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On the Structure of Contraction Operators with Applications to Invariant Subspaces

B. CHEVREAU

U. E. R. de Mathématiques, Université de Bordeaux I, 33405 Talence, France

AND

C. PEARCY

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

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1. INTRODUCTION

In the past few years operator theorists have been studying the problem of solving systems of simultaneous equations of a particular type in the predual of certain dual operator algebras, and the knowledge gained thereby has led to significant advances in those areas of operator theory concerned with invariant subspaces, dilation theory, and reflexivity. (See the bibliography for a partial list of pertinent articles. For a more extensive bibliography, see [8].) In particular, the theory of the class $\mathbb{A}_{\aleph_0}$ (to be defined below) has been quite successful, in the sense that several rather general sufficient conditions for membership in the class have been obtained [2, 3, 8], and, moreover, quite a lot of information (concerning, in particular, dilation theory and invariant-subspace lattices) about operators in $\mathbb{A}_{\aleph_0}$ has been found [6, 5, 8]).

In this paper, which is most naturally regarded as a continuation of the sequence [6, 5, 2, 3, 8], we improve some sufficient conditions for membership in $\mathbb{A}_{\aleph_0}$ from [2, 8], and in so doing, we obtain as easy corollaries some new sufficient conditions for a contraction on Hilbert space whose spectrum contains the unit circle to have nontrivial invariant subspaces (see Sect. 4).

The notation and terminology employed herein agree with that in [8] and the sequence of papers listed above. Nevertheless we begin by reviewing a few pertinent definitions and important earlier results. Let $\mathscr{H}$ be a

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separable, infinite dimensional, complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. If $T \in \mathscr{L}(\mathscr{H})$, the spectrum of T will be denoted by $\sigma(T)$, and the essential (Calkin) spectrum of T by $\sigma_e(T)$. It is well known (cf. [16, p. 40]) that $\mathscr{L}(\mathscr{H})$ is the dual space of the Banach space (and ideal) $\mathscr{C}_1(\mathscr{H})$ of trace-class operators on $\mathscr{H}$ equipped with the trace norm $\| \ \|_1$. This duality is implemented by the bilinear functional

$$\langle T, L \rangle = \operatorname{tr}(TL), \qquad T \in \mathscr{L}(\mathscr{H}),\ L \in \mathscr{C}_1(\mathscr{H}).$$

A subalgebra $\mathfrak{A}$ of $\mathscr{L}(\mathscr{H})$ that contains $1_{\mathscr{H}}$ and is closed in the weak* topology on $\mathscr{L}(\mathscr{H})$ is called a *dual algebra*. It follows from general principles (cf. [13]) that if $\mathfrak{A}$ is a dual algebra, then $\mathfrak{A}$ can be identified with the dual space of $Q_{\mathfrak{A}} = \mathscr{C}_1(\mathscr{H})/{}^{\perp}\mathfrak{A}$, where ${}^{\perp}\mathfrak{A}$ is the preannihilator in $\mathscr{C}_1(\mathscr{H})$ of $\mathfrak{A}$, under the pairing

$$\langle T, [L] \rangle = \operatorname{tr}(TL), \qquad T \in \mathfrak{A},\ [L] \in Q_{\mathfrak{A}}. \tag{1}$$

(Here and throughout the paper we write $[L]_{\mathfrak{A}}$, or simply $[L]$, where no confusion will result, for the coset in $Q_{\mathfrak{A}}$ containing the operator $L \in \mathscr{C}_1(\mathscr{H})$.) It is also easy to see (cf. [13]) that the weak* topology that accrues to $\mathfrak{A}$ by virtue of being the dual space of $Q_{\mathfrak{A}}$ is identical with the relative weak* topology that $\mathfrak{A}$ inherits as a subspace of $\mathscr{L}(\mathscr{H})$.

If x and y are vectors in $\mathscr{H}$, then the associated rank-one operator $x \otimes y$, defined as usual by $(x \otimes y)(u) = (u, y)\, x$, $u \in \mathscr{H}$, belongs to $\mathscr{C}_1(\mathscr{H})$ and satisfies

$$\operatorname{tr}(x \otimes y) = (x, y). \tag{2}$$

Thus if $\mathfrak{A}$ is a given dual subalgebra of $\mathscr{L}(\mathscr{H})$, $[x \otimes y] \in Q_{\mathfrak{A}}$. As is well known, every operator L in $\mathscr{C}_1(\mathscr{H})$ can be written as $L = \sum_{i=1}^{\infty} x_i \otimes y_i$ for certain square-summable sequences $\{x_i\}$ and $\{y_i\}$ (with convergence in the norm $\| \ \|_1$), and it follows easily that every element of $Q_{\mathfrak{A}}$ has the form $\sum_{i=1}^{\infty} [x_i \otimes y_i]$. A dual algebra $\mathfrak{A} \subset \mathscr{L}(\mathscr{H})$ is said to have property $(\mathbb{A}_1)$ if for every element $[L]$ of $Q_{\mathfrak{A}}$ there exist vectors x and y in $\mathscr{H}$ satisfying $[L] = [x \otimes y]$. More generally, if n is any nonzero cardinal number not exceeding $\aleph_0$, and if for every doubly indexed family $\{[L_{ij}]\}_{0 \leqslant i,j < n}$ of elements of $Q_{\mathfrak{A}}$ there exists a pair of sequences $\{x_i\}_{0 \leqslant i < n}$ and $\{y_j\}_{0 \leqslant j < n}$ of vectors from $\mathscr{H}$ such that

$$[L_{ij}] = [x_i \otimes y_j], \qquad 0 \leqslant i, j < n,$$

then $\mathfrak{A}$ is said to have property $(\mathbb{A}_n)$.

Let $\mathbb{N}$ denote the set of positive integers, let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$, and let $\mathbb{T} = \partial\mathbb{D}$. A set $\Lambda \subset \mathbb{D}$ is said to be *dominating* for $\mathbb{T}$ if almost every point of $\mathbb{T}$ is a nontangential limit of a sequence of points from Λ. The spaces $L^p = L^p(\mathbb{T})$ and $H^p = H^p(\mathbb{T})$, $1 \leqslant p \leqslant \infty$, are the usual function spaces.

If T is an absolutely continuous contraction in $\mathscr{L}(\mathscr{H})$ (i.e., a contraction whose maximal unitary direct summand is either absolutely continuous or acts on the space (0)), we denote by $\mathcal{A}_T$ the dual algebra generated by T and we write Q_T for the predual $Q_{\mathcal{A}_T}$. (If $[L] \in Q_T$, we will write $[L]_T$ for $[L]_{\mathcal{A}_T}$ when there is a possibiity of confusion.) For such T, as is well known (cf. [22, p. 114]), the Sz.-Nagy–Foias functional calculus Φ_T is a weak* continuous, norm decreasing, algebra homomorphism of H^∞ onto a weak* dense subalgebra of $\mathcal{A}_T$, and we define the class $\mathbb{A} = \mathbb{A}(\mathscr{H})$ to be the set of all absolutely continuous contractions T for which Φ_T is an isometry of H^∞ onto $\mathcal{A}_T$. If $T \in \mathbb{A}(\mathscr{H})$, then one knows (cf. [13]) that Φ_T is a weak* homomorphism between H^∞ and $\mathcal{A}_T$ and that there exists a linear isometry ϕ_T of $Q_T = Q_{\mathcal{A}_T}$ onto L^1/H^1_0 (the predual of H^∞) such that $\phi_T^* = \Phi_T$. When $T \in \mathbb{A}$, the pair of spaces $\{\mathcal{A}_T, Q_T\}$ can be identified with the pair $\{H^\infty, L^1/H^1_0\}$ via the pair of isometries $\{\Phi_T, \phi_T\}$ (for more detail see [8]). We recall also (cf. [8, Proposition 4.6]) that if $T \in \mathbb{A}$, then $\sigma(T) \supset \mathbb{T}$.

If n is any cardinal number satisfying $1 \leqslant n \leqslant \aleph_0$, we define the class $\mathbb{A}_n$ to consist of all those T in $\mathbb{A}$ for which the dual algebra $\mathcal{A}_T$ has property $(\mathbb{A}_n)$. In particular, then, $\mathbb{A}_{\aleph_0}$, which is the central object of study in this paper, consists of all T in $\mathbb{A}$ such that $\mathcal{A}_T$ has property $(\mathbb{A}_{\aleph_0})$.

We turn now to review briefly some earlier results concerning this class. Recall that $C_{0\cdot}$ (resp. $C_{\cdot 0}$) is defined to be the class of all (completely non-unitary) contractions T acting on some Hilbert space of dimension at most $\aleph_0$ with the property that the sequence $\{T^n\}$ (resp. $\{T^{*n}\}$) converges to 0 in the strong operator topology, and C_{00} is defined as $C_{0\cdot} \cap C_{\cdot 0}$. In [6] a very useful dilation theory was developed for operators T belonging to some class $\mathbb{A}_n$, and in [20] it was shown that the (BCP)-operators (first studied in [13]) belong to the class $\mathbb{A}_{\aleph_0}$. Furthermore, in [5] it was proved that if $T \in \mathbb{A}_{\aleph_0}$, then T is reflexive (and thus has a huge lattice of invariant subspaces), and $\mathcal{A}_T$ is closed in the weak operator topology (cf. [8] for definitions). A better result about reflexivity was obtained later in [9]. In [8] it was shown that $\bigcap_{n=1}^{\infty} \mathbb{A}_n = \mathbb{A}_{\aleph_0}$, and in [2] the important formula $\mathbb{A}_1 \cap C_{00} = \mathbb{A}_{\aleph_0} \cap C_{00}$ was established. Also in [2] some additional sufficient conditions for membership in $\mathbb{A}_{\aleph_0}$ were found, two of which we enumerate here because of their pertinence to what follows.

THEOREM 1.1. *Suppose $T \in C_{00}(\mathscr{H})$ and there exists a set $\Lambda \in \mathbb{D}$ dominating for $\mathbb{T}$ such that each point of Λ belongs either to $\sigma_e(T)$ or to the derived set of $\sigma(T)$. Then $T \in \mathbb{A}_{\aleph_0}$.*

THEOREM 1.2. *Suppose* $T \in C_{00} \cap \mathbb{A}$ *and the (relative) weak operator topology coincides with the weak* topology on* $\mathcal{A}_T$. *Then* $T \in \mathbb{A}_{\aleph_0}$.

The main idea of this paper is as follows: It is well known that the invariant subspace problem for operators T on Hilbert space reduces to the case that $T \in C_{0\cdot}$. Furthermore, if $\sigma(T) \supset \mathbb{T}$, then by a theorem of Apostol [1], one may further assume that $T \in \mathbb{A}$. Therefore we establish some sufficient conditions for contractions T belonging to $C_{0\cdot} \cap \mathbb{A}$ and satisfying $\sigma(T) \supset \mathbb{T}$ to belong to $\mathbb{A}_{\aleph_0}$, similar to Theorems 1.1 and 1.2, and as easy corollaries we obtain some new invariant-subspace theorems for contractions T satisfying $\sigma(T) \supset \mathbb{T}$.

The main results of this paper were announced in [15].

2. PRELIMINARIES

In this section we first recall some pertinent definitions and terminology that will be useful in establishing our new sufficient conditions for membership in $\mathbb{A}_{\aleph_0}$. Next, for the reader's convenience, we state the main results from [8] that we shall need. Finally, we prove four original propositions, containing some new ideas, and thereby facilitate the proofs of our main results in Section 3.

We will suppose that the reader is familiar with the Fredholm theory for operators in $\mathcal{L}(\mathcal{H})$. In particular, we denote the set of Fredholm operators in $\mathcal{L}(\mathcal{H})$ by $\mathcal{F}(\mathcal{H})$, the set of semi-Fredholm operators in $\mathcal{L}(\mathcal{H})$ by $\mathcal{S}\mathcal{F}(\mathcal{H})$, and the continuous (Fredholm) index function (taking values in $\mathbb{Z} \cup \{+\infty, -\infty\}$) defined on $\mathcal{S}\mathcal{F}(\mathcal{H})$ by $i(\cdot)$. By a *hole* in a compact subset K of $\mathbb{C}$ we mean a bounded (connected) component of $\mathbb{C} \setminus K$. Thus if $T \in \mathcal{L}(\mathcal{H})$, then each hole H in $\sigma_e(T)$ is associated with a unique finite index $i(H)$, defined by choosing any λ in H and setting $i(H) = i(T - \lambda)$. (This results, of course, from the facts that $T - \lambda \in \mathcal{F}(\mathcal{H})$ if and only if $\lambda \notin \sigma_e(T)$, and $i(\cdot)$ is constant on connected sets.) If H is a hole in $\sigma_e(T)$ such that $i(H) \neq 0$, then, of course, $H \subset \sigma(T)$. On the other hand, if H is a hole in $\sigma_e(T)$ with $i(H) = 0$, then either $H \subset \sigma(T)$ or $H \cap \sigma(T)$ consists of a countable (possibly empty) set of isolated points (cf. [19, Chap. I]).

DEFINITION 2.1. For each T in $\mathcal{L}(\mathcal{H})$ we shall write $\mathcal{F}_-(T)$ (resp. $\mathcal{F}_+(T)$) for the (possibly empty) union of all holes in $\sigma_e(T)$ such that $i(H) \leqslant 0$ (resp. $i(H) \geqslant 0$) and $H \subset \sigma(T)$. Thus $\mathcal{F}_-(T) \cap \mathcal{F}_+(T)$ consists of the union of all holes in H in $\sigma_e(T)$ such that $i(H) = 0$ and $H \subset \sigma(T)$.

Concerning this circle of ideas we shall need the following lemma, which is no doubt known, and could be proved, for instance, as in the second paragraph of the proof of [8, Theorem 6.8]. We give a different short proof.

LEMMA 2.2. *Suppose* $T \in \mathscr{L}(\mathscr{H})$ *and* H *is a hole in* $\sigma_e(T)$ *such that* $i(H)=0$ *and* $H \subset \sigma(T)$. *Then, for every* $\lambda \in H$, *the sequence of subspaces* $\{\operatorname{Ker}(T-\lambda)^n\}_{n=1}^{\infty}$ *(as well, of course, as the sequence* $\{\operatorname{Ker}(T-\lambda)^{*n}\}$*) is strictly increasing.*

Proof. Suppose there exists $\lambda_0 \in H$ such that $\{\operatorname{Ker}(T-\lambda_0)^n\}$ is not strictly increasing. By translation we may assume that $\lambda_0=0$, and thus that $\operatorname{Ker} T^n = \operatorname{Ker} T^{n+1}$ for some $n \in \mathbb{N}$. To obtain a contradiction we will show that 0 is an isolated point of $\sigma(T)$. It is easy to see that $\operatorname{Ker} T^n = \operatorname{Ker} T^{n+k}$ for all $k \in \mathbb{N}$ and that $\operatorname{Ker} T^n \cap \operatorname{Ran} T^n = (0)$, so T^n is one-to-one on the invariant subspace $\operatorname{Ran} T^n$. Since $\dim(\operatorname{Ker} T^n) = \dim(\operatorname{Ker} T^{n*}) = \dim(\{\operatorname{Ran} T^n\}^{\perp})$ is finite, it follows that $\mathscr{H}$ is the topological direct sum $\mathscr{H} = \operatorname{Ker} T^n \dotplus \operatorname{Ran} T^n$ of the invariant subspaces $\operatorname{Ker} T^n$ and $\operatorname{Ran} T^n$ of T^n. Since $\operatorname{Ran} T^n = T^n\mathscr{H} = T^n(\operatorname{Ker} T^n \dotplus \operatorname{Ran} T^n) = T^n(\operatorname{Ran} T^n)$, it follows from the open mapping theorem that $T^n \mid (\operatorname{Ran} T^n)$ is an invertible operator on $\operatorname{Ran} T^n$. Thus for $|\lambda|>0$ and sufficiently small,

$$\begin{aligned} T^n - \lambda &= (T^n-\lambda) \mid \operatorname{Ker} T^n \dotplus (T^n-\lambda) \mid \operatorname{Ran} T^n \\ &= \lambda_{\operatorname{Ker} T^n} \dotplus \{(T^n \mid \operatorname{Ran} T^n) - \lambda_{\operatorname{Ran} T^n}\} \end{aligned}$$

is obviously an invertible operator (being a direct sum of invertible operators), and it follows immediately from the spectral mapping theorem that 0 is an isolated point of $\sigma(T)$, the contradiction we were seeking.

Suppose next that $T \in \mathscr{L}(\mathscr{H})$, $\mathscr{M}$ is an invariant subspace for T (notation: $\mathscr{M} \in \operatorname{Lat}(T)$), and $A = T \mid \mathscr{M}$. Then the operator $A \in \mathscr{L}(\mathscr{M})$ is called a *part* of T. If $\mathscr{M}, \mathscr{N} \in \operatorname{Lat}(T)$ with $\mathscr{M} \supset \mathscr{N}$, then the subspace $\mathscr{M} \ominus \mathscr{N}$ is called a *semi-invariant* subspace for T and the *compression* $T_{\mathscr{M} \ominus \mathscr{N}}$ of T to $\mathscr{M} \ominus \mathscr{N}$, defined by $T_{\mathscr{M} \ominus \mathscr{N}} = P_{\mathscr{M} \ominus \mathscr{N}} T \mid (\mathscr{M} \ominus \mathscr{N})$, where $P_{\mathscr{M} \ominus \mathscr{N}}$ is the (orthogonal) projection of $\mathscr{H}$ onto $\mathscr{M} \ominus \mathscr{N}$, is easily seen to satisfy $(T^n)_{\mathscr{M} \ominus \mathscr{N}} = (T_{\mathscr{M} \ominus \mathscr{N}})^n$ for every $n \in \mathbb{N}$. Furthermore, if $n \in \mathbb{N}$, we denote by $\mathscr{H}^{(n)}$ the Hilbert space consisting of the direct sum of n copies of $\mathscr{H}$ and by $T^{(n)}$ the n-fold *ampliation* of T acting on $\mathscr{H}^{(n)}$ defined by

$$T^{(n)}(x_1 \oplus \cdots \oplus x_n) = Tx_1 \oplus \cdots \oplus Tx_n.$$

Moreover, if $\mathcal{A} \subset \mathscr{L}(\mathscr{H})$ is a dual algebra, we denote by $\mathcal{A}^{(n)}$ the ampliated dual algebra acting on $\mathscr{H}^{(n)}$ defined by $\mathcal{A}^{(n)} = \{T^{(n)}: T \in \mathcal{A}\}$. That $\mathcal{A}^{(n)}$ is indeed a dual algebra on $\mathscr{H}^{(n)}$ follows from [8, Proposition 2.5]. For each T in $\mathscr{L}(\mathscr{H})$ it is clear that $(\mathcal{A}_T)^{(n)} = \mathcal{A}_{T^{(n)}}$.

Suppose now that $T \in \mathbb{A}(\mathscr{H})$, so the Sz.-Nagy–Foias functional calculus

$\Phi_T\colon H^\infty \to \mathcal{A}_T$ is an isometry and a weak* homeomorphism onto $\mathcal{A}_T$. If $\lambda \in \mathbb{D}$ and $p_\lambda(e^{it}) = (1-|\lambda|^2)\ |1-\bar{\lambda}e^{it}|^{-2}$, $e^{it} \in \mathbb{T}$, is the Poisson kernel function in L^1 corresponding to λ, then $[p_\lambda] \in L^1/H_0^1$, and we denote by $[C_\lambda]$ (or $[C_\lambda]_T$ when there is a possibility of confusion) the element of Q_T defined by $[C_\lambda] = \phi_T^{-1}[p_\lambda]$. Then we have

$$\langle h(T), [C_\lambda] \rangle = h(\lambda), \qquad h \in H^\infty,\ \lambda \in \mathbb{D}, \tag{3}$$

as is proved by the one-line calculation

$$\begin{aligned}\langle h(T), [C_\lambda] \rangle &= \langle \Phi_T(h), [C_\lambda] \rangle = \langle h, \phi_T[C_\lambda] \rangle \\ &= \langle h, [p_\lambda] \rangle = h(\lambda).\end{aligned}$$

Concerning this circle of ideas we will need the following lemmas.

LEMMA 2.3. *Suppose* $T \in \mathbb{A}(\mathscr{H})$, $\lambda \in \mathbb{D}$, $k \in \mathbb{N}$, *and* $[L] \in Q_{\mathcal{A}}$ *satisfies*

$$\langle 1_{\mathscr{H}}, [L] \rangle = 1 \qquad \textit{and} \qquad \langle (T-\lambda)^n, [L] \rangle = 0, \qquad n \in \mathbb{N}.$$

Then $[L] = [C_\lambda]$. *Moreover, if* $A = T_{\mathscr{M}}$ *is a compression of* T *to a semi-invariant subspace* $\mathscr{M}$, *and* e *is a unit vector in*

$$\begin{gathered}\operatorname{Ker}(A-\lambda)^{*k+1} \ominus \operatorname{Ker}(A-\lambda)^{*k} \\ (= \{\operatorname{Ran}(A-\lambda)^k\}^- \ominus \{\operatorname{Ran}(A-\lambda)^{k+1}\}^-),\end{gathered}$$

then $[e \otimes e]_T = [C_\lambda]_T$.

Proof. The polynomials in $(T-\lambda)$ are clearly weak* dense in $\mathcal{A}_T$. Since every element of Q_T corresponds to a unique weak* continuous linear functional on $\mathcal{A}_T$, and the functionals corresponding to $[L]$ and $[C_\lambda]$ agree on a weak* dense subset of $\mathcal{A}_T$, we have $[L] = [C_\lambda]$. To prove the second statement, observe that for all $j \in \mathbb{N}$, $(A-\lambda)^{*j}e$ belongs to $\operatorname{Ker}(A-\lambda)^{*k}$ and hence is orthogonal to e. Thus from (1) and (2) we obtain

$$\begin{aligned}\langle (T-\lambda)^j, [e \otimes e] \rangle &= \operatorname{tr}((T-\lambda)^j (e \otimes e)) = ((T-\lambda)^j e, e) \\ &= ((A-\lambda)^j e, e) = (e, (A-\lambda)^{*j} e) = 0, \qquad j \in \mathbb{N},\end{aligned}$$

and since

$$\langle 1_{\mathscr{H}}, [e \otimes e] \rangle = \operatorname{tr}(e \otimes e) = (e, e) = 1,$$

the result follows from what was already proved.

LEMMA 2.4. *Suppose* $T \in \mathbb{A}(\mathscr{H})$, $n \in \mathbb{N}$, *and* $\lambda \in \mathbb{D}$. *Then the mapping* Θ *defined by* $\Theta(S) = S^{(n)}$, $S \in \mathcal{A}_T$, *is a linear, isometric, weak* homeomorphism* *of $\mathfrak{A}_T$ onto $\mathfrak{A}_T{}^{(n)}$ that is the adjoint of a linear isometry θ of $Q_T{}^{(n)}$ onto Q_T. If $\tilde{x} = x_1 \oplus \cdots \oplus x_n$ and $\tilde{y} = y_1 \oplus \cdots \oplus y_n$ are vectors in $\mathscr{H}^{(n)}$, then $\theta([\tilde{x} \otimes \tilde{y}]_{T^{(n)}}) = \sum_{i=1}^{n} [x_i \otimes y_i]_T$, and $\theta([C_\lambda]_{T^{(n)}}) = [C_\lambda]_T$.*

Proof. Only the last statement is not contained in [8, Proposition 2.5], and it follows easily from Lemma 2.3..

We turn now to record some known sufficient conditions for membership in $\mathbb{A}_{\aleph_0}$ from [2, 6, and 8] that will be needed later.

PROPOSITION 2.5. *Let T be an absolutely continuous contraction in $\mathscr{L}(\mathscr{H})$, and suppose there exists an $n \in \mathbb{N}$ and an infinite dimensional semi-invariant subspace $\mathscr{K}$ for $T^{(n)}$ such that $(T^{(n)})_{\mathscr{K}} \in \mathbb{A}_{\aleph_0}(\mathscr{K})$. Then $T \in \mathbb{A}_{\aleph_0}$.*

Proof. That $T^{(n)} \in \mathbb{A}_{\aleph_0}(\mathscr{H}^{(n)})$ follows from [6, Proposition 4.11] and that $T \in \mathbb{A}_{\aleph_0}$ if (and only if) $T^{(n)} \in \mathbb{A}_{\aleph_0}$ is a consequence of [8, Theorems 3.8 and 6.3].

PROPOSITION 2.6. *Suppose $T \in \mathbb{A}(\mathscr{H})$. Then $T \in \mathbb{A}_{\aleph_0}$ if there exists a set $\Lambda \subset \mathbb{D}$ dominating for $\mathbb{T}$ such that for each $\lambda \in \Lambda$ one can find a sequence $\{x_n^\lambda\}_{n=1}^{\infty} = \{x_n\}$ is the closed unit ball of $\mathscr{H}$ satisfying*

(a) $\|[C_\lambda] - [x_n \otimes x_n]\|_{Q_T} \to 0$, *and*

(b) $\|[x_n \otimes w]\|_{Q_T} + \|[w \otimes x_n]\|_{Q_T} \to 0$, $w \in \mathscr{H}$.

Proof. If $T \in \mathbb{A}$ and satisfies (a) and (b) for λ in Λ, then by virtue of [8, Proposition 1.21], $\mathfrak{A}_T$ has property $X_{0,1}$ (cf. [8, Definition 2.8]), and thus by [8, Theorem 6.3], $T \in \mathbb{A}_{\aleph_0}$.

This proposition will be our fundamental tool used to prove that certain contractions belong to $\mathbb{A}_{\aleph_0}$. Lemma 2.3 already gives a hint as to how (a) might be satisfied, and the next proposition shows one way that (b) can be satisfied.

PROPOSITION 2.7. *Suppose $T \in \mathbb{A}(\mathscr{H})$ and there exists a semi-invariant subspace $\mathscr{M}$ for T such that the compression $T_{\mathscr{M}}$ belongs to $C_{0\cdot}$ (resp. $C_{\cdot 0}$). Then for any fixed w in $\mathscr{M}$ and sequence $\{x_n\}_{n=1}^{\infty}$ from $\mathscr{M}$ converging weakly to zero, we have*

$$\|[w \otimes x_n]\|_{Q_T} \to 0 \qquad (\text{resp. } \|[x_n \otimes w]\|_{Q_T} \to 0). \tag{4}$$

In particular, if $T_{\mathscr{M}} \in C_{00}$, then both sequences in (4) converge to zero.

Proof. An easy calculation shows that $\|[u \otimes v]\|_{Q_T} = \|[v \otimes u]\|_{Q_{T^*}}$ for all $u, v \in \mathscr{H}$, so it suffices to consider the case that $T_{\mathscr{M}} \in C_{0\cdot}$. We recall from [14, Theorem 7.2] that if $w \in \mathscr{M}$ and $\{k_n\}$ is any sequence of functions from H^∞ that is weak* convergent to zero, then $\|k_n(T_{\mathscr{M}})\, w\| \to 0$. By a corollary of the Hahn–Banach theorem, for each $n \in \mathbb{N}$ there exists an operator $B_n \in \mathfrak{A}_T$ of norm one such that $\|[w \otimes x_n]\|_{Q_T} = \langle B_n, [w \otimes x_n] \rangle$. Moreover, since $T \in \mathbb{A}$, each B_n has the form $h_n(T)$ for some h_n in H^∞ satisfying $\|h_n\| = 1$. Thus

$$\|[w \otimes x_n]\| = \langle h_n(T), [w \otimes x_n] \rangle = (h_n(T)\, w, x_n),$$

and since w and the x_n all belong to $\mathcal{M}$, we have

$$\|[w \otimes x_n]\| = (h_n(T_{\mathcal{M}})\, w, x_n), \qquad n \in \mathbb{N}.$$

Now if the sequence $\{\|[w \otimes x_n]\|\}$ does not converge to zero, then there exists a subsequence $\{\|[w \otimes x_{n_j}]\|\}$ and a number $l > 0$ such that $\|[w \otimes x_{n_j}]\| \geqslant l$ for all $j \in \mathbb{N}$. Since the unit ball in H^∞ is sequentially weak* compact, we may extract a subsequence $\{f_k = h_{n_{j_k}}\}$ of the sequence $\{h_{n_j}\}$ that is weak* convergent—say to f. Thus, upon setting $u_k = x_{n_{j_k}}$, we have

$$\|[w \otimes u_k]\| = ((f_k - f)(T_{\mathcal{M}})\, w, u_k) + (f(T_{\mathcal{M}})\, w, u_k). \tag{5}$$

Since the sequence $\{f_k - f\}$ is weak* convergent to zero, it follows from the above remark that the sequence $\{\|(f_k - f)(T_{\mathcal{M}})\, w\|\}$ tends to zero, and since $\{u_k\}$ converges weakly to zero, the right-hand side of (5) converges to zero as $k \to \infty$, contradicting $\|[w \otimes u_k]\| \geqslant l$ for all $k \in \mathbb{N}$. Thus the proof is complete.

PROPOSITION 2.8. *Suppose T is a contraction, $\varnothing \neq \Lambda \subset \mathbb{D}$, and $\varnothing \neq M \subset \mathbb{N}$. Then the restriction $T \mid \mathcal{M}$ of T to the invariant subspace $\mathcal{M} = \bigvee_{\lambda \in \Lambda, n \in M} \operatorname{Ker}(T - \lambda)^n$ belongs to C_0.*

Proof. Clearly $\mathcal{M} \in \operatorname{Lat}(T)$. Since $\mathcal{N} = \{x \in \mathcal{M} : \|(T \mid \mathcal{M})^n x\| \to 0\}$ is a subspace of $\mathcal{M}$, it clearly suffices to show that for any fixed $\lambda_0 \in \Lambda$ and $n_0 \in \mathbb{N}$, $\mathcal{K} = \operatorname{Ker}(T - \lambda_0)^{n_0} \subset \mathcal{N}$. But $T\mathcal{K} \subset \mathcal{K}$ and $(T \mid \mathcal{K} - \lambda_0)^{n_0} = 0$, so $\sigma(T \mid \mathcal{K}) = \{\lambda_0\}$. Thus the spectral radius of $T \mid \mathcal{K}$ is less than 1, so $\|(T \mid \mathcal{K})^n\| \to 0$ and $\mathcal{K} \subset \mathcal{N}$, as desired.

The last two propositions of this section move us a step closer to being able to employ Proposition 2.6.

PROPOSITION 2.9. *Suppose $T \in \mathbb{A}(\mathcal{H})$, A is a compression of T to an infinite dimensional semi-invariant subspace, and $\mathcal{J}$ is a nonempty subset of $\mathcal{F}_-(A)$. Then there exists a subspace $\mathcal{N}$ invariant for A^* (and hence semi-invariant for T) such that*

(a) *The compression $T_{\mathcal{N}} \in C_{\cdot 0}$, and*

(b) *For every $\lambda \in \mathcal{J}$ there exists an orthonormal sequence $\{e_n^\lambda\}_{n=1}^\infty$ in $\mathcal{N}$ satisfying $[C_\lambda]_T = [e_n^\lambda \otimes e_n^\lambda]_T$ for every $n \in \mathbb{N}$.*

Proof. Set $\mathcal{N} = \bigvee_{\lambda \in \mathcal{J}, n \in \mathbb{N}} \operatorname{Ker}(A-\lambda)^{*n}$; clearly $\mathcal{N}$ is invariant for A^* and thus semi-invariant for A and T. By Proposition 2.8, $A^* \mid \mathcal{N} \in C_{0\cdot}$, and hence its adjoint $(A^* \mid \mathcal{N})^* = A_{\mathcal{N}} = T_{\mathcal{N}} \in C_{\cdot 0}$. Now consider $\lambda \in \mathcal{J}$. If $i(A-\lambda) < 0$ we deduce easily from the definition of the index and the equality $i((A-\lambda)^n) = ni(A-\lambda)$ that the sequence $\{\operatorname{Ker}(A-\lambda)^{*n}\}$ is strictly increasing, and if $i(A-\lambda) = 0$ the same property holds by virtue of Lemma 2.2. Thus for each $\lambda \in \mathcal{J}$ there exists an orthonormal sequence $\{e_n^\lambda\}_{n=1}^\infty$ in $\mathcal{N}$ such that $e_n^\lambda \in \operatorname{Ker}(A-\lambda)^{*n+1} \ominus \operatorname{Ker}(A-\lambda)^{*n}$ for each $n \in \mathbb{N}$, and hence by Lemma 2.3 we have $[e_n^\lambda \otimes e_n^\lambda]_T = [C_\lambda]_T$ for each $n \in \mathbb{N}$, as desired.

The following proposition is an analog of Proposition 2.9, valid under the assumption that certain weak* continuous linear functionals on $\mathcal{A}_T$ are also continuous in the weak operator topology.

PROPOSITION 2.10. *Suppose $T \in \mathbb{A}(\mathcal{H})$, J is either $\mathbb{N}$ or some nonempty finite subset of $\mathbb{N}$, $\{H_j\}_{j \in J}$ is a collection of (distinct) holes in $\sigma(T)$, and $\{\lambda_j\}_{j \in J}$ is a sequence satisfying $\lambda_j \in H_j$, $j \in J$, and $\sum_{j \in J}(1-|\lambda_j|) < \infty$. Suppose also that there exists a summable sequence $\{\alpha_j\}_{j \in J}$ of nonzero complex numbers such that $\sum_{j \in J} \alpha_j [C_{\lambda_j}] = [u \otimes v]$ for some vectors u, v in $\mathcal{H}$. Then there exists a semi-invariant subspace $\mathcal{M}$ for T such that*

(a) $T_{\mathcal{M}} \in C_{\cdot 0}$, *and*

(b) *for every λ in $\bigcup_{j \in J} H_j$ there exists an orthonormal sequence $\{e_n^\lambda\}_{n=1}^\infty$ in $\mathcal{M}$ satisfying $[e_n^\lambda \otimes e_n^\lambda]_T = [C_\lambda]_T$ for each $n \in \mathbb{N}$.*

Proof. Observe that since each $\|[C_{\lambda_j}]\| = 1$ and $\sum_{j \in J} |\alpha_j| < \infty$, the element $[L] = \sum_{j \in J} \alpha_j [C_{\lambda_j}]$ is well defined in Q_T. Since $[L] = [u \otimes v]$, we conclude from (1), (2), and (3) that

$$(h(T)\,u, g(T)^*\,v) = \sum_{j \in J} \alpha_j\, g(\lambda_j)\, h(\lambda_j), \qquad g, h \in H^\infty. \tag{6}$$

We introduce now the Blaschke products $b(\xi)$ and $b_j(\xi)$, $j \in J$, in H^∞ defined by

$$b(\xi) = \prod_{k \in J} \frac{\bar{\lambda}_k}{\lambda_k} \left(\frac{\xi - \lambda_k}{1 - \bar{\lambda}_k \xi} \right),$$

$$b_j(\xi) = b(\xi) \Big/ \frac{\bar{\lambda}_j}{\lambda_j} \left(\frac{\xi - \lambda_j}{1 - \bar{\lambda}_j \xi} \right), \qquad j \in J,$$

where, of course, it is understood that if $\lambda_k = 0$, then the corresponding factor becomes ξ. (Recall that $J \neq \varnothing$ and these functions are well defined when $J = \mathbb{N}$ because of the condition $\sum_{j \in J}(1 - |\lambda_j|) < \infty$; cf. [17, p. 63].) From (6) we deduce immediately that

$$(u, b_j(T)^*\, v) = \alpha_j b_j(\lambda_j) \neq 0, \qquad j \in J, \tag{7}$$

and

$$((T-\lambda_j)^k u,\ b_j(T)^* v)=0, \qquad k\in\mathbb{N}, j\in J. \tag{8}$$

We now define

$$\mathscr{H}_1=\bigvee_{k\geqslant 0}(T-\lambda_1)^k u \qquad \left(=\bigvee_{k\geqslant 0}(T-\lambda_j)^k u,\quad j\in J\right), \tag{9}$$

and note that $\mathscr{H}_1\in\operatorname{Lat}(T)$. Furthermore, it is immediate from (7) and (8) that

$$(T-\lambda_j)\,\mathscr{H}_1 \subsetneqq \mathscr{H}_1, \qquad j\in J. \tag{10}$$

On the other hand, since $\lambda_j\notin\sigma(T)$, $(T-\lambda_j)$ is invertible, and hence bounded below on $\mathscr{H}_1$. This clearly implies (via (10)) that $\mathscr{H}_1$ is infinite dimensional, and (via (9)) that $T\mid\mathscr{H}_1-\lambda_j\in\mathscr{SF}(\mathscr{H}_1)$, $j\in J$, and satisfies $i(T\mid\mathscr{H}_1-\lambda_j)=-1$. Since for any $\lambda\in\bigcup_{j\in J}H_j$, $T-\lambda$ is invertible and hence $(T-\lambda)\mid\mathscr{H}_1=T\mid\mathscr{H}_1-\lambda\in\mathscr{SF}(\mathscr{H}_1)$, the continuity of the index gives us, upon defining $A=T\mid\mathscr{H}_1$, $i(A-\lambda)=-1$ for every $\lambda\in\bigcup_{j\in J}H_j$. Thus the desired conclusion follows from Proposition 2.9 with $\mathscr{J}=\bigcup_{j\in J}H_j$.

3. Membership in $\mathbb{A}_{\aleph_0}$

In this section we establish some new sufficient conditions for membership in $\mathbb{A}_{\aleph_0}$. Our results differ from most of those in [2; 8] in that the setting for our results is the class $C_{0\cdot}$ as opposed to the class C_{00}. Our first theorem contains a new idea—that of being able to "work piecewise."

THEOREM 3.1. *Suppose $T\in\mathbb{A}(\mathscr{H})$, $m\in\mathbb{N}$, $\Lambda\subset\mathbb{D}$ is dominating for $\mathbb{T}$ and can be written as $\Lambda=\bigcup_{1\leqslant i\leqslant m}\Lambda_i$, where $\Lambda_1\subset\sigma_e(T)$ and (in case $m>1$) for $i=2,\dots,m$, there exists a semi-invariant subspace $\mathscr{M}_i$ of T such that $T_{\mathscr{M}_i}\in C_{00}$ and for every $\lambda\in\Lambda_i$ there exists a sequence $\{x_n^\lambda\}_{n=1}^\infty$ of unit vectors in $\mathscr{M}_i$ converging weakly to zero and satisfying*

$$\|[C_\lambda]_T-[x_n^\lambda\otimes x_n^\lambda]_T\|\to 0. \tag{11}$$

Then $T\in\mathbb{A}_{\aleph_0}$.

Proof. Suppose first that $m=1$. Then $\sigma_e(T)\cap\mathbb{D}$ is dominating for $\mathbb{T}$, so by definition T is a (BCP)-operator (cf. [20]), and that $T\in\mathbb{A}_{\aleph_0}$ follows from [20] or [7]. If $m>1$, we note that Λ_1 may be void and we consider the ampliation $T^{(m)}$ acting on $\mathscr{H}^{(m)}$. It is easy to see that

$$\mathscr{K}=\mathscr{H}\oplus\mathscr{M}_2\oplus\cdots\oplus\mathscr{M}_m$$

is a semi-invariant subspace for $T^{(m)}$ whose compression $(T^{(m)})_{\mathscr{K}}$ is (unitarily equivalent to)

$$\tilde{T} = T \oplus T_{\mathscr{M}_2} \oplus \cdots \oplus T_{\mathscr{M}_m}.$$

Thus, according to Proposition 2.5, to show that $T \in \mathbb{A}_{\aleph_0}$ it suffices to show that $\tilde{T} \in \mathbb{A}_{\aleph_0}$. Moreover, since $\tilde{T}$ has T as a direct summand, $\tilde{T} \in \mathbb{A}$, and thus it will be enough to show that, for every $\lambda \in \Lambda$, there exists a sequence of vectors $\{x_n^\lambda\}_{n=1}^\infty$ in the unit ball of $\mathscr{K}$ satisfying (a) and (b) of Proposition 2.6 with respect to $\tilde{T}$ (for every $\tilde{w}$ in $\mathscr{K}$). If $\lambda \in \Lambda_1$ then $\lambda \in \sigma_e(\tilde{T})$, and thus by [20, Lemmas 3.2 and 3.4] such a sequence exists. If $\lambda \in \Lambda_i$ for $i > 1$, then by hypothesis there exists a sequence $\{x_n^\lambda\}$ of unit vectors in $\mathscr{M}_i$ converging weakly to zero such that (11) is satisfied. Consider now the sequence of vectors in $\mathscr{K}$ defined by

$$\tilde{x}_n^\lambda = \underbrace{0 \oplus \cdots \oplus x_n^\lambda}_{i} \oplus \cdots \oplus 0, \qquad n \in \mathbb{N}, \tag{12}$$

whose only nonzero components lie in the space $\mathscr{M}_i$. It is easily checked that the linear isometry $\phi_{T^{(m)}}^{-1} \cdot \phi_{\tilde{T}}$ of $Q_{\tilde{T}}$ onto $Q_{T^{(m)}}$ maps $[C_\lambda]_{\tilde{T}}$ to $[C_\lambda]_{T^{(m)}}$ and $[\tilde{x} \otimes \tilde{y}]_{\tilde{T}}$ to $[\tilde{x} \otimes \tilde{y}]_{T^{(m)}}$ for all λ in $\mathbb{D}$ and all $\tilde{x}$, $\tilde{y}$ in $\mathscr{K}$. (Use the facts $\phi_{\tilde{T}}^* = \Phi_{\tilde{T}}$, $\phi_{T^{(m)}}^* = \Phi_{T^{(m)}}$, $\mathscr{K}$ is a semi-invariant subspace, and Lemma 2.4.) Combining this with another application of Lemma 2.4, we have

$$\begin{aligned} \|[C_\lambda]_{\tilde{T}} - [\tilde{x}_n^\lambda \otimes \tilde{x}_n^\lambda]_{\tilde{T}}\| &= \|[C_\lambda]_{T^{(m)}} - [\tilde{x}_n^\lambda \otimes \tilde{x}_n^\lambda]_{T^{(m)}}\| \\ &= \|[C_\lambda]_T - [x_n^\lambda \otimes x_n^\lambda]_T\| \to 0, \end{aligned}$$

and, furthermore, for any $\tilde{w} = w_1 \oplus \cdots \oplus w_m$ in $\mathscr{K}$,

$$\begin{aligned} \|[\tilde{w} \otimes \tilde{x}_n^\lambda]_{\tilde{T}}\| + \|[\tilde{x}_n^\lambda \otimes \tilde{w}]_{\tilde{T}}\| &= \|[\tilde{w} \otimes \tilde{x}_n^\lambda]_{T^{(m)}}\| + \|[\tilde{x}_n^\lambda \otimes \tilde{w}]_{T^{(m)}}\| \\ &= \|[w_i \otimes x_n^\lambda]_T\| + \|[x_n^\lambda \otimes w_i]_T\| \to 0 \end{aligned}$$

by virtue of (12), Lemma 2.4, and Proposition 2.7. Thus the sequence $\{x_n^\lambda\}$ satisfies the appropriate versions of conditions (a) and (b) of Proposition 2.6, from which it follows that $\tilde{T} \in \mathbb{A}_{\aleph_0}$ as desired.

We are now ready to give more concrete criteria for membership in $\mathbb{A}_{\aleph_0}$. The first is of a purely spectral nature.

THEOREM 3.2. *Suppose $T \in C_{0\cdot}(\mathscr{H})$ and there exists an infinite dimensional semi-invariant subspace $\mathscr{M}$ (possibly $\mathscr{H}$) for T such that $(\sigma_e(T) \cap \mathbb{D}) \cup \mathscr{F}_-(T_{\mathscr{M}})$ is dominating for $\mathbb{T}$. Then $T \in \mathbb{A}_{\aleph_0}$.*

Proof. We first show that $T \in \mathbb{A}$, i.e., that $\|h(T)\| = \|h\|_\infty$ for every h in H^∞. Since $T \in C_{0\cdot}$, T is a completely nonunitary contraction, and thus Φ_T

is well defined. Set $\Lambda_1 = \sigma_e(T) \cap \mathbb{D}$ and $\Lambda_2 = \mathscr{F}_-(T_{\mathscr{M}})$. Since $\Lambda = \Lambda_1 \cup \Lambda_2$ is dominating for $\mathbb{T}$, to prove that $T \in \mathbb{A}$ it suffices to show (cf. [8, Definition 4.5] that

$$|h(\lambda)| \leqslant \|h(T)\|, \qquad \lambda \in \Lambda,\ h \in H^{\infty}. \tag{13}$$

Since $\Lambda_1 \subset \sigma(T) \cap \mathbb{D}$, by [14, Lemma 3.1] we have $h(\Lambda_1) \subset \sigma(h(T))$ for every h in H^{∞}, and thus (13) is satisfied for all λ in Λ_1. If $\lambda \in \Lambda_2$, there exists a unit vector x_λ in $\mathscr{M}$ such that $T^*_{\mathscr{M}} x_\lambda = \bar{\lambda} x_\lambda$. Thus

$$\begin{aligned} \|h(T)\| &\geqslant \|h(T)\, x_\lambda\| \geqslant |(h(T)\, x_\lambda, x_\lambda)| = |(h(T_{\mathscr{M}})\, x_\lambda, x_\lambda)| \\ &= |(x_\lambda, \tilde{h}(T^*_{\mathscr{M}})\, x_\lambda)| = |\tilde{h}(\bar{\lambda})| = |h(\lambda)|, \end{aligned}$$

which shows that (13) is satisfied for all λ in Λ_2. Therefore $T \in \mathbb{A}$ as asserted. Now let $\mathscr{M}_2$ be the semi-invariant subspace for T obtained via Proposition 2.9 with $A = T_{\mathscr{M}}$ and $\mathscr{J} = \Lambda_2$. Since T belongs to $C_{0\cdot}$, clearly the compression $T_{\mathscr{M}_2} \in C_{00}$, and that T belongs to $\mathbb{A}_{\aleph_0}$ now follows immediately from Proposition 2.9 and Theorem 3.1.

Remark 3.3. Of course Theorem 3.2 has a dual version corresponding to the case in which $T \in C_{\cdot 0}$ and there exists a semi-invariant subspace $\mathscr{M}$ for T such that $(\sigma_e(T) \cap \mathbb{D}) \cup \mathscr{F}_+(T_{\mathscr{M}})$ is dominating for $\mathbb{T}$ (Indeed, a semi-invariant subspace for an operator is also semi-invariant for its adjoint.) Furthermore, a special case of those theorems is the case in which $\mathscr{M} = \mathscr{H}$ and thus $T_{\mathscr{M}} = T$.

Remark 3.4. Theorem 3.2 should be compared with Theorem 1.1, noting that the open set $\mathscr{F}_-(T)$ is contained in the derived set of $\sigma(T)$. However, the familiar example of the unilateral backward shift of multiplicity one, which belongs to $C_{0\cdot} \cap \mathbb{A}_1$ but not to $\mathbb{A}_2$ ([6, Theorem 3.7]), shows that there are real obstructions to be dealt with in trying to replace the hypothesis "$T \in C_{00}$" in Theorem 1.1 by "$T \in C_{0\cdot}$."

The presence of "good hidden compressions" of an operator that would permit the application of Theorem 3.2 is not unusual, as is illustrated by the following example.

EXAMPLE 3.5. Suppose W_ω is a weighted bilateral shift of multiplicity one in $\mathscr{L}(\mathscr{H})$ with weight sequence $\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots)$ (see [8, Chap. 10] for definitions) such that $\|W_\omega\| = 1$ and $\sigma(W_\omega) = \mathbb{T}$. Then either $W_\omega \in \mathbb{A}_{\aleph_0}$ or W_ω is similar to the unweighted bilateral shift W_1 and belongs to $\mathbb{A}_1 \setminus \mathbb{A}_2$. The main ideas of the proof, most of which are contained in the proof of [8, Theorem 10.5], are as follows. Since $\sigma(W_\omega) = \mathbb{T}$, it follows easily from [21, Theorem 4] that all weights ω_i are nonzero, so W_ω is injective. Furthermore, if either of the infinite products $\prod_{i>0} \omega_i$ or $\prod_{i<0} \omega_i$ diverges to zero, then W_ω has a compression to a semi-invariant subspace that satisfies the hypotheses of Theorem 3.2 (or its dual version), and thus W_ω belongs to $\mathbb{A}_{\aleph_0}$. On the other hand, if neither of these infinite products equals zero, then W_ω is similar to W_1 [21, Theorem 2], and it is easy to see that $W_1 \in \mathbb{A}_1 \setminus \mathbb{A}_2$ from [8, Theorem 10.5 and Corollary 4.14].

THEOREM 3.6. *Suppose $T \in C_{0\cdot} \cap \mathbb{A}(\mathscr{H})$, J is either $\mathbb{N}$ or some nonempty finite subset of $\mathbb{N}$, $\{H_j\}_{j\in J}$ is a collection of holes in $\sigma(T)$, and there exists a sequence $\{\lambda_j\}_{j\in J}$ satisfying $\lambda_j \in H_j$, $j \in J$, and $\sum_{j\in J}(1-|\lambda_j|) < \infty$. Suppose also that there exists a summable sequence $\{\alpha_j\}_{j\in J}$ of nonzero complex numbers such that the linear functional induced on $\mathfrak{A}_T$ by the element $\sum_{j\in J}\alpha_j[C_{\lambda_j}]$ of Q_T is continuous in the weak operator topology. Finally, suppose that the set*

$$(\sigma_e(T) \cap \mathbb{D}) \cup \mathscr{F}_-(T) \cup \Big(\bigcup_{j\in J} H_j\Big) \tag{14}$$

is dominating for $\mathbb{T}$. Then $T \in \mathbb{A}_{\aleph_0}$.

Proof. Set $[L] = \sum_{j\in J}\alpha_j[C_{\lambda_j}]$. Since $[L]$ induces a linear functional on $\mathfrak{A}_T$ that is continuous in the weak operator topology, there exists an $m \in \mathbb{N}$ and vectors $x_1, \ldots, x_m, y_1, \ldots, y_m$ from $\mathscr{H}$ such that $[L] = \sum_{i=1}^m [x_i \otimes y_i]$ (cf. [8, Proposition 1.7]).

We consider the ampliation $T^{(m)}$, and define

$$\tilde{x} = x_1 \oplus \cdots \oplus x_m, \qquad \tilde{y} = y_1 \oplus \cdots \oplus y_m.$$

Then, by Lemma 2.4, $\theta^{-1}([L]) = [\tilde{x} \otimes \tilde{y}] \in Q_{T^{(m)}}$. Of course, by Proposition 2.5, it suffices to show that $T^{(m)} \in \mathbb{A}_{\aleph_0}$, and to accomplish this, via Theorem 3.1, we note several relations: $T^{(m)} \in C_{0\cdot} \cap \mathbb{A}$, $\sigma(T^{(m)}) = \sigma(T)$ (so the holes H_j, $j \in J$, are also holes in $\sigma(T^{(m)})$), $\mathscr{F}_-(T^{(m)}) = \mathscr{F}_-(T)$, and $\sigma_e(T^{(m)}) = \sigma_e(T)$. Thus

$$\Lambda_1 \cup \Lambda_2 \cup \Lambda_3 = (\sigma_e(T^{(m)}) \cap \mathbb{D}) \cup \Big(\bigcup_{j\in J} H_j\Big) \cup \mathscr{F}_-(T^{(m)}),$$

being equal to the set in (14), remains dominating for $\mathbb{T}$. To complete the proof, using Theorem 3.1, it suffices to exhibit semi-invariant subspaces $\mathscr{M}_2$ and $\mathscr{M}_3$ (in case $\Lambda_3 \neq \varnothing$) having the appropriate properties. By Lemma 2.4, $\sum_{j\in J}\alpha_j[C_{\lambda_j}]_{T^{(m)}} = [\tilde{x} \otimes \tilde{y}]_{T^{(m)}}$, so by Proposition 2.10 there exists a semi-invariant subspace $\mathscr{M}_2$ for $T^{(m)}$ satisfying $(T^{(m)})_{\mathscr{M}_2} \in C_{\cdot 0}$ and (b) of Proposition 2.10. Since $T^{(m)} \in C_{0\cdot}$, $(T^{(m)})_{\mathscr{M}_2} \in C_{00}$, so $\mathscr{M}_2$ satisfies the appropriate conditions of Theorem 3.1. If $\Lambda_3 = \mathscr{F}_-(T^{(m)})$ is void, then $\Lambda_1 \cup \Lambda_2$ is dominating for $\mathbb{T}$, and the result follows from Theorem 3.1. If $\Lambda_3 \neq \varnothing$, then there exists a semi-invariant subspace $\mathscr{M}_3$ for $T^{(m)}$ satisfying the appropriate conditions of Theorem 3.1 by Proposition 2.9, with $A = T^{(m)}$ and $\mathscr{J} = \Lambda_3$, so the proof is complete.

Our next result, which is an easy corollary of Theorem 3.6, shows that the hypothesis "$T \in C_{00}$" in Theorem 1.2 can be replaced by "$T \in C_{0\cdot}$," again modulo the addition of a condition of a spectral nature. For T in $\mathbb{A}$ we denote by $\mathscr{F}'_+(T)$ the (perhaps empty) union of all holes in H in $\sigma_e(T)$ such that $i(H) > 0$.

THEOREM 3.7. *Suppose $T \in C_{0\cdot} \cap \mathbb{A}(\mathscr{H})$, $\mathbb{D} \setminus \mathscr{F}'_+(T)$ is dominating for $\mathbb{T}$, and the weak* and (relative) weak operator topologies coincide on $\mathcal{A}_T$. Then $T \in \mathbb{A}_{\aleph_0}$.*

Proof. The main idea here is (*): since $\mathbb{D} \setminus \mathscr{F}'_+(T)$ is dominating for $\mathbb{T}$, one can construct a (possibly empty) collection $\{H_j\}_{j \in J}$ of holes in $\sigma(T)$ and a collection $\{\lambda_j\}_{j \in J}$ with $\lambda_j \in H_j$, $j \in J$, and $\sum_{j \in J} (1 - |\lambda_j|) < \infty$, such that the set in (14) is dominating for $\mathbb{T}$. Suppose this has been done. If $J = \varnothing$, then $T \in \mathbb{A}_{\aleph_0}$ by Theorem 3.2, and if $J \neq \varnothing$, let $\{\alpha_j\}_{j \in J}$ be any summable family of nonzero complex numbers. By hypothesis, the element $\sum_{j \in J} \alpha_j [C_{\lambda_j}]$ of Q_T induces a linear functional on $\mathcal{A}_T$ that is continuous in the weak operator topology, and the result then follows from Theorem 3.6. For the convenience of the reader, we now sketch a proof of (*). It is an easy consequence of spectral theory (cf. [19, Chap. I]) that $\mathbb{D} \setminus \mathscr{F}'_+(T)$ is the disjoint union

$$\mathbb{D} \setminus \mathscr{F}'_+(T) = (\sigma_e(T) \cap \mathbb{D}) \cup \mathscr{F}_-(T) \cup \rho_b(T) \cup \sigma_{if}(T), \tag{15}$$

where $\rho_b(T)$ is the union of the holes in $\sigma(T)$ and

$$\sigma_{if}(T) = \{\mu \in \sigma(T) \setminus (\sigma_e(T) \cup \mathscr{F}_-(T)) : i(T - \mu) = 0\}.$$

One knows that the isolated Fredholm spectrum $\sigma_{if}(T)$ of T consists at most of a countable set of isolated points λ of $\sigma(T)$ each of which has a punctured neighborhood contained in some hole H in $\sigma(T)$ (so $H \subset \rho_b(T)$). It thus follows from trivial geometric considerations and (15) that since $\mathbb{D} \setminus \mathscr{F}'_+(T)$ is dominating for $\mathbb{T}$, then so is

$$(\sigma_e(T) \cap \mathbb{D}) \cup \mathscr{F}_-(T) \cup \rho_b(T). \tag{16}$$

Next observe that if $U \subset \rho_b(T)$ is the union of all holes H in $\sigma(T)$ such that $\partial H \cap \mathbb{T} = \varnothing$, then since the set in (16) is dominating for $\mathbb{T}$, so is the set

$$(\sigma_e(T) \cap \mathbb{D}) \cup \mathscr{F}_-(T) \cup (\rho_b(T) \setminus U). \tag{17}$$

Since $\rho_b(T)$ is the union of at most countable many holes H in $\sigma(T)$, and all such holes $H \subset \rho_b(T) \setminus U$ satisfy $\partial H \cap \mathbb{T} \neq \varnothing$, it is easy to write

$$\rho_b(T) \setminus U = \bigcup_{j \in J} H_j \tag{18}$$

where (in case $J \neq \varnothing$) each H_j is a hole in $\sigma(T)$ and to choose a collection

$\{\lambda_j\}_{j\in J}$ of points satisfying $\lambda_j \in H_j$, $j\in J$, and $\sum_{j\in J}(1-|\lambda_j|)<\infty$. Putting together (17) and (18), we have now established (*), so the theorem is proved.

Remark 3.8. Of course there is a dual version of Theorem 3.7 obtained by replacing "$T\in C_{0\cdot}$" by "$T\in C_{\cdot 0}$" and the set $\mathscr{F}'_+(T)$ by its counterpart $\mathscr{F}'_-(T)$.

COROLLARY 3.9. *Suppose $T\in C_{0\cdot}\cap \mathbb{A}(\mathscr{H})$ (resp. $T\in C_{\cdot 0}\cap \mathbb{A}(\mathscr{H})$], the weak* and (relative) weak operator topologies coincide on $\mathfrak{A}_T$, and every hole H in $\sigma_e(T)$ with $i(H)>0$ (resp. $i(H)<0$) satisfies* $\operatorname{dist}(\partial H, \mathbb{T})>0$. *Then $T\in \mathbb{A}_{\aleph_0}$.*

It is well known (cf. [8, Theorem 6.3 and Proposition 2.09]) that if $T\in\mathbb{A}_{\aleph_0}$ then $\mathfrak{A}_T$ is closed in the weak operator topology and the weak* and weak operator topologies coincide on $\mathfrak{A}_T$. Thus it is natural to try to replace, in Theorem 3.7, the hypothesis that these two topologies coincide by the hypothesis that $\mathfrak{A}_T=\mathscr{W}_T$, where $\mathscr{W}_T$ is defined (for any $T\in\mathscr{L}(\mathscr{H})$) to be the closure of $\mathfrak{A}_T$ in the weak operator topology. Our next results are in this direction.

If $T\in\mathscr{L}(\mathscr{H})$, $[L]\in Q_T$, and there exists an $m\in\mathbb{N}$ with the properties that (a) there exist vectors $x_1,\dots,x_m$, $y_1,\dots,y_m$, in $\mathscr{H}$ satisfying $[L]=\sum_{i=1}^m [x_i\otimes y_i]$ and (b) there exists no smaller $m'\in\mathbb{N}$ with the corresponding property, then $[L]$ will be said to have *length m* (in Q_T). (Thus the zero element of Q_T has length one.) If, for a given $[L]$ in Q_T, there exists no such integer m, then $[L]$ will be said to have *infinite length.* It follows (cf. [8, Proposition 1.7]) that the elements of Q_T that induce linear functionals on $\mathfrak{A}_T$ that are continuous in the weak operator topology are exactly the elements of finite length.

The following two lemmas are of independent interest.

LEMMA 3.10. *Suppose $T\in\mathbb{A}(\mathscr{H})$ and H is a hole in $\sigma(T)$. Then either* (a) *there exists some $m\in\mathbb{N}$ such that for every $\lambda\in H$, $[C_\lambda]$ has length m (and hence induces a linear functional on $\mathfrak{A}_T$ that is continuous in the weak operator topology), or* (b) *for every $\lambda\in H$, $[C_\lambda]$ has infinite length (and the linear functional induced on $\mathfrak{A}_T$ by $[C_\lambda]$ fails to be continuous in the weak operator topology). Furthermore,* (a) *and* (b) *are equivalent, respectively, to* (a′) *for every (resp. some) $\lambda\in H$, $(T-\lambda)^{-1}\notin\mathscr{W}_T$, and* (b′) *for every (resp. some) $\lambda\in H$, $(T-\lambda)^{-1}\in\mathscr{W}_T$.*

Proof. If (b) is false, then there exists some $\lambda\in H$ such that $[C_\lambda]$ has finite length. Let $m\in\mathbb{N}$ be the minimum length of any element $[C_\lambda]$, as λ ranges over H, suppose that $[C_{\lambda_0}]$ has length m, and let

$$x_1,\dots,x_m,\qquad y_1,\dots,y_m,$$

be vectors in $\mathscr{H}$ satisfying $[C_{\lambda_0}]=\sum_{i=1}^m [x_i\otimes y_i]$. Now consider the ampliation $T^{(m)}$ of T, and define

$$\tilde{x}=x_1\oplus\cdots\oplus x_m,\qquad \tilde{y}=y_1\oplus\cdots\oplus y_m.$$

Then, by Lemma 2.4, $[C_{\lambda_0}]_{T^{(m)}}=[\tilde{x}\otimes\tilde{y}]_{T^{(m)}}$, and from Proposition 2.10, applied to $T^{(m)}$ with J a singleton, we see that for every $\lambda\in H$, $[C_\lambda]_{T^{(m)}}$ has length one. Applying Lemma 2.4 again, we learn that for every $\lambda\in H$. $[C_\lambda]_T$ has length at most m. But by the way m was defined, clearly every $[C_\lambda]$ must have length equal to m. Since elements of Q_T of finite length clearly induce linear functionals on $\mathcal{A}_T$ that are continuous in the weak operator topology ([10, Proposition 1.7]), the first statement of the lemma is proved. To prove the second statement, observe that for a fixed λ in H, the linear functional f_λ induced on $\mathcal{A}_T$ by $[C_\lambda]$ fails to be continuous in the weak operator topology if and only if $\operatorname{Ker}(f_\lambda)$ is dense in $\mathcal{A}_T$ in that topology, which happens if and only if $1_{\mathscr{H}}$ belongs to the closure of $\operatorname{Ker}(f_\lambda)$ in the weak operator topology. Since

$$\operatorname{Ker}(f_\lambda)=\{(T-\lambda)\,h(T): h\in H^\infty\},$$

we see that if $1_{\mathscr{H}}$ belongs to this closure, then there exists a net $\{h_\alpha(T)(T-\lambda)\}$ converging to $1_{\mathscr{H}}$ in the weak operator topology, and hence the net $\{h_\alpha(T)\}$ converges to $(T-\lambda)^{-1}$ in the same topology. Thus $(T-\lambda)^{-1}\in\mathscr{W}_T$. On the other hand, if $(T-\lambda)^{-1}\in\mathscr{W}_T$, then there exists a net $\{k_\alpha(T)\}$ contained in $\mathcal{A}_T$ and converging to $(T-\lambda)^{-1}$ in the weak operator topology. Thus the net $\{(T-\lambda)\,k_\alpha(T)\}$ converges to $1_{\mathscr{H}}$ in that topology, so $1_{\mathscr{H}}$ belongs to the closure of $\operatorname{Ker}(f_\lambda)$ in the weak operator topology. This clearly completes the proof.

LEMMA 3.11. *Suppose $T\in\mathbb{A}$ and $\mathcal{A}_T=\mathscr{W}_T$. If H is any hole in $\sigma(T)$, then there exists an integer $m\in\mathbb{N}$ such that for every $\lambda\in H$, $[C_\lambda]$ has length m (and hence induces a linear functional on $\mathcal{A}_T$ that is continuous in the weak operator topology).*

Proof. According to Lemma 3.10, it suffices to show that for some $\lambda_0\in H$, $(T-\lambda_0)^{-1}\notin\mathcal{A}_T=\mathscr{W}_T$. But if $(T-\lambda_0)^{-1}\in\mathcal{A}_T$, then, since $T\in\mathbb{A}$, there exists $h\in H^\infty$ such that $\Phi_T(h)=(T-\lambda_0)^{-1}$ and consequently $h(\xi)(\xi-\lambda_0)\equiv 1$, an obvious absurdity.

We can now establish some sufficient conditions for membership in $\mathbb{A}_{\aleph_0}$ for operators T in $\mathbb{A}$ that generate a dual algebra which is closed in the weak operator topology. The following theorem should be compared with [8, Theorem 6.10].

THEOREM 3.12. *Suppose $T\in C_{0\cdot}\cap\mathbb{A}(\mathscr{H})$ and $\mathcal{A}_T=\mathscr{W}_T$. Then $T\in\mathbb{A}_{\aleph_0}$ in each of the following cases:*

(a) *there exist a finite number of holes* $H_1, \ldots, H_k$ *in* $\sigma(T)$ *such that the set*

$$(\sigma_e(T) \cap \mathbb{D}) \cup \mathscr{F}_-(T) \cup \left(\bigcup_{1 \leqslant j \leqslant k} H_j \right)$$

is dominating for $\mathbb{T}$.

(b) *each hole in* $\sigma_e(T)$ *of positive index is at a positive distance from* $\mathbb{T}$ *and there are only finitely many holes* H *in* $\sigma(T)$ *such that the* (*Lebesgue*) *measure of* $\partial H \cap \mathbb{T}$ *is positive.*

Proof. Easy geometric considerations show that (b) is a particular case of (a), so it suffices to establish the theorem when T satisfies (a). In this situation, choose $\lambda_i \in H_i$, $i = 1, \ldots, k$, and set $[L] = \sum_{i=1}^{k} \alpha_i [C_{\lambda_i}]$, where the α_i are any nonzero complex numbers. By Lemma 3.11, each $[C_{\lambda_i}]$ induces a linear functional on $\mathcal{A}_T$ that is continuous in the weak operator topology, and thus the same is true of $[L]$. The result is now an immediate consequence of Theorem 3.6.

Of course, Theorem 3.12 has a dual version. Slight variations on this theme are the following.

COROLLARY 3.13. *Suppose* $T \in C_{\cdot 0} \cap \mathbb{A}$ *is a quasitriangular operator,* $\mathcal{A}_T = \mathscr{W}_T$, *and there are only finitely many holes* H *in* $\sigma(T)$ *such that the* (*Lebesgue*) *measure of* $\partial H \cap \mathbb{T} \neq 0$. *Then* $T \in \mathbb{A}_{\aleph_0}$.

Proof. It is well known (cf. [19, Chap. 4]) that if T is quasitriangular and H' is a hole in $\sigma_e(T)$, then $i(H') \geqslant 0$, so the corollary follows from the dual version of Theorem 3.12(b).

COROLLARY 3.14. *Suppose* $T \in (C_{0\cdot} \cup C_{\cdot 0}) \cap \mathbb{A}$, $\sigma(T) = \mathbb{T}$, *and* $\mathcal{A}_T = \mathscr{W}_T$. *Then* $T \in \mathbb{A}_{\aleph_0}$.

This corollary is an improvement of [8, Corollary 6.11].

4. INVARIANT SUBSPACES

In this section we derive the promised corollaries concerning the invariant subspace problem for contractions on Hilbert space.

THEOREM 4.1. *Suppose that* T *is a contraction in* $\mathscr{L}(\mathscr{H})$ *such that* $\sigma(T) \supset \mathbb{T}$ *and the weak* and* (*relative*) *weak operator topologies coincide on* $\mathcal{A}_T$. *Then* T *has a nontrivial invariant subspace.*

Proof. There are a number of well-known reductions that one can make without loss of generality when looking for invariant subspaces. In the first place, one may suppose that $\sigma(T) = \sigma_e(T)$ (for, otherwise, either T or T^* has an eigenvalue). Second, since T is a contraction, the sets

$$\mathcal{M} = \{x \in \mathcal{H}: \|T^n x\| \to 0\} \qquad \text{and} \qquad \mathcal{M}_* = \{x \in \mathcal{H}: \|T^{*n} x\| \to 0\}$$

are both subspaces of $\mathcal{H}$, and it is easy to see that $\mathcal{M}$ and $\mathcal{M}_*^\perp$ are invariant subspaces for T. When one takes into account the various possibilities and recalls that if $\mathcal{M} = \mathcal{M}_* = (0)$, then T is quasisimilar to a unitary operator and thus has a good supply of invariant subspaces (cf. [22, pp. 76–79]), the only cases that remain to be dealt with are $T \in C_{0\cdot}$ and $T \in C_{\cdot 0}$. Therefore, by taking adjoints if necessary, one may suppose that $T \in C_{0\cdot}$. Finally, since $\sigma(T) \supset \mathbb{T}$, one may suppose that $T \in \mathbb{A}$ by [1, Theorem 2.2]. Thus these reductions combined give us $T \in C_{0\cdot} \cap \mathbb{A}$ and $\sigma_e(T) = \sigma(T)$, so $T \in \mathbb{A}_{\aleph_0}$ by Corollary 3.9, and that operators in $\mathbb{A}_{\aleph_0}$ (even $\mathbb{A}_1$) have nontrivial invariant subspaces is well known (see, for example, [8, Proposition 4.8]).

The following corollary of Theorem 4.1 answers a question that we were unable to answer for several years.

COROLLARY 4.2. *Suppose T is a contraction in $\mathcal{L}(\mathcal{H})$ and m is a positive integer such that $T^{(m)} \in \mathbb{A}_1$ (equivalently, in the terminology of [8], $T \in \mathbb{A}_{1/m}$). Then T has a nontrivial invariant subspace.*

Proof. Since $T^{(m)} \in \mathbb{A}_1$, $T \in \mathbb{A}$, and therefore $\sigma(T) \supset \mathbb{T}$. It is an easy consequence of Lemma 2.4 that each $[L]$ in Q_T has length at most m, and hence induces on $\mathcal{A}_T$ a linear functional that is continuous in the weak operator topology. Thus the weak* and (relative) weak operator topologies coincide on $\mathcal{A}_T$.

Once again we may sometimes replace the coincidence of the topologies on $\mathcal{A}_T$ by the equality of the algebras $\mathcal{A}_T$ and $\mathcal{W}_T$.

THEOREM 4.3. *Suppose that T is a contraction in $\mathcal{L}(\mathcal{H})$ such that $\sigma(T) \supset \mathbb{T}$, $\mathcal{A}_T = \mathcal{W}_T$, and there are only a finite number of holes H in $\sigma(T)$ such that the Lebesgue measure of $\partial H \cap \mathbb{T}$ is positive. Then T has a nontrivial invariant subspace.*

Proof. As in the proof of Theorem 4.1, we may suppose that $T \in C_{0\cdot} \cap \mathbb{A}$ and that $\sigma_e(T) = \sigma(T)$. That $T \in \mathbb{A}_{\aleph_0}$ now follows immediately from Theorem 3.12(b).

Remark 4.4. It was recently shown by Westwood [23] that there do exist operators T in $\mathcal{L}(\mathcal{H})$ such that $\mathcal{A}_T = \mathcal{W}_T$ but the weak* and weak operator topologies do not coincide on $\mathcal{A}_T$. Moreover, very recently it was shown by W. Wogen that there exist operators T in $\mathscr{L}(\mathscr{H})$ such that $\mathfrak{A}_T \neq \mathscr{W}_T$, and it was shown by the first author and J. Esterle that if $\mathfrak{A}$ is any dual algebra such that the weak* and weak operator topologies coincide on $\mathfrak{A}$, then $\mathfrak{A}$ is closed in the weak operator topology. The authors conjecture that every operator T in $\mathbb{A} \cap C_{0\cdot}$ has the property that the weak* and weak operator topologies coincide on $\mathfrak{A}_T$. A positive answer to this conjecture would imply, of course, together with Theorem 4.1, that every contraction T in $\mathscr{L}(\mathscr{H})$ satisfying $\sigma(T) \supset \mathbb{T}$ has a nontrivial invariant subspace.

Remark 4.5. The following consequence of Theorem 3.6 and Lemma 3.10 seems worth pointing out: If T is a contraction in $\mathscr{L}(\mathscr{H})$, $\sigma(T) = \mathbb{T}$, and $T^{-1} \notin \mathscr{W}_T$, then T has a nontrivial invariant subspace.

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