. Then the following statements are equivalent:*

(i) *M has DICC.*

(ii) *M contains a submodule N such that*

(α) *M/N is Noetherian;*

(β) *N has DCC;*

(γ) *for all $x \in M - N$, $N/Rx \cap N$ has finite length.*

Proof. (ii) $\Rightarrow$ (i) Note that if $N=0$ or M, then M would be Noetherian or Artinian, respectively, and hence DICC. Let us therefore assume that $N \neq 0, M$ and that M is not DICC. Denote by (*) $\cdots M_{-2} \subset M_{-1} \subset M_0 \subset M_1 \subset \cdots$ a dic (=doubly infinite chain) of submodules of M which makes (i) fail. Condition (α) implies that $M_t + N = M_{t+1} + N = \cdots$ from some t on. As (*) is strictly increasing, $M_t \cap N \subset M_{t+1} \cap N \subset \cdots \subseteq N$, for otherwise $M_t = M_{t+1} = \cdots$, which would contradict the assumption on (*). Hence $N/M_t \cap N$ does not have ACC. This is a contradiction because condition (β) implies $M_t \not\subset N$ as (*) is strictly decreasing, and therefore $N/M_t \cap N$ has ACC since it is a homomorphic image of $N/Rx \cap N$, where $x \in M_t - N$, which has ACC. Now (i) $\Rightarrow$ (ii). Let us assume that M is DICC but neither Noetherian nor Artinian. Set $\mathfrak{M} = \mathrm{Ass}(M) \cap \mathrm{Max\,Spec}(R)$, and let $\mathfrak{M}' = \mathrm{Ass}(M) - \mathfrak{M}$. In this set-up there exists a submodule N of M such that $\mathrm{Ass}(N) = \mathfrak{M}$ and $\mathrm{Ass}(M/N) = \mathfrak{M}'$ (see [Bak, IV, Sect. 1.2, Proposition 4]). Hence by Lemma 2 M/N is Noetherian. The submodule N is not Noetherian, for otherwise M would be Noetherian, a contradiction. Thus N has DCC by Lemma 2. Let us prove that (γ) holds also. Observe that, for any $x \in M - N$, Rx does not have DCC because $\mathrm{Ass}(Rx) \cap \mathrm{Max\,Spec}(R) = \varnothing$, hence $Rx + N/Rx$ must have ACC. If not, there would be a dic inside $Rx + N$, a contradiction. The isomorphism $Rx + N/Rx \cong N/Rx \cap N$ gives the conclusion since $N/Rx \cap N$ has DCC.

There follows

COROLLARY 2.5. *For a DICC R-module M, R being a commutative ring with unity,* $\mathrm{Supp}(M) = \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{q} \supseteq \mathfrak{p}$ *and* $\mathfrak{p} \in \mathrm{Ass}(M)\}$.

Proof. Let all notation be as in Theorem 4. It is enough to observe that $\mathrm{Supp}(M) = \mathrm{Supp}(N) \cup \mathrm{Supp}(M/N)$ since the assertion is true for Noetherian and Artinian modules.

Also, the following holds.

COROLLARY 2.6. *If M is a DICC module over R and not DCC, then M is finitely generated.*

Proof. Let $M \neq N$ and let $x \in M - N$. Then form the short exact sequence

$$0 \to M/N \cap Rx \to M/Rx \to M/N \to 0.$$

Since M/N is finitely generated by say $\bar{m}_1, ..., \bar{m}_k$, and since $M/N \cap Rx$ has finite length and so is finitely generated by $\bar{n}_1, ..., \bar{n}_l$, $n_1, ..., n_l$, $m_1, ..., m_k$ and x will generate M.

Remark. If M is DCC, it is countably generated by [A].

Our last result is about the support of a non-Noetherian, non-Artinian, DICC module and will be preceded by

LEMMA 2.7. *Let M be a DICC non-Noetherian R-module and let N be the Artinian submodule of M such that $\bar{M}=M/N$ is Noetherian. Then*

(1) *For $x \notin \operatorname{Ann}(\bar{M})$, $N/xM \cap N$ has finite length $\Leftrightarrow N/xN$ has finite length;*

(2) *for $y \notin \operatorname{rad}(\operatorname{Ann}(\bar{M}))$, $M/yM \cap N$ has finite length.*

Proof 1. ($\Leftarrow$) Set $M^x = N:_M x = \{m \in M \mid xm \in N\}$. Then (*) $M \supsetneqq M^x \supseteq N$ (note that $M \neq M^x$ because $x \notin \operatorname{Ann}(\bar{M}) \Leftrightarrow xM \not\subset N$), which implies $N \supseteq xM^x = Mx \cap N \supseteq xN$, and therefore the conclusion since $N/xN \xrightarrow{\pi} N/xM^x \to 0$. ($\Rightarrow$) We remark that N/xN has finite length $\Leftrightarrow \operatorname{Ker} \pi = xM^x/xN$ has finite length. Now (*) $\mapsto M^x/N \subset M/N$, hence M^x/N is finitely generated. Let $\bar{m}_1,\dots,\bar{m}_k$ be a set of generators. Then $M^x=\sum_{i=1}^k Rm_i + N$, where $m_1,\dots,m_k$ are the preimages of $\bar{m}_i$ ($i=1,\dots,k$) in M, which implies $xM^x=\sum_{i=1}^k Rxm_i + xN$ and hence xM^x/xN is finitely generated and DCC. Thus it has finite length. 2. $y \notin \operatorname{rad}(\operatorname{Ann}(\bar{M})) \Leftrightarrow y^n\bar{M} \neq \bar{0}$ all $n \geqslant 1 \Leftrightarrow y^nM \not\subset N$ all $n \geqslant 1$. We claim that $\{y^nM\}_{n\geqslant 1}$ is a strictly decreasing sequence. Assume not. Then $y^tM=y^{t+1}M=\dots$ for some $t \mapsto y^tM/N = y^{t+1}M/N=\dots \mapsto y^t\bar{M}=\bigcap_{n\geqslant 1} y^n\bar{M}=\bar{0}$ because $\bar{M}$ is Noetherian and this implies that $y^tM \subset N$, a contradiction. This proves that $yM+N/yM$ has ACC, for otherwise a dic would exist in M, which would yield a contradiction. The isomorphism $yM+N/yM \cong N/yM \cap N$ and the right-hand side having DCC imply the desired conclusion.

DEFINITION. An R-module M has no min/max ideals if no maximal ideal of R is minimal in the support of M.

THEOREM 2.8. *The support of a DICC, non-Noetherian non-Artinian, R-module with no* min/max *ideals is an irreducible closed subset of* $\operatorname{Spec}(R)$.

Proof. Let M be such an R-module. By Theorem 4 there exists an Artinian submodule N of M such that $\bar{M}=M/N$ is Noetherian and $\operatorname{Ass}(\bar{M}) \cap \operatorname{Max}\operatorname{Spec}(R)=\varnothing$. Let us observe that $\operatorname{Supp}(N)=\operatorname{Ass}(N)=\{\mathfrak{m}_1,\dots,\mathfrak{m}_k\} \subseteq \operatorname{Max}\operatorname{Spec}(R)$ is a closed subset of $\operatorname{Spec}(R)$, and let us also observe that $\operatorname{Supp}(\bar{M})=V(\operatorname{Ann}(\bar{M}))$ is a closed subset of $\operatorname{Spec}(R)$ as well. It follows that $\operatorname{Supp}(M)=\operatorname{Supp}(N)\cup\operatorname{Supp}(\bar{M})$ is a closed subset of $\operatorname{Spec}(R)$ and $\operatorname{Min}(\operatorname{Supp}(M))=\operatorname{Min}(\operatorname{Supp}(\bar{M}))=\operatorname{Min}(\operatorname{Ass}(\bar{M}))$ is finite. The disjointness of $\operatorname{Supp}(N)=\operatorname{Ass}(N)$ from $\operatorname{Ass}(\bar{M})$, which is contained in $\operatorname{Supp}(\bar{M})$, implies that the ring $R/\bigcap_{\mathfrak{p}\in\operatorname{Min}(\operatorname{Supp}(M))}\mathfrak{p}$ has no min/max ideals (see [C, Definition 2]). Let us remark that N does not have finite length (see Theorem 4) and hence in the direct decomposition $N=\bigoplus_{j=1}^k(\bigcup_{h\geqslant 1}\operatorname{Ann}_N \mathfrak{m}_j^h)=\bigoplus_{j=1}^k N_j$, where $\mathfrak{m}_j \in \operatorname{Ass}(N)$ and $N_j=\bigcup_{h\geqslant 1}\operatorname{Ann}_N \mathfrak{m}_j^h$, there exists at least one component N_j which has infinite length. Pick the maximal ideal $\mathfrak{m}_j$ corresponding to N_j and let $\mathfrak{p}$ be a minimal prime ideal of $\operatorname{Supp}(\bar{M})$ contained in $\mathfrak{m}_j$ (One exists because $\mathfrak{m}_j \supset \bigcap_{\mathfrak{p} \in \operatorname{Min}(\operatorname{Supp}(\bar{M}))} \mathfrak{p}$ and this intersection is finite!) We claim that $\mathfrak{p}$ is the only minimal prime ideal in $\operatorname{Supp}(\bar{M})$. Let us assume, on the contrary, that there exists another $\mathfrak{q}$. Then for any $y \in \mathfrak{p} - \mathfrak{q}$, we have $y \notin \operatorname{rad}(\operatorname{Ann}(\bar{M}))$, which implies that N/yN has finite length, by Lemma 7, and hence N/y^tN has finite length for all $t \geqslant 1$. Let us remark that y^tN of finite length would imply N of finite length, a contradiction. As y^tN is supported only at maximal ideals not containing $\mathfrak{p}$ (if $\mathfrak{m} \supseteq \mathfrak{p}$, $\mathfrak{p}^t N_{\mathfrak{m}} = 0$ for large t), we infer that y^tN is not supported at $\mathfrak{m}_j$ and hence $N/y^tN \supset N_j$, which has infinite length. This contradiction concludes the proof.

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