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FUNCTIONS ON UNIVERSAL ALGEBRAS

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In the classifying topos for any variety $\mathbb{V}$ of algebras, all functions into the universal $\mathbb{V}$-algebra U from its finite powers U^n are given by polynomials.

We use the usual terminology of universal algebra [2] except that we use the word 'universal' in the sense of category theory and that we do not require functionally complete algebras to be finite. If $\mathbb{V}$ is any variety of algebras and A is any algebra in $\mathbb{V}$, then we write $A[X_1, \ldots, X_n]$ or $A[X]$ for the algebra of polynomials over A in n variables X_i, i.e., the coproduct in $\mathbb{V}$ of A and n copies of the free $\mathbb{V}$-algebra F on one generator. Each element of $A[X]$ can be interpreted as an n-ary function on A by substitution of n-tuples from A for the n-tuple X of variables. If every function $A^n \to A$ arises in this way, i.e., if the function

$$A[X] \to A^{(A^n)}$$

exponentially adjoint to the substitution operation

$$A[X] \times A^n \to A$$

is surjective, then A is said to be *functionally complete*. These concepts can be interpreted internally in any topos, and we shall use them in the classifying topos for $\mathbb{V}$-algebras.

Theorem. *Let $\mathbb{V}$ be any variety of algebras, $\mathscr{E}$ the classifying topos for $\mathbb{V}$-algebras, and U the universal $\mathbb{V}$-algebra in $\mathscr{E}$. Then U is functionally complete in $\mathscr{E}$.*

Proof. We begin by recalling some well-known facts. First, the classifying topos $\mathscr{E}$ for $\mathbb{V}$-algebras is the category $\mathscr{S}^{\mathscr{C}}$ of set-valued functors on the category $\mathscr{C}$ of finitely presented $\mathbb{V}$-algebras, and the universal $\mathbb{V}$-algebra U is the underlying-set functor, also describable as the functor represented by the free $\mathbb{V}$-algebra F on one generator, on $\mathscr{C}$ with the evident $\mathbb{V}$-algebra structure [4, Theorem 5.21]. Second, the inverse-image functors of geometric morphisms preserve $\mathbb{V}$-algebras, coproducts of $\mathbb{V}$-algebras and F [4, Lemma 5.8], so they commute (up to natural isomorphism) with the functors adjoining indeterminates $A \mapsto A[X]$. In particular, this second fact can be applied to the evaluation functor $\mathcal{S}^{\mathscr{C}} \to \mathcal{S}$: $A \mapsto A(C)$ for any object C of $\mathscr{C}$; it yields that

$$U[X](C) \cong U(C)[X] = C[X].$$

(naturally in C).

We compute $U^{(U^n)}$ and compare it with $U[X]$. We have the following natural isomorphisms:

$$\begin{aligned} U^{(U^n)}(C) &\cong \mathcal{S}^{\mathscr{C}}(\mathscr{C}(C,-),\ U^{(U^n)}) \\ &\cong \mathcal{S}^{\mathscr{C}}(\mathscr{C}(C,-)\times U^n,\ U) \\ &\cong \mathcal{S}^{\mathscr{C}}(\mathscr{C}(C,-)\times(\mathscr{C}(F,-))^n,\ U) \\ &\cong \mathcal{S}^{\mathscr{C}}(\mathscr{C}(C[X],-),\ U) \\ &\cong U(C[X]) \cong C[X]. \end{aligned}$$

Thus, the functors $U[X]$ and $U^{(U^n)}$ are naturally isomorphic. Straightforward but tedious chasing of definitions and natural equivalences shows that this isomorphism is the usual conversion of a polynomial in n variables into an n-ary function, so U is functionally complete. □

Remarks. (1) The proof actually establishes more than the theorem asserts, for it shows that $U[X] \to U^{(U^n)}$ is not merely an epimorphism but an isomorphism in $\mathscr{E}$. In other words, every n-ary function on U is given by a *unique* polynomial.

(2) The proof works in somewhat greater generality than asserted. Instead of a variety of algebras, we could have used the class of models of any universal Horn theory. It is shown in [1, Theorem 1] that the classifying topos for such a theory and the universal model can be described just as for varieties of algebras. (In particular, the classifying topos is a functor category $\mathcal{S}^{\mathscr{C}}$ rather than a sheaf subcategory.) This observation suffices to transfer the proof to the more general situation.

(3) For a non-trivial variety $\mathbb{V}$, the universal algebra U is, of course, non-trivial, i.e., not isomorphic to 1, for any isomorphism $U \cong 1$ would be preserved by all inverse-image functors and would thus imply, by universality, that all $\mathbb{V}$-algebras are trivial. However, since U satisfies

(1) $$\neg\ \exists x\, \exists y\ \neg(x=y)$$

in the internal logic of $\mathscr{E}$, it seems dangerously close to being a trivial algebra, which makes the theorem seem less interesting. It may therefore be reassuring to notice that U is, in another sense, not very small, for it satisfies, internally,

$$\neg\ \exists x_1 \cdots \exists x_k\ \forall y\ (y=x_1 \text{ or } y=x_2 \text{ or } \cdots \text{ or } y=x_k)$$

for all k. (This is why we deviated from the usual definition [2, p. 176] of functional completeness by not requiring finiteness.)

Examples. (1) Let $\mathbb{V}$ be defined by no operations and no equations, so $\mathbb{V}$-algebras are just sets. To freely adjoin n indeterminates means simply to adjoin n distinct new points. The theorem asserts that, internally in the object classifier topos, the only n-ary functions from the universal object U to itself are the projections and the constants.

(2) By a *combinatory algebra* we mean a set with a binary operation (written as juxtaposition and thought of as application of a function to an argument) and two constants, S and K (thought of as the combinators usually given those names in combinatory logic [3, Chapter 2]), subject to the equations

$$(Kx)y = x \quad \text{and} \quad ((Sx)y)z = (xz)(yz).$$

These equations are designed to ensure that every polynomial in one variable X over a combinatory algebra A has the form aX for some $a \in A$. Indeed, (see [3, Chapter 2]) for the polynomial X we can take $a = (SK)K$, for constant polynomials $b \in A$ (including $b = S$ or K) we can take $a = Kb$, and for pq, where p and q are already of the desired from $p = bX$ and $q = cX$, we can take $a = (Sb)c$. It follows, by induction on n, that any polynomial over A in n variables $X_1, \ldots, X_n$ has the form $(\cdots((aX_1)X_2)\cdots X_{n-1})X_n$. If we think of the operation in a combinatory algebra as application, so that each element a of the algebra is viewed as a function $x \mapsto ax$, whose values are again viewed as functions so that a acts as a multiplace function, then the preceding arguments say that all polynomial functions on such an algebra are elements of the algebra. These arguments can be formulated in any topos, in particular in the classifying topos for combinatory algebras, where they combine with the theorem to tell us that all n-ary functions from (the underlying object of) the universal combinatory algebra U into itself are given by elements of U. Thus, U behaves like the intended interpretation of untyped lambda calculus: a collection of objects, which serve simultaneously as functions, arguments, and values, such that every mapping from this collection to itself is an element of the collection. The existence of such a collection was shown to be consistent with intuitionistic set theory by D. Scott [5], using a more complicated construction than the present one. His result says that intuitionistic set theory does not demand the existence of any more functions than combinatory logic does.

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