A Unified Approach to the Synthesis of the Irrational Immitance $\sqrt{(Z(s))}$ *and the Fractional-step Delay Operator* $z^{-1/2}$

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ABSTRACT: *A simple method for the synthesis of irrational immittances* $\sqrt{(Z(s))}$, *based on infinite cascades of balanced symmetric lattices, is presented. This approach is applicable to both analog and digital implementations. The method is illustrated by physical realizations of* $\sqrt{(Ls)}$, $\sqrt{(1/Cs)}$ *and the irrational number* $\sqrt{a}$. *As an obvious extension, a physical realization for the fractional-step delay* $z^{-1/2}$ *in a single-rate digital-filter structure is developed. Also, expressions for the input immittance of a truncated realization of* $\sqrt{(Z(s))}$ *and approximate realization of* $z^{-1/2}$ *are derived.*

I. Introduction

Numerous detailed approaches have been discussed in the literature for the synthesis of irrational immittances in the form of infinite networks (**1–3**). This is due to the many applications of such network functions in areas like modelling of diffusion processes, compensation of servo-systems, termination of high-frequency cables, etc. For analysis, synthesis and implementation of such systems, the need often arises for a rational approximation of $Z(s)$ which yields simple lumped network realizations with known component values.

It should also be noted that a number of applications are found for the digital counterpart of s, namely the fractional-step delay operator $z^{-1/2}$, in areas such as two-dimensional fan filters for geoseismic data (**4**) and array beamforming (**5**). During the past two decades, a number of investigators have proposed different procedures for rational approximations of some irrational immittances through infinite networks. Networks have previously been constructed to simulate the operators $\sqrt{s}$, $\sqrt{(1/s)}$ (**3**) and the irrational number $\sqrt{a}$ where a is a positive number greater than unity ($a > 1$) (**6**). Moreover, there now exists a rigorous theory for infinite electrical networks (**7**).

This paper presents a unified approach to the synthesis and realization of irrational functions. While the similarity between the analog and digital realizations is indicated, the algorithms presented here are significant because of their simplicity and utility. Furthermore, the fidelity of the approximations can be made arbitrarily good without significant computational cost during the synthesis procedure.

II. Derivation of the Algorithm

We begin with a derivation of the algorithm for the synthesis of the one-dimensional continuous time immittance $\sqrt{(Z(s))}$. Let $Z(s)$ be a positive real (PR) function. Properties of such functions can be found in (**8**). It is our intention to present an algorithm for the expression of $\sqrt{(Z(s))}$ in terms of the rational function $Z(s)$. To start, let us write

$$\sqrt{(Z(s))} = 1 + \frac{Z(s)-1}{\sqrt{(Z(s))}+1} \tag{1}$$

which can alternatively be written as follows:

$$Z(s) = 1 + \cfrac{1}{\cfrac{2}{Z(s)-1} + \cfrac{1}{1+\sqrt{(Z(s))}}}. \tag{2}$$

This can be expanded to result in the following continued fraction expansion:

$$Z(s) = 1 + \cfrac{1}{\cfrac{2}{Z(s)-1} + \cfrac{1}{1+1+\cfrac{1}{\cfrac{2}{Z(s)-1} + \cfrac{1}{1+\ddots}}}}. \tag{3}$$

Now, identify (2) by either the impedance

$$Z_{in}(s) = Z_a(s) + \cfrac{1}{\cfrac{2}{Z_b(s)-Z_a(s)} + \cfrac{1}{Z_a(s) + \cfrac{1}{Z_{in}(s)}}} \tag{4}$$

or by the admittance

$$Y_{in}(s) = Y_b(s) + \cfrac{1}{\cfrac{2}{Y_a(s)-Y_b(s)} + \cfrac{1}{Y_b(s) + \cfrac{1}{Y_{in}(s)}}}. \tag{5}$$

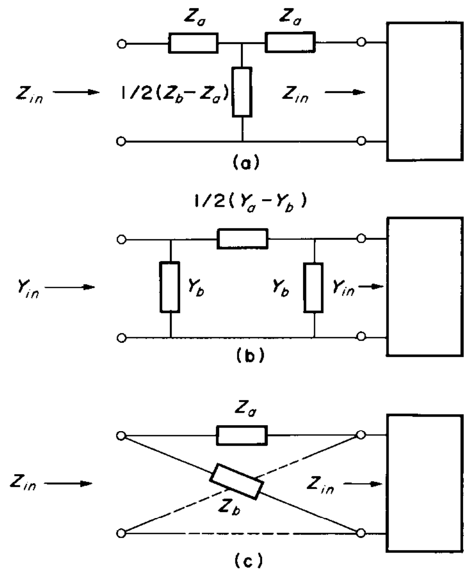

FIG. 1. An infinite cascade realization of the irrational immittance $\sqrt{(Z(s))}$. (a) An infinite ladder realization of the impedance $\sqrt{(Z(s))}$; (b) an infinite ladder realization of the admittance $\sqrt{(Y(s))}$ and (c) an infinite lattice realization of the immittance $\sqrt{(Z(s))}$.

These immittances can be realized in infinite ladder forms as shown in Fig. 1(a) and (b), respectively. Note that the T and π sections in Fig. 1(a) and (b) can be transformed to a lattice section as shown in Fig. 1(c). This transformation will of course eliminate the unrealizability condition which may arise if $Z_a(s) > Z_b(s)$ or $Y_a(s) > Y_b(s)$.

Thus, the immittance function (3) can now easily be realized as an infinite cascade of balanced symmetric lattices with unit resistors in the parallel arms and rational immittances $Z(s)$ or $Y(s)$ in the cross arms, as illustrated in Fig. 2(a) and (b). It is

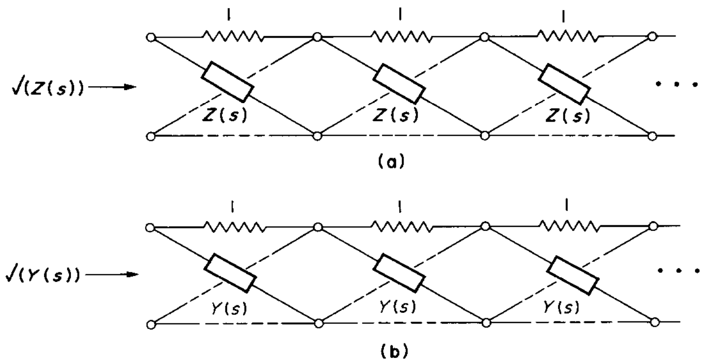

FIG. 2. An infinite cascade lattice realization of (a) the irrational impedance $\sqrt{(Z(s))}$ and (b) the irrational admittance $\sqrt{(Y(s))}$.

obvious that convergence of (3) is required in practice in order to have a physically realizable structure. Therefore, truncation of the continued fraction expansion (3) at a suitable point is mandatory. This will result in a rational approximation of the irrational immittance $\sqrt{(Z(s))}$ which leads to a finite cascade lattice realization.

Let $Z_n(s) = N_n(s)/D_n(s)$ denote the rational function obtained by truncating Eq. (3) at the $(n+1)$th lattice, so that $Z_0(s) = \infty$, and $Z_1(s) = (Z(s)+1)/2$ and so on. Then obviously we have

$$Z_n(s) = \frac{Z_{n-1}(s)[Z(s)+1]+2Z(s)}{2Z_{n-1}(s)+[Z(s)+1]}, \quad n \geqslant 1 \tag{6}$$

which gives the following recurrence relations:

$$N_n(s) = [Z(s)+1]N_{n-1}(s)+2Z(s)D_{n-1}(s) \tag{7a}$$

$$D_n(s) = [Z(s)+1]D_{n-1}(s)+2N_{n-1}(s). \tag{7b}$$

Solving for $N_n(s)$ and $D_n(s)$, we get

$$N_n(s)-2[Z(s)+1]N_{n-1}(s)+[Z(s)+1]^2N_{n-2}(s) = 0 \tag{8a}$$

and

$$D_n(s) = N_n(s). \tag{8b}$$

Solving these difference equations with the respective initial conditions $N_0 = 1$, $N_1 = Z(s)+1$, and $D_0 = 0$, $D_1 = 2$, we get

$$Z_n(s) = \frac{N_n(s)}{D_n(s)} = \frac{\sum_{r=0}^{n}\binom{2n}{2r}Z^r(s)}{\sum_{r=0}^{n-1}\binom{2n}{2r+1}Z^r(s)}. \tag{9}$$

For example, for $Z(s) = s$ with $n = 4$, Eq. (9) can be written as

$$\sqrt{s} \simeq Z_4(s) = \frac{s^4+28s^3+70s^2+28s+1}{8s^3+56s^2+56s+1}. \tag{10}$$

To show the fidelity of the approximation, the amplitude and phase plots of $\sqrt{s}$ and those of Eq. (10) are depicted in Figs. 3 and 4, respectively.

III. Realization

The synthesis of the irrational inductor $\sqrt{(Ls)}$ and irrational capacitor $1/\sqrt{(C\,s)}$ using the method described in Section II is shown in Fig. 5(a) and (b), respectively. It should be noted that the synthesis of a more general irrational immittance $Z(p)$ where $p = \sqrt{s}$ is also possible. In this case, the synthesis of $Z(p)$ is carried out using the Bott-Duffin method. Then the complete realization of $Z(\sqrt{s})$ is obtained by replacing every rational reactive element in the structure by irrational reactances as given by Eq. (3) and as depicted in Fig. 5.

Note that the given algorithm along with Eq. (9) suggest a lattice realization

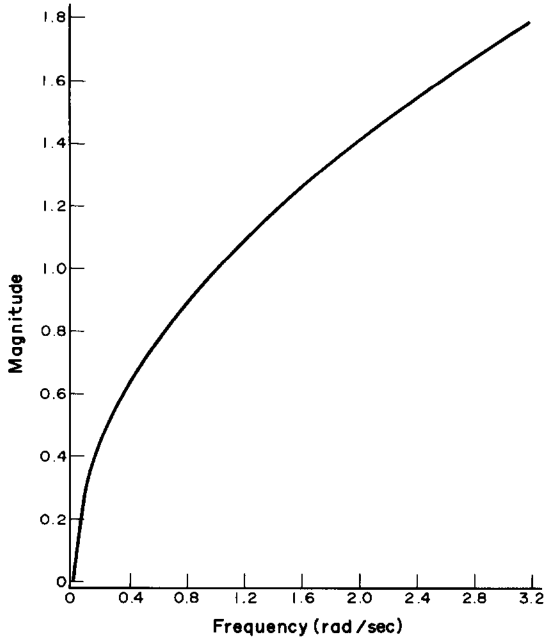

FIG. 3(a). Amplitude plot of Sqrt.(*s*).

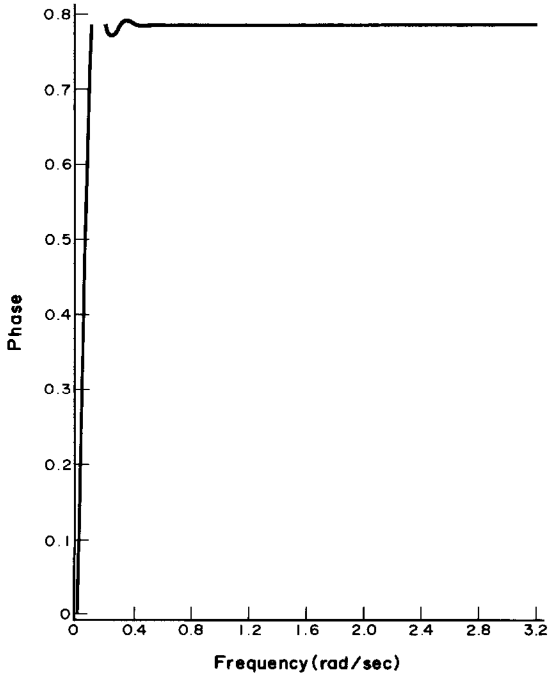

FIG. 3(b). Phase plot of Sqrt.(*s*).

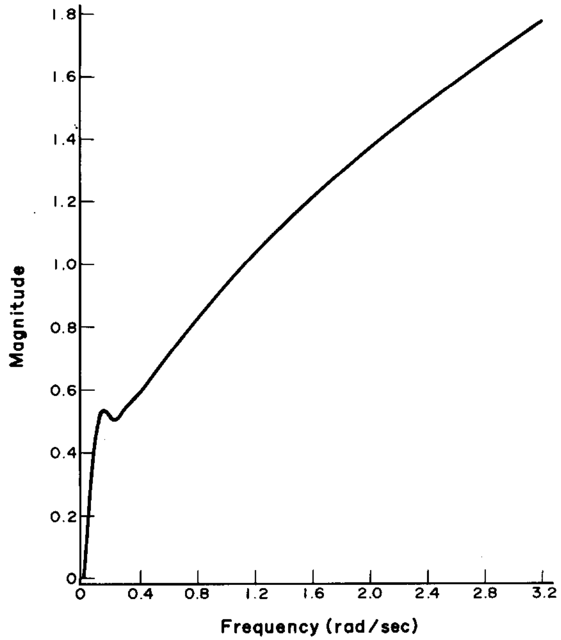

FIG. 4(a). Amplitude response of the approximation circuit for Sqrt.(*s*).

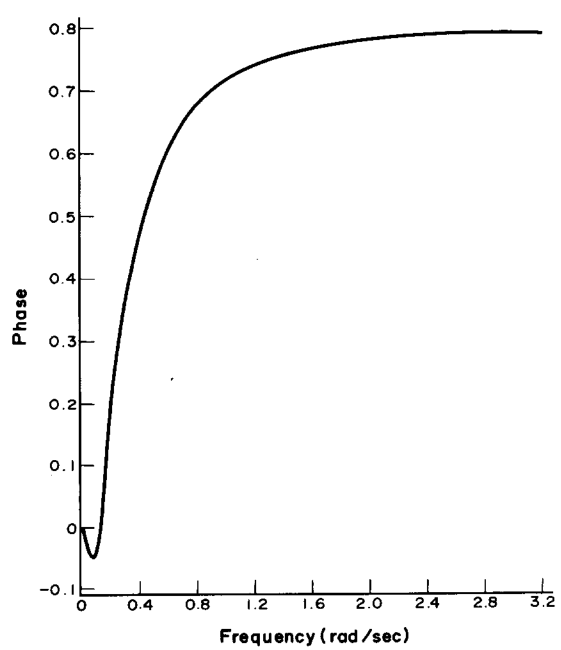

FIG. 4(b). Phase response of the approximation circuit for Sqrt.(*s*).

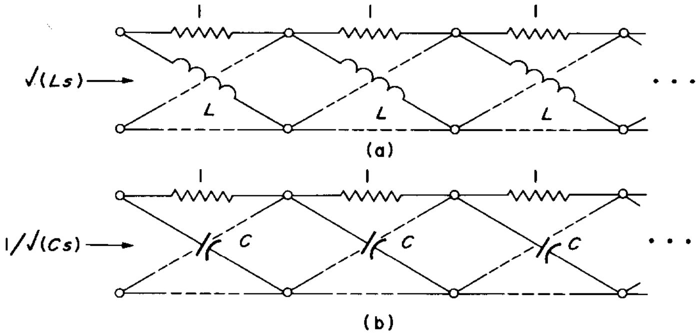

FIG. 5. An infinite cascade lattice realization of (a) the fractional operator $\sqrt{(Ls)}$ and (b) the fractional operator $1/\sqrt{(Cs)}$.

scheme for irrational numbers $\sqrt{a}$, where a is any positive number ($a > 0$). In fact, we have shown this in the following example.

Example

Let a be any positive number. Then the irrational number $\sqrt{a}$ can be approximated by the ratio of two rational numbers as

$$\sqrt{a} \simeq \left[\sum_{r=0}^{n}\binom{2n}{2r}a^r\right]\Big/\left[\sum_{r=0}^{n-1}\binom{2n}{2r+1}a^r\right] \tag{11}$$

where n is taken to be large enough.

Using the given algorithm, the irrational resistor $\sqrt{a}$ is realized as infinite cascades of symmetrical lattices with unit resistors in the parallel arms and rational resistors a in the cross-arms as illustrated in Fig. 6. This can be considered as a generalization and an alternative realization of a previously reported infinite ladder realization of $\sqrt{a}$, where $a > 1$ (**6**).

IV. Digital Simulation

In this section a signal-flow graph realization of the continued fraction expansion of the fractional operator $s^{\pm 1/2}$ is given. If digital simulations of these operators are desired, this realization can easily be transformed to the z-domain using the bilinear transformation:

$$s \rightarrow \frac{1-z^{-1}}{1+z^{-1}}. \tag{12}$$

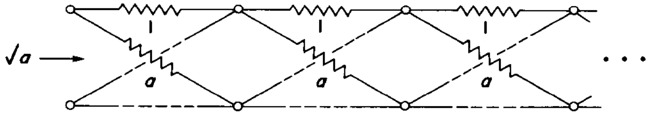

FIG. 6. Lattice realization of $\sqrt{a}$.

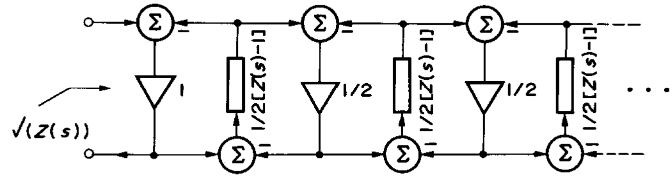

FIG. 7. Signal-flow graph representation of the continued fraction expansion given by Eq. (3).

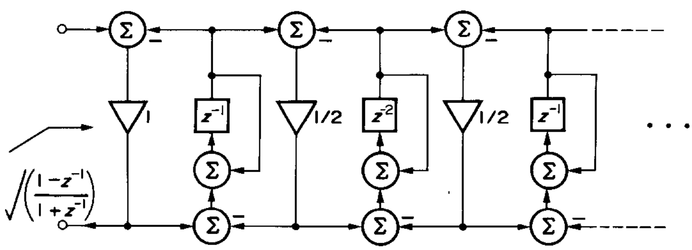

FIG. 8. Digital simulation of the fractional operator $\sqrt{s}$.

To obtain the proper realization, note that the irrational immittance $\sqrt{(Z(s))}$ can be expanded into a continued fraction expansion, as given by Eq. (3) and as shown in Fig. 7, which may be realized as a leapfrog structure using the feedback nature of the continued fraction expansion of Eq. (3). Setting $Z(s) = s$, and applying the bilinear transformation (12), the resulting digital ladder simulation of $\sqrt{s}$ is as shown in Fig. 8.

Similar results can be obtained for the digital simulation of the fractional operator $1/\sqrt{s}$. In fact, replacing z^{-1} by $-z^{-1}$ in the structure of Fig. 8 yields the desired realization. Obviously, when the continued fraction expansion (3) is truncated, a rational function approximation of the fractional operator $\sqrt{((1-z^{-1})/(1+z^{-1}))}$ is obtained which leads to a finite cascade ladder realization.

V. Physical Realization of the Fractional-step Delay Operator $z^{-1/2}$

It is well known that 2-D fan filters can be designed by transforming a 1-D reference digital filter into a 2-D filter using the index transformation reported in (4) as:

$$z = \sqrt{(z_1 z_2)}. \tag{13}$$

In general, this transformation will result in a 2-D z-transfer function with the variables z_1 and z_2 having rational non-integer powers. In the case of the above transformation, terms such as $z_1^{-1/2}$ and $z_2^{-1/2}$ will appear in the 2-D z-transfer function, which makes implementation of the derived filter in the first quadrant plane difficult. In this section, we extend the result of the previous section to realize the fractional-step delay operator $z^{-1/2}$ so as to obviate this difficulty.

Analogous to the continued fraction expansion of irrational functions discussed in Section II, note that the fractional operator $\sqrt{(G(z))}$ can be expressed as follows:

$$\begin{aligned} G(s) &= 1+\frac{G(z)-1}{1+\sqrt{(G(z))}} \\ &= 1+\cfrac{G(z)-1}{2+\cfrac{G(z)-1}{2+\cfrac{G(z)-1}{2+\cfrac{G(z)-1}{2+\ddots}}}}. \end{aligned} \tag{14}$$

Equation (14) can easily be implemented as an infinite cascade connection of symmetrical lattices as shown in Fig. 9. If $G(z)$ is set to be the unit delay z^{-1}, the lattice structure in Fig. 9 would result in physical realization of the fractional-step delay $z^{-1/2}$.

It is easily seen that a cascade connection of an infinite number of two-port lattice networks is not realistic. Let $G_n(z) = P_n/Q_n(z)$ denote the rational function obtained by truncating Eq. (14) at the $(n+1)$th lattice, so that $G_0(z) = 0$, and $G_1(z) = G(z)$. Then, we can write:

$$G_n(z) = \frac{G(z)+G_{n-1}(z)}{1+G_{n-1}(z)} = \frac{G(z)Q_{n-1}(z)+P_{n-1}(z)}{Q_{n-1}(z)+P_{n-1}(z)}. \tag{15}$$

Solving (15) for $P_n(z)$ and $Q_n(z)$, we obtain

$$P_n(z)-2P_{n-1}(z)+[1-G(z)]P_{n-2}(z) = 0 \tag{16a}$$

and

$$Q_n(z) = P_n(z). \tag{16b}$$

Solving these difference equations with the respective initial conditions $P_0(z) = 0$, $P_1(z) = G(z)$ and $Q_0(z) = 1$, $Q_1(z) = 1$, we obtain the following formula for the nth convergent:

$$G_n(z) = \frac{P_n(z)}{Q_n(z)} = \frac{\sum_{r=0}^{p}\binom{n}{2r+1}[G(z)]^{r+1}}{\sum_{r=0}^{q}\binom{n}{2r}[G(z)]^{r}} \quad \left| \begin{array}{l} p = q = \dfrac{n-1}{2} \quad \text{for} \quad n \quad \text{odd} \\ \text{and} \\ p = \dfrac{n}{2}-1, \\ q = \dfrac{n}{2} \quad \text{for} \quad n \quad \text{even.} \end{array} \right. \tag{17}$$

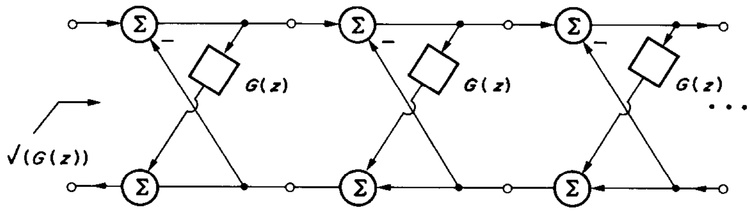

FIG. 9. Implementation of $[G(z)]^{1/2}$ using a cascaded lattice structure.

In our experience, we have obtained a good approximation of the fractional-step operator $\sqrt{(G(z))}$ with only a finite number of cascaded lattices. For example, letting $G(z) = z^{-1}$ with $n = 8$, from Eq. (17) we obtain

$$z^{-1/2} \simeq G_8(z) = \frac{\sum_{r=0}^{3} \binom{8}{2r+1} z^{-(r+1)}}{\sum_{r=0}^{4} \binom{8}{2r} z^{-r}} = \frac{8z^{-4}+56z^{-3}+56z^{-2}+8z^{-1}}{z^{-4}+18z^{-3}+70z^{-2}+28z^{-1}+1}. \quad (18)$$

Figure 10(a) shows the phase response of Eq. (18) obtained by cascading eight symmetrical lattices which is a very good approximation of $z^{-1/2}$. It should be noted that if the number of cascade sections is even, the phase response will be closer to that of $z^{-1/2}$ while the amplitude response will be an approximation to that of $z^{-1/2}$ which is an all-pass function. If, however, the number of sections chosen is an odd number, then the transfer function of the approximation will be an all-pass function which yields an amplitude response exactly the same as that of $z^{-1/2}$ while the phase response will be an approximation to that of $z^{-1/2}$. As an example, consider n to be equal to seven, then we have

$$z^{-1/2} G_7(z) = \frac{\sum_{r=0}^{3} \begin{bmatrix} 7 \\ 2r+1 \end{bmatrix} z^{-(r+1)}}{\sum_{r=0}^{3} \begin{bmatrix} 7 \\ 2r \end{bmatrix} z^{-r}} = \frac{7z^{-1}+35z^{-2}+21z^{-3}+z^{-4}}{1+21z^{-1}+35z^{-2}+7z^{-3}} \quad (19)$$

which is an all-pass function. Figure 11(a) and (b) shows the phase and amplitude response of $G_7(z)$, respectively. As can be seen from Fig. 11(a) and (b), the amplitude response of $G_7(z)$ is exactly the same as that of $z^{-1/2}$ while the phase response is an approximation to that of $z^{-1/2}$. Adding to the number of cascaded sections will improve the closeness of the approximation to that of $z^{-1/2}$. As an example, the amplitude and phase plots of fourteen cascades are depicted in Fig. 12(a) and (b) which clearly shows the closeness of the amplitude approximation to that of $z^{-1/2}$.

VI. Conclusions

We have presented a general approach to the realization of irrational functions based on a continued fraction expansion, which applies equally well to analog and

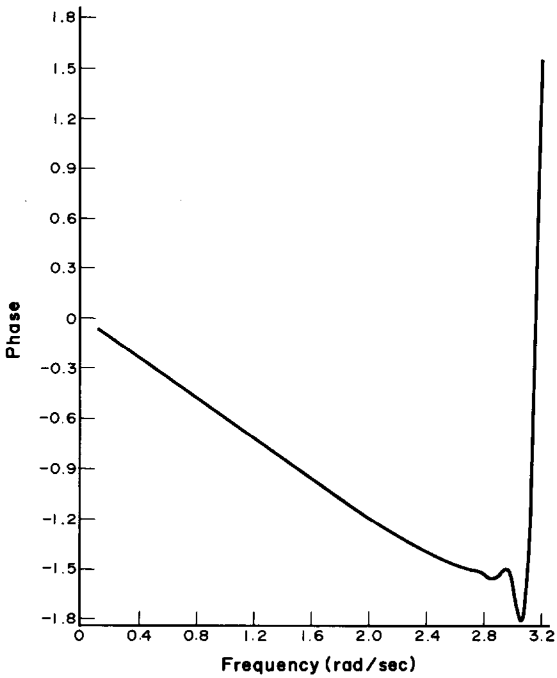

FIG. 10(a). Phase plot of the approximation circuit for 1/Sqrt.(z) with n even (n = 8).

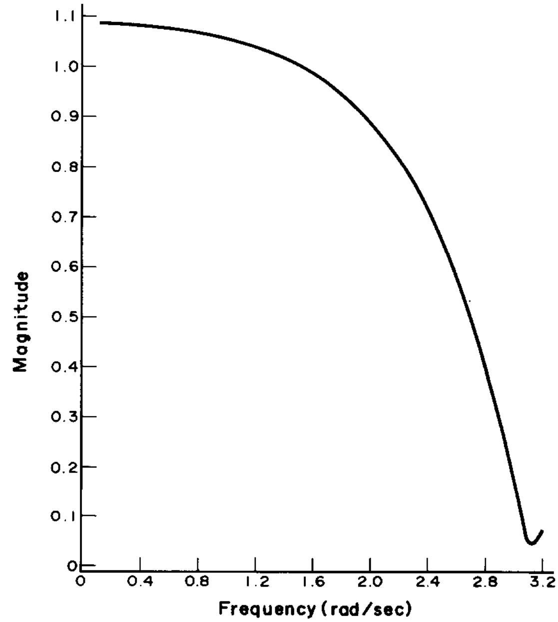

FIG. 10(b). Amplitude plot of the approximation circuit for 1/Sqrt.(z) with n even (n = 8).

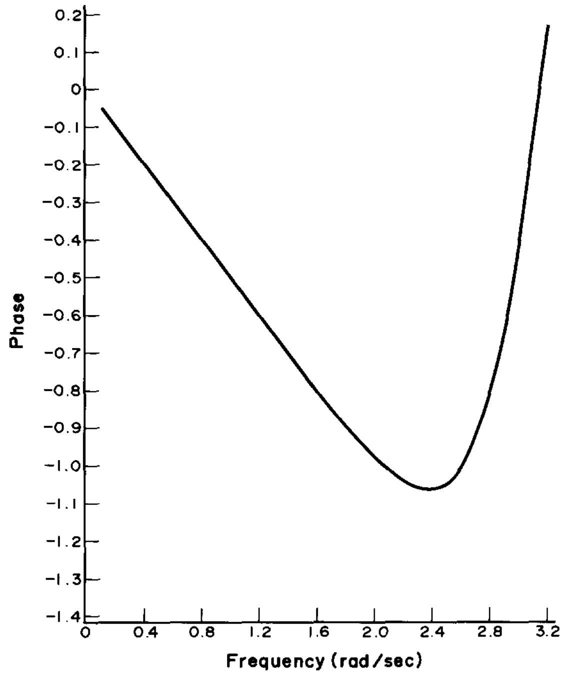

FIG. 11(a). Phase plot of the approximation circuit for 1/Sqrt.(z) with n odd ($n = 7$).

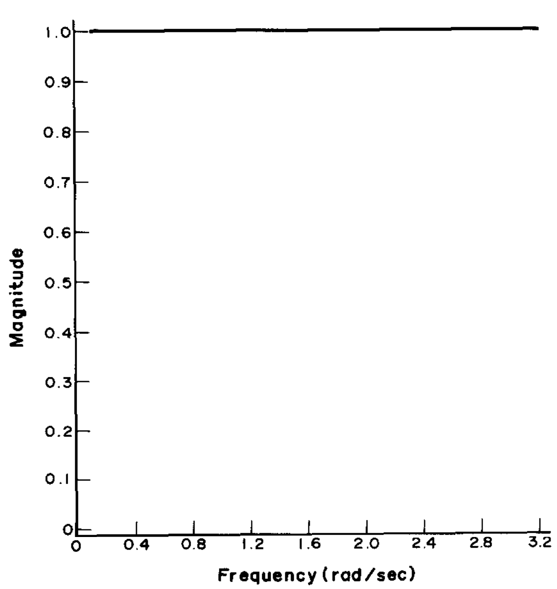

FIG. 11(b). Amplitude plot of the approximation circuit for 1/Sqrt.(z) with n odd ($n = 7$).

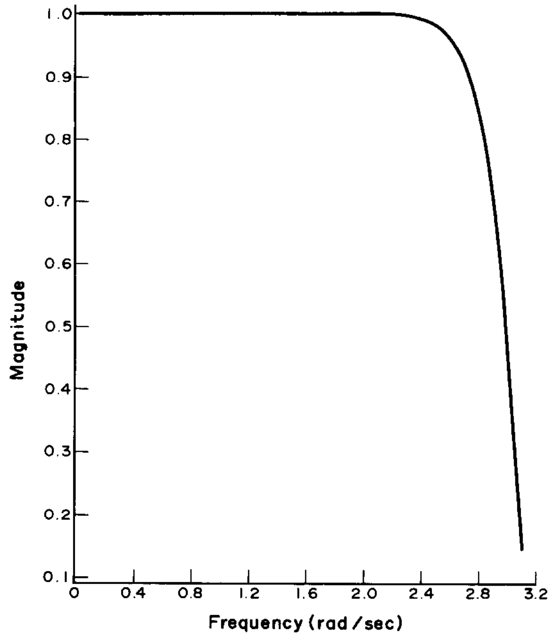

FIG. 12(a). Amplitude plot of approximation circuit for for 1/Sqrt.(z) with $n = 14$.

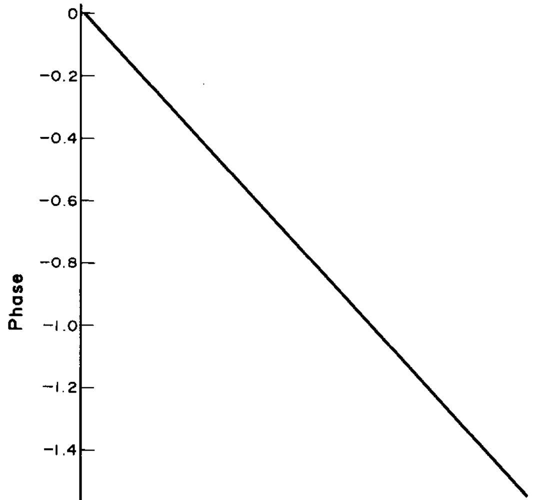

FIG. 12(b). Phase plot of approximation circuit for 1/Sqrt.(z) with $n = 14$.

digital filters. A simple lattice structure for the synthesis of $\sqrt{(Z(s))}$ is given. Applications of the proposed structure to the synthesis of special cases of the irrational operator $\sqrt{s}$, and the irrational immittances $\sqrt{(Ls)}$ and $1/\sqrt{(Cs)}$, and $\sqrt{a}$ (where $a > 0$), and the more general case of $Z(\sqrt{s})$ are also given. Explicit formulas for truncated realizations are developed. Also, a multiplierless lattice realization for the fractional-step delay operator $z^{-1/2}$ is presented. The significance of such fractional-step delays in the design of 2-D fan filters is evident. The main advantage of the proposed realization for the fractional-step operator $x^{-1/2}$ is that no sample-rate changes are required in the system, thereby yielding a computationally efficient system.

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