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SUMMABILITY METHODS FOR HERMITE FUNCTIONS

JOHN P. BOYD

Department of Atmospheric and Oceanic Science, University of Michigan, 2455 Hayward Avenue, Ann Arbor, MI 48109 (U.S.A.)

DENNIS W. MOORE

Joint Institute for Marine and Atmospheric Research (JIMAR), University of Hawaii, 1000 Pope Road, Honolulu, HI 98622 (U.S.A.)

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ABSTRACT

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Many problems in equatorial oceanography can be analytically solved via series of Hermite functions, but unfortunately these expansions converge very poorly. In this note, we describe two simple tricks for accurately evaluating such series. The first, due to Moore, is to apply numerical weighting factors to the last four terms in the series. This is a special case of a more powerful technique known as the 'Euler–Abel' method which is almost as easy to apply. Our numerical examples show that both methods are very effective. The Euler–Abel method gives an error which decreases *exponentially* fast as N, the number of terms retained in the truncated series, increases. Moore's method only reduced the error by a factor of $0(1/N^2)$ in comparison to the original series, but this is more than enough for most practical purposes, and this trick is simpler and distributes the error more uniformly in latitude than the Euler–Abel transformation. A conservative rule-of-thumb is that both methods give errors too small to observe on a graph on the range $|y| \leqslant (1/3)(2N+1)^{1/2}$ where N is the number of terms in the Hermite series.

1. INTRODUCTION

The normal modes of an equatorial ocean can be described as sums of one or two Hermite functions (Moore and Philander, 1976), so it is hardly surprising that infinite series of Hermite functions have played a major role in analytical theories of low-latitude dynamics. Many examples are given in the forthcoming monograph (Boyd, 1986).

Unfortunately, a serious problem with Hermite series is that whenever either the forcing decays algebraically with latitude or when there are zonal

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boundaries, the coefficients $\{a_n\}$ of the Hermite series decrease algebraically with n. Summability methods to improve the slow convergence of Hermite series are therefore helpful and sometimes essential in obtaining useful results without including hundreds or even thousands of terms.

Moore (1968) had great success with an 'iterated averaging' process that we shall henceforth refer to simply as 'Moore's sum method'. Unfortunately, his technique is not described even in his thesis, so this note is its first appearance in print. Nonetheless, his device has been used by a number of others who learned the trick through word of mouth.

Moore's sum method has the advantage of great simplicity since it alters only the last four terms of the truncated series, but this implies the disadvantage that the improved sequence of partial sums still decreases only algebraically with n. For purposes of comparison, it is therefore useful to discuss a more complicated procedure, the 'Euler–Abel' method, which in principle converges *exponentially* with N, where N is the highest term retained in the truncated series.

In the rest of this note, we briefly describe these two methods in the next two sections, compare their numerical effectiveness for four representative Hermite series in section 4, explain the methods' limitations, and summarize our conclusions.

MOORE'S SUM METHOD: ITERATED AVERAGING

The authors have not been able to find any references to Moore's algorithm in the precise form in which he used it, but the basic idea is classical, and was stated succinctly by Morse and Feshbach (1953) in a discussion of the asymptotic series for the exponential integral. The key observation is that this expansion is an *alternating* series, i.e., successive terms are of opposite sign. An elementary theorem (Kaplan, 1952) states that the error in truncating an alternating series after N terms is bounded in absolute value by the first neglected term, $|A_{n+1}|$. Another way of stating the same theorem is that if we define the N-th partial sum via

$$S_N \equiv \sum_{n=0}^{N} A_n \tag{2.1}$$

then successive partial sums will alternately overshoot and undershoot the sum of the infinite series. It follows that a better sequence of approximations to the infinite sum can be obtained by the *averaged* partial sums T_N defined by

$$T_N \equiv (S_{N-1} + S_N)/2 = S_{N-1} + (1/2)A_N \tag{2.2}$$

This averaging trick is useful for a wide range of alternating series. For series of the form

$$A_n = (-1)^{n+1}[1/n^2 + O(1/n^3)] \tag{2.3}$$

one can easily show with a pocket calculator that for large N

$$|S - S_N| \sim (1/2)[1/N^2] + O(1/N^3) \tag{2.4}$$

so that weighting the last retained term in the series, as done in the sequence T_N, eliminates the dominant error term so that

$$|S - T_N| \sim O(1/N^3) \tag{2.5}$$

Thus, applying the averaging operation has increased the algebraic order of convergence by one so that the error is $O(1/N^3)$ instead of $O(1/N^2)$.

Morse and Feshbach (1953) discussed only a single application of this idea. If the sequence T_N is itself alternating, however, then further improvement to $O(1/N^4)$ can be obtained by averaging the averaged partial sums T_N, i.e., by approximating the infinite sum S by the new sequence $V_N \equiv (T_{N-1} + T_N)/2$. Moore's algorithm iterates the simple averaging four times to obtain

$$f(y) \approx M_N \tag{2.6}$$

where

$$M_N \equiv \sum_{n=0}^{N-4} a_n \psi_n(y) + \tfrac{15}{16} a_{N-3}\psi_{N-3}(y) + \tfrac{11}{16} a_{N-2}\psi_{N-2}(y) + (5/16) a_{N-1}\psi_{N-1}(y) + (1/16) a_N \psi_N(y) \tag{2.7}$$

where $\psi_n(y)$ is the n-th normalized Hermite function. When the series contains only Hermite functions of even or odd subscript, which is often the case, then the weight factors are applied to the last four *non-zero* terms of the Hermite series.

When applied to a series like $a_n = (-1)^{n+1}/n^2$, the four-times-averaged sequence of partial sums has an error $O(1/N^6)$, i.e., we have reduced the error by $O(1/N^4)$ in comparison to the original sum. Unfortunately, the coefficients a_n of most Hermite expansions have asymptotic expansions in inverse powers of the *square root* of n, so Moore's method normally improves the convergence of a Hermite series by $O(1/N^2)$. This is enough, however, to give quite impressive results as shown in section 5.

EULER–ABEL SUMMATION METHOD

It is possible to obtain a summability procedure which makes the error decrease *exponentially* rather than algebraically with N by using an approximation which weights *all* the terms of the truncated series, not just the last four. This trick is actually a combination of two separate ideas.

The first was popularized by Abel, who pointed out that even if the series for an infinite sum S converges slowly—or perhaps does not converge at all—the series for the 'extended sum'

$$S(r) \equiv \sum_{n=0}^{\infty} r^n A_n \tag{3.1}$$

must converge like a geometric series for all $|r| < 1$ provided that the coefficients A_n are bounded by some algebraic function of n. If the limit exists, then Abel showed that S is the limit of $S(r)$ as $r \to 1$.

As stated in mathematics texts like Hardy (1949), for example, Abel summation is a useful theoretical idea of no practical usefulness whatsoever. It becomes valuable, however, as soon as we recognize that if the coefficients A_n are *alternating*, then the algebraic decrease of the coefficients with n implies that $S(r)$ is singular at $r = -1$—but the sum we want is the value of S for the same absolute value of r, but the opposite sign. To compute $S(1)$, we can therefore use any one of a number of numerical techniques for evaluating a function on and beyond its circle of convergence. One possibility is to form Pade approximants in r (Bender and Orszag, 1978) and then evaluate them at $r = 1$. This should work very well and give exponential convergence in N.

A simpler procedure, however, is to use Euler's transformation, which is a change of variable that replaces r by a new variable ζ such that the singularity is moved to infinity in the ζ-plane. The coefficients of the transformed series can be given in symbolic form as follows. Define the averaging operator via

$$\delta A_n \equiv (A_n + A_{n+1})/2 \tag{3.2}$$

(3.3)

and the new variable ζ to be

$$\zeta = 2r/(1 + r) \tag{3.4}$$

Then the sum $S(r)$ can be expressed as a power series in ζ via ($\delta^\circ A_0 \equiv A_0$)

$$S(r) = [(2 - \zeta)/2] \sum_{n=0}^{\infty} [\delta^n A_0] \zeta^n \tag{3.5}$$

which at $r = 1$—the only value we are actually interested in—gives

$$S = (1/2) \sum_{n=0}^{\infty} \delta^n A_0 \tag{3.7}$$

Since the radius of convergence of (3.5) in ζ is > 1, it follows that (3.7) must converge like a geometric series. This in turn implies that the errors in the Euler partial sums defined by

$$E_N \equiv (1/2) \sum_{n=0}^{N} \delta^n A_0 \tag{3.8}$$

must decrease exponentially with N.

The coefficients in (3.8) can be easily computed by recursion. Initialize by setting

$$\alpha_n = A_n \qquad n = 0, 1, \ldots N \tag{3.9}$$

and then successively compute

$$j = 1, 2, \ldots N$$

$$\beta_n = (1/2)(\alpha_n + \alpha_{n+1}) \qquad n = 0, 1, \ldots N - j \tag{3.10}$$

$$\delta^j A_0 = \beta_0 \tag{3.11}$$

$$\alpha_n = \beta_n \qquad n = 0, 1, \ldots N - j \tag{3.12}$$

Proof of these results (with different notation) is given in Morse and Feshbach (1953).

The interested reader can easily show that E_3 is *identical* with Moore's sum method applied to the same four terms. This in turn implies that regardless of how many terms are kept in the series, Moore's trick is identical with Euler's method applied to the sum $(S - S_{N-4})$. From the opposite perspective, it seems likely, although we have not bothered with a rigorous proof, that the Abel–Euler method is equivalent to the averaging method when the number of iterations of the averaging is equal to the total number of terms retained in the truncated series. Thus, Moore's algorithm and the Euler–Abel procedure are very closely related.

In many applications, the Hermite series contain only terms of even degree (for functions symmetric about the equator) or only odd degree. In such cases, we set

$$A_n \equiv a_{2n} \psi_{2n}(y) \text{ [even] or } A_n \equiv a_{2n+1} \psi_{2n+1}(y) \text{ [odd]} \tag{3.13}$$

and apply the Euler–Abel method as described above.

The constant function $f(y) \equiv 1$ is a useful example because its Abel extension (3.1) is

$$f(y;r) = [2/(1+r)]^{1/2} e^{-(1/2) y^2 [(1-r)/(1+r)]} \tag{3.14}$$

(Cane and Sarachik, 1981). Applying Euler's transformation gives

$$f(y;\zeta[r]) = 2^{1/2} e^{-(1/2) y^2} \left\{ (1 - \zeta/2)^{1/2} e^{(1/2) y^2 \zeta} \right\} \tag{3.15}$$

The coefficients $\{a_n\}$ of $f(y)=1$ decrease as $n^{-1/4}$. One can show via the asymptotic expansions of the normalized Hermite functions (Boyd, 1986) that $\psi_{2n}(0)$ decreases as $n^{-1/4}$ also, so the expected accuracy of the ordinary partial sum S_N is $O(N^{-1/2})$. In startling contrast, the Euler sum of $f(y)$ is simply the power series expansion of (3.15) evaluated for $\zeta=1$. Since the only singularity of $f(y;\zeta)$ is at $\zeta=2$, it follows that, ignoring algebraic factors in N, each term in the series in ζ will be only half as large as its predecessor. Taking 35 terms of the Euler series will give a 10 decimal place accuracy for $y=0$!

4. NUMERICAL EXAMPLES

Table I lists the terms in the Hermite series for $f(y)=1$ at $y=1$ along with the terms of the corresponding Euler series and the errors in the partial sums S_N of the original series, the partial sums E_N of the Euler series, and the sequence M_N obtained via Moore's approximation. (Note that since this

TABLE I

Series coefficients and errors for the Hermite expansion of $f(y)\equiv 1$ for $y=1$. The second column gives the terms of the direct sum, $A_N=a_{2n}\psi_{2n}(1)$. The third column gives the coefficients of the Euler transformed series (without the overall multiplicative factor of 1/2). The fourth, fifth, and sixth columns give the absolute errors in the partial sums S_N, the Euler partial sums E_N, and Moore's approximations M_N.

n	A_n	$\delta^n A_0$	e_S	e_E	E_M
0	0.8578	0.8578	0.1422	0.5711	–
1	0.4289	0.6433	−0.2866	0.2495	–
2	−0.5361	0.2949	0.2495	0.1020	–
3	0.4110	0.1184	−0.1616	0.0428	0.0428
4	−0.2301	0.0482	0.0685	0.0188	−0.0053
5	0.0574	0.0206	0.0111	0.0085	0.0017
6	0.0817	0.0091	−0.0706	0.0039	−7.33E-4
7	−0.1793	0.0042	0.1087	0.0018	3.58E-4
8	0.2362	0.0019	−0.1274	8.55E-4	−1.81E-4
9	−0.2573	9.00E-4	0.1299	4.05E-4	8.36E-5
10	0.2496	4.24E-4	−0.1197	1.92E-4	−3.06E-5
11	−0.2202	2.01E-4	0.1005	9.18E-5	−1.84E-6
12	0.1762	9.61E-5	−0.0756	4.38E-5	1.81E-5
13	−0.1236	4.60E-5	0.0480	2.08E-5	−2.84E-5
14	0.0679	2.21E-5	−0.0199	9.70E-6	3.08E-5
15	−0.0132	1.07E-5	−0.0067	4.37E-6	−3.26E-5
16	−0.0371	5.15E-6	0.0305	1.79E-6	2.96E-5
17	0.0809	2.50E-6	−0.0505	5.45E-7	−2.81E-5
18	−0.1165	1.21E-6	0.0661	−6.25E-8	2.32E-5

function is symmetric about the equator, S_N includes all Hermite functions up to and including $\psi_{2N}(y)$.) Because we have moved away from the equator, the original series is not strictly alternating, but the sign does change every one or two terms, and both methods give tremendous improvement over direct summation. As expected, the terms of the Euler series are decreasing like $N^{-1/2}\,(1/2)^N$.

The lowest Moore approximation, M_3, is equal to E_3, but as N increases, the Moore terms oscillate about the correct answer, which explains why additional iterations of the averaging process, as embodied in the Abel–Euler methods, improve the convergence still further for *large* N. The qualifier 'large N' has to be added because Moore's method gives *smaller* errors than the Euler method for $3 < N < 12$. The relationship between the two summation methods is reminiscent of Aesop's fable about the race between the tortoise and the hare: the averaging method starts out faster but is always overtaken in the end by the Euler–Abel algorithm as $N \to \infty$. The slow decrease of the error for Moore's algorithm for $N > 12$ shows that the expected asymptotic behavior does eventually occur.

We also examined the known Hermite series of the steady zonal current u, latitudinal flow v, and pressure p of the so-called 'Yoshida' jet (Yoshida, 1959; Moore and Philander, 1977). These three fields decrease for large $|y|$ as $1/y^4$, $1/y$, and $1/y^2$, respectively, and one can prove that their Hermite coefficients a_n decrease algebraically with n as $n^{-9/4}$, $O(n^{-3/4})$, and $O(n^{-5/4})$, respectively (Boyd, 1984). Figure 1 shows that for u, which decays most rapidly for $|y|$, it matters little whether a summability method is even

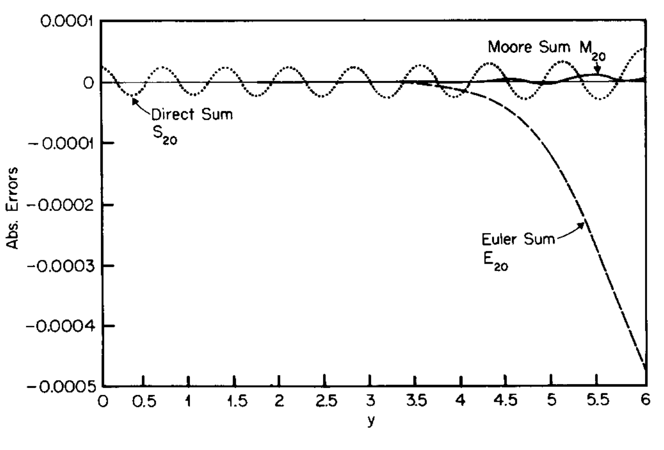

Fig. 1. The errors in the partial sums S_{20} (dotted line), Euler–Abel partial sums E_{20} (dashed line), and Moore's sequence M_{20} (solid line) for the zonal velocity u of the steady component of the 'Yoshida jet'. (Note that because $u(y)$ is symmetric about the equator, only the even Hermite terms up to and including $\psi_{40}\,(y)$ are retained in the truncation.)

used. Both the Moore and Abel–Euler algorithms do give much better accuracy; for small $|y|$, the error with $N = 20$ is less than the thickness of the curve for both. The ordinary partial sum S_{40}, however, has an error of < 0.00005 and taking just the lowest term (that proportional to $\psi_0[y]$) in fact gives an absolute error which is only 7% of the maximum value of the function.

One striking feature of the graph is that the error of the Abel–Euler method grows very rapidly for $|y| > 4$. We can understand similar behavior for the Euler sums of $f(y) = 1$ (not illustrated) by looking at (3.15): the Euler transformation involves a factor of $\exp[(1/2)y^2\zeta]$. As $|y|$ increases, it is clear that more and more terms of the power series in ζ—more terms in the Euler-transformed series for $f(y)$—are needed to obtain an accurate representation of the function.

This flaw of non-uniformity in y, however, is actually a property of *all* Hermite expansions including both the original series and the Moore sequence formed from it. The reason is that the n-th Hermite function decays exponentially (like $\exp[-(1/2)Y^2]$) beyond the 'turning points' given by $|y_t| = (2n+1)^{1/2}$. It follows that if we truncate the series after N terms and then consider a value of y outside the turning points of all the modes included in the sum, we cannot possibly obtain a good approximation to a function which is only slowly decreasing with $|y|$. Although barely visible in Fig. 1, the ripples in the error curves for the partial sums and for Moore's sequence grow with y, too, but at a slower rate than for the Euler error.

The price that must be paid for the greater uniformity in y is poorer accuracy for small y. Figure 2 compares the errors for $N = 20$ (terms

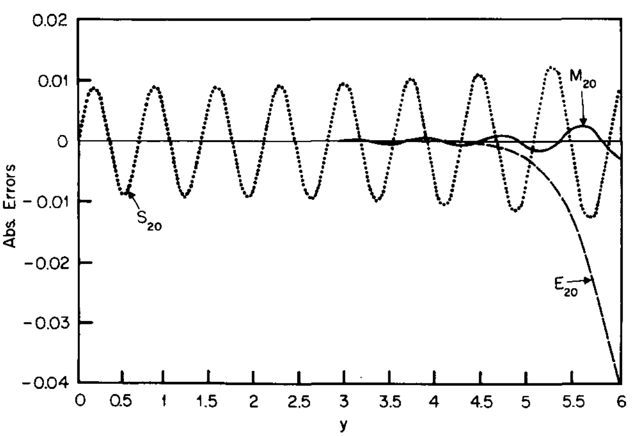

Fig. 2. The errors in S_{20} (dotted line), E_{20} (dashed line), and M_{20} (solid line) for the north–south steady flow in the Yoshida jet for $y \leqslant 6$.

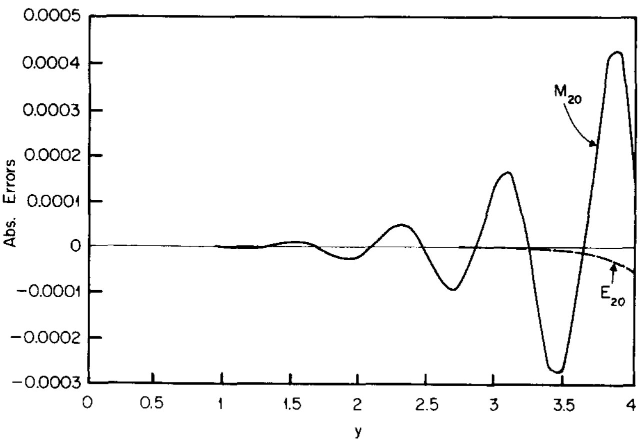

Fig. 3. The errors in E_{20} (dashed line) and M_{20} (solid line) for the north–south velocity v as in Fig. 2, but on the smaller interval $y \leqslant 4$.

through $\psi_{41}(y)$) on the interval $|y| \leqslant 6$ for the north–south velocity v, which is the most slowly converging series of Yoshida's solution. Figure 3 compares just the two summability methods on the smaller interval $|y| \leqslant 4$. The first graph shows that the ordinary partial sums give an error which is quite uniformly distributed in y over the interval shown—but the approximation is poor. The two summability methods, as shown in Fig. 3, give a superb approximation for small y (note the change in error scale from Fig. 2), but the errors grow rapidly for large y. Moore's method does not give the *exponential* divergence with $|y|$ which is so evident for the Euler sum in Fig. 2; the growth is slower for Moore's trick. It also does not give the exponential accuracy of the Euler algorithm; Fig. 3 shows that the Euler approximation is indistinguishable from the exact answer to within the thickness of the curve over most of the interval whereas the error in the Moore sequence is noticeable for $|y|$ as small as 1.5, and has become very large on the scale of Fig. 3 for $y = 4$.

5. LIMITATIONS OF SUMMABILITY METHODS AND ALTERNATIVES

Both Moore's sequence and the Euler sum work best for strictly alternating series; when applied to a series whose terms are all positive, they invariably increase rather than decrease the error. However, as noted by Wimp (1981): 'Only weak methods (Cesaro summability, for example [which gives an error no better than $O(1/N)$ for *any* sum] are regular for large classes of sequences'. In other words, no useful summability method works for all possible sums. However, Moore's method (and therefore its generalization, the Euler–Abel transformation) have never been known to fail in equatorial oceanography. This is as much as one can ever hope to say.

The acid test, of course, is to evaluate the same sum with different truncations and to accept the result only when the two closely agree. This is not only necessary because of the remote possibility that the summation method may not improve convergence, but also because Hermite expansions —however evaluated—do have this inherent nonuniformity in y. It is also impossible to predict in advance exactly what value of N is the minimum needed to give acceptable accuracy, so comparison of results for different N is essential.

Anderson (1973) discussed an alternative to both summability methods: taking a running mean in y. The reason that this trick—which operates in coordinate space rather than in the space of Hermite coefficients—is also successful is evident in Fig. 2: the error in the partial sums is an oscillatory function in y with a wavenumber approximately equal to $(2N+1)^{1/2}$, which is the local wavenumber of the N-th Hermite function. The running mean therefore damps out these wiggles in y quite effectively. Another perspective is provided by noting that a running mean or any form of artificial diffusion will most strongly damp the highest coefficients in a Fourier series of Hermite expansion while leaving the lower coefficients almost unchanged. Moore's method, which reduces the amplitude of the coefficients of just the four Hermite functions of highest degree, is clearly something along the same line.

In general, however, the summability methods are better because the theory behind them is more rigorous and they do not introduce an artificial damping or smoothing, but instead are simply a short-cut to the *exact* sum of the infinite series. Since viscosity is included in most numerical models of the ocean, however, it is important to note that it will perform much the same function as the non-dissipative algorithms discussed here: eliminating the small scale ripples.

6. CONCLUSIONS

In this note, we have compared two summability methods for obtaining smooth, accurate solutions from slowly converging Hermite series. Moore's method, which weights the last four terms of the truncated series, is simple to apply and very effective. For both methods (and for the partial sums of the original series, too), the error grows rapidly with $|y|$, and both procedures give useless results when $|y|$ is too large in comparison to N, the number of Hermite terms retained in the truncation. The second advantage of Moore's method is that its error grows much more slowly with y than that for the Abel–Euler method.

The alternative procedure, which is based upon applying Euler's transformation of a series to the Abel extension of the original sum, is more complicated, but the coefficients of the new series can be easily calculated through a double DO loop as described in section 3. The strength of the Abel–Euler method is that the error decreases *exponentially* rather than algebraically fast with N as N increases for fixed y. The weakness is that Moore's method is more uniform in y.

With the cautions noted in section 5—one should always reevaluate the sum with a different number of terms and compare—we offer an approximate rule-of-thumb. As noted earlier, the Hermite functions become exponential rather than oscillatory for $|y| \geqslant (2N+1)^{1/2}$. This gives an upper bound on the interval over which the partial sum up to ψ_N will give even a crude approximation. For $N = 40$, the largest Hermite function included in the sums in Table I and Fig. 1, $(2N+1)^{1/2} = 9$. Both summation methods are accurate to within 15% (Euler) and 1% (Moore) for $|y| \leqslant 6$, for the examples shown in Figs. 1–3. Because of the nonuniformity in y, however, the summation methods are at least an order-of-magnitude more accurate on the smaller interval $|y| \leqslant (1/3)\,(2N+1)^{1/2}$, especially the Euler procedure. We therefore express our advice as a pair of rules: (6.1) for optimists and (6.2) for the more cautious.

RULE-OF-THUMB: The Moore and Euler sums of the Hermite series up to and including ψ_N will be accurate for the range of y bounded by

$$(1)\ \ |y| \leqslant (2/3)(2N+1)^{1/2} \qquad \text{[errors of a few per cent]} \tag{6.1}$$

$$(2)\ \ |y| \leqslant (1/3)(2N+1)^{1/2} \qquad \text{[errors invisible on a graph]} \tag{6.2}$$

even for sums whose coefficients are decreasing as small negative powers of n.

We warn that we have not tested these guidelines for very large N. Since Moore's method gives fewer decimal places, but is at least crudely accurate for a wider range of y than Euler's method, one should probably use Moore's technique with (6.1) and the Euler algorithm with (6.2) or whenever high accuracy is more important than graphing a large range in latitude.

For both methods, N must increase as roughly the *square* of $|y|$ to maintain a good approximation at that latitude. Nonetheless, these summability methods make it possible, using only a moderate number of terms, to accurately evaluate analytical solutions for equatorial flows even when the original Hermite series converges only conditionally.

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