\( \phi \)-Summing Operators in Banach Spaces

R. Khalil*

University of Michigan, Department of Mathematics, Ann Arbor, Michigan

AND

W. Deeb

Kuwait University, Department of Mathematics, Kuwait, Kuwait

Submitted by V. Lakshmikantham

Received November 20, 1985

Let \( \phi: [0, \infty) \rightarrow [0, \infty) \) be a continuous subadditive strictly increasing function and \( \phi(0) = 0 \). Let \( E \) and \( F \) be Banach spaces. A bounded linear operator \( A: E \rightarrow F \) will be called \( \phi \)-summing operator if there exists \( \lambda > 0 \) such that

\[
\sum_{i=1}^{n} \|Ax_i\| \leq \lambda \sup_{\|x^*\| = 1} \sum_{i=1}^{n} \phi(|\langle x_i, x^* \rangle|),
\]

for all sequences \( \{x_1, ..., x_n\} \subseteq E \).

We study the basic properties of the space \( \Pi^\phi(E, F) \). In particular, we prove that \( \Pi^\phi(H, H) = \Pi^\phi(H, H) \) for \( 0 < p < 1 \), where \( H \) is a Banach space with the metric approximation property.

0. INTRODUCTION

Let \( \phi: [0, \infty) \rightarrow [0, \infty) \) be a continuous function. The function \( \phi \) is called a modulus function if

(i) \( \phi(x + y) \leq \phi(x) + \phi(y) \)

(ii) \( \phi(0) = 0 \)

(iii) \( \phi \) is strictly increasing.

The functions \( \phi(x) = x^p \), \( 0 < p \leq 1 \) and \( \phi(x) = \ln(1 + x) \) are examples of modulus functions.

For Banach spaces \( E \) and \( F \), a bounded linear operator \( A: E \rightarrow F \) is called \( p \)-summing, \( 0 < p < \infty \), if there exists \( \lambda > 0 \) such that

\[
\sum_{i=1}^{n} \|Ax_i\|^p \leq \lambda \sup_{\|x^*\| = 1} \sum_{i=1}^{n} |\langle x_i, x^* \rangle|^p,
\]

for all sequences \( \{x_1, ..., x_n\} \subseteq E \). For \( p = 1 \), this definition is due to

* Current address: Kuwait University, Department of Mathematics, Kuwait.

577
Grothendieck [3], and for \( p \neq 1 \), the definition was given by Pietsch [6]. If \( \prod^p(E, F) \) is the space of all \( p \)-summing operators from \( E \) to \( F \), then it is well known [3, p. 293] that \( \prod^p(E, F) = \prod^q(E, F) \) for \( 0 < p, q \leq 1 \). If \( E \) and \( F \) are Hilbert spaces then \( \prod^p(E, F) = \prod^q(E, F) \) for \( 0 < p < q < \infty \) [6, p. 302].

The object is to introduce \( \phi \)-summing operators for modulus functions \( \phi \). The basic properties of these operators are studied. We, further, prove that \( \phi \)-summing operators are \( p \)-summing for \( 0 < p \leq 1 \), for Banach spaces having the metric approximation property.

Throughout this paper, \( L(E, F) \) denotes the space of all bounded linear operators from \( E \) to \( F \). The dual of \( E \) is \( E^* \). The compact elements in \( L(E, F) \) will be denoted by \( K(E, F) \). The unit sphere of a Banach space \( E \) is denoted by \( S(E) \). The set of complex numbers is denoted by \( \mathbb{C} \).

1. \( \prod^p(E, F) \)

Let \( E \) and \( F \) be two Banach spaces and \( \phi \) be a modulus function on \([0, \infty)\). Consider the following two spaces:

(i) \( l^\phi(E) = \{(x_n) : \sup_{\|x\| \leq 1} \sum_n \phi |\langle x_n, x^*\rangle| < \infty, x_n \in E\} \).

(ii) \( \ell^\phi(F) = \{(x_n) : \sum_n \phi \|x_n\| < \infty, x_n \in E\} \).

For \( x = (x_n) \in l^\phi(E) \), we define

\[
\|x\|_\varepsilon = \sup_{\|x^*\| \leq 1} \sum_n \phi |\langle x_n, x^*\rangle|,
\]

and for \( y = (y_n) \in \ell^\phi(F) \) we define

\[
\|y\|_\pi = \sum_n \phi \|y_n\|.
\]

It is a routine matter to verify the following result:

**Theorem 1.1.** The spaces \( (l^\phi(E), \|\cdot\|_\varepsilon) \) and \( (\ell^\phi(F), \|\cdot\|_\pi) \) are complete metric linear spaces.

**Remark 1.2.** The spaces \( l^\phi(E) \) and \( \ell^\phi(E) \) are generalizations of the spaces \( l^p(E) \) and \( l^p(F) \) for \( 0 < p < 1 \). We refer to [6, Chap. 16; 1] for a discussion of such spaces.

A linear operator \( T: l^\phi(E) \to \ell^\phi(F) \) will be called *metrically bounded* if there is a \( \lambda > 0 \) such that

\[
\|Tx\|_\pi \leq \lambda \|x\|_\varepsilon.
\]
for all \( x = (x_n) \in l^p \langle E \rangle \). Clearly every metrically bounded operator is continuous. We let \( L^\phi(E, F) \) denote the space of all metrically bounded operator from \( l^p \langle E \rangle \) into \( l^p \langle F \rangle \). For \( T \in L^\phi(E, F) \), we set \( \| T \|_\phi = \inf \{ \lambda : \| Tx \|_\phi \leq \lambda \| x \|_\phi, x \in l^p \langle E \rangle \} \). The proof of the following result is similar to the proof in case of Banach spaces, [7, p. 185], and it will be omitted.

**Theorem 1.3.** The space \( (L^\phi(E, F), \| \cdot \|_\phi) \) is a complete metric linear space.

**Definition 1.4.** Let \( E \) and \( F \) be two Banach spaces. Then, a bounded linear operator \( T : E \to F \) is called \( \phi \)-summing if there is \( \lambda > 0 \) such that

\[
\sum_1^N \phi \| Tx_n \| \leq \lambda \sup_{\| x^* \|_1 < 1} \sum_1^N \phi | \langle x_n, x^* \rangle | \tag{\ast}
\]

for all sequences \( \{ x_1, \ldots, x_n \} \subseteq E \).

The definition is a generalization of the definition of \( p \)-summing operators for \( 0 \leq p \leq 1 \). We refer to [6] for a full study of \( p \)-summing operators \( 0 < p < \infty \).

Let \( \prod^\phi(E, F) \) be the set of all \( \phi \)-summing operators from \( E \) to \( F \). Every \( T \in \prod^\phi(E, F) \) defines an element \( \hat{T} \in L^\phi(E) \) via:

\[
\hat{T} : l^p \langle E \rangle \to l^p \langle E \rangle \\
\hat{T}((x_n)) = ((Tx_n)).
\]

For \( T \in \prod^\phi(E, F) \) we define the \( \phi \)-summing metric of \( T \) as: \( \| T \|_\phi = \| \hat{T} \|_\phi \). Hence \( \| T \|_\phi = \inf \{ \lambda : * \text{ holds} \} \). The definition of \( \phi \)-summing operators together with Theorem 1.2 implies:

**Theorem 1.5.** \( (\prod^\phi(E, F), \| \cdot \|_\phi) \) is a complete metric linear space.

**Theorem 1.6.** Let \( A \in \prod^\phi(G, E) \), \( B \in L(G, E) \), and \( D \in L(F, H) \). Then \( AB \in \prod^\phi(G, E) \) and \( DA \in \prod^\phi(E, H) \). Further, \( \| AB \|_\phi \leq (\| B \| + 1) \| A \|_\phi \) and \( \| DA \|_\phi \leq (\| D \| + 1) \| A \|_\phi \).

**Proof.** The proof follows from the fact that for all \( a > 0 \), \( \phi(at) \leq (a + 1) \phi(t) \) which is a consequence of the monotonicity and subadditivity of \( \phi \). Q.E.D.

Let \( B_1(E^*) \) be the unit ball of \( E^* \) equipped with the \( w^* \)-topology, and \( M \) be the space of all regular Borel measures on \( B_1(E^*) \). The unit sphere of \( M \) is denoted by \( S(M) \).
Theorem 1.6. Let $A \in L(E, F)$. The followings are equivalent:

(i) $A \in \prod^\phi(E, F)$.

(ii) There exists $\lambda > 0$ and $\nu \in S(M)$ such that

$$\phi \|Ax\| \leq \lambda \int_{B(E^*)} \phi |\langle x, x^* \rangle| \, dv(x^*).$$

Proof. (ii) $\rightarrow$ (i) This is evident.

(i) $\rightarrow$ (ii) Let $A \in \prod^\phi(E, F)$ and $\lambda = \|A\|_{\phi}$. For every finite sequence \(\{x_1, \ldots, x_N\} \subseteq E\), define the map:

$$Q: S(M) \rightarrow \mathbb{C}$$

$$Q(\mu) = \sum_{n=1}^{N} \phi \|Ax_n\| - \lambda \sum_{n=1}^{N} \int_{B(E^*)} \phi |\langle x_n, x^* \rangle| \, d\mu \cdots \quad (**)$$

Clearly, the function $Q$ is convex. Further, there is a point $\mu_0 \in S(M)$ such that $Q(\mu_0) < 0$. Indeed choose $\mu_0$ = the dirac measure at $x_0^*$, where

$$\sum_{n=1}^{N} \phi |\langle x_n, x_0^* \rangle| = \sup_{\|x^*\| \leq 1} \sum_{n=1}^{N} \phi |\langle x_n, x^* \rangle|.$$ 

Further, if \(\{Q_1, \ldots, Q_r\}\) is a collection of such functions defined by (**) then for any $a_1, \ldots, a_r, \sum a_k = 1$, there is $Q$ defined in a similar way, such that $\sum a_k Q_k(\mu) \leq Q(\mu)$ for all $\mu \in S(M)$. Hence the collection of functions on $S(M)$ defined by (**) satisfies Fan's lemma [6, p. 401]. Consequently there is a measure $\nu$ in $S(M)$ such that $Q(\nu) \leq 0$ for all $Q$ defined by (**) . In particular if $Q$ is defined by (**) with associated sequence \(\{x\}, x \in E\), we get

$$\phi \|Ax\| \leq \lambda \int_{B(E^*)} \phi |\langle x, x^* \rangle| \, dv. \quad \text{Q.E.D.}$$

Remark 1.7. The proof of Theorem 1.6 is similar to the proof of Theorem 17.3.2. in [6], where $\phi(t) = t^p$, $0 < p \leq 1$. We included the detailed proof here for completeness and to include modulus functions.

2. $\prod^\phi(H, H) = \prod^\phi(H, H)$, $0 \leq p \leq 1$

Let $m$ be the Lebesgue measure on $I = [0, 1]$. For the modulus function $\phi$, set $L^\phi$ to denote the space of all measurable functions $f$ on $[0, 1]$ for which $\int_0^1 |f(t)|^p \, dm(t) < \infty$. For $f \in L^\phi$ we define $\|f\|_{\phi} = \phi^{-1} \int_0^1 |f(t)| \, dm(t)$. The function $\|\|_{\phi}$ is not a metric on $L^\phi$. However, we can define a topology via: $f_n \rightarrow f$ in $L^\phi$ if
\(\phi^{-1} \int \phi |f_n - f| \, dm(t) \to 0\). It is not difficult to prove that such a topology makes \(L^\phi\) a topological vector space. In case \(\phi(t) = t^p, 0 < p \leq 1\), \(L^\phi\) is a quasi-normed space [4, p. 159]. If \(\phi(t) = t/(1 + t)\), we write \(L^0\) for \(L^\phi\).

The concept of \(\phi\)-summing operators is still valid for operators \(T: E \to L^\phi\), where \(E\) is a Banach space.

**Definition 2.1.** Let \(E\) be a Banach space. A linear map \(T: E \to L^\phi\) is called \(\phi\)-decomposable if there is a function \(\psi: [0, 1] \to F^*\) such that

(i) The function \(\langle x, \psi(t) \rangle\) is \(m\)-measurable and

\[
(Tx)(t) = \langle x, \psi(t) \rangle \text{ a.e.m. for all } x \in E.
\]

(ii) There exists \(f \in L^1\) such that \(\|\psi(t)\| \leq f(t)\) a.e.m.

This definition is due to Kwapien [5] for \(\phi(t) = t^p\). In [5], the function \(f\) in (ii) is assumed to belong to \(L^p\). Since \(L^\phi \subseteq L^0\) for all modulus functions \(\phi\), the following lemma is immediate:

**Lemma 2.2.** Every \(\phi\)-decomposable map \(T: E \to L^\phi\) is \(0\)-decomposable.

**Theorem 2.3.** Let \(E\) be any Banach space. If a linear map \(T: E \to L^\phi\) is \(\phi\)-decomposable, then \(T\) is \(\phi\)-summing.

**Proof.** Let \(\psi: [0, 1] \to E^*\) be as in Definition 2.1 and \(\{x_1, ..., x_N\} \subseteq E\). Then

\[
\sum_{i=1}^{N} \phi \|Tx_n\|_{\phi} = \sum_{i=1}^{N} \phi \left[ \phi^{-1} \int_{0}^{1} \phi |\langle x_n, \psi(t) \rangle| \, dm(t) \right]
\]

\[
\leq \sum_{i=1}^{N} \int_{0}^{1} (\|\psi(t)\| + 1) \phi \left| \langle x_n, \frac{\psi(t)}{\|\psi(t)\|} \rangle \right| \, dm(t)
\]

\[
\leq (\|f\|_1 + 1) \sup_{\|x\| \leq 1} \sum_{i=1}^{N} \phi |x_n, x^*|.
\]

Q.E.D.

Before we state the next theorem, we should remark that the topology on \(L^\phi\) generated by the gauge \(\|f\|_{\phi} = \phi^{-1} \int \phi |f| \, dm\), is equivalent to the topology generated by the metric \(\|f\|_{\phi} = \int \phi |f| \, dm\). Consequently, the bounded sets in both topologies coincide.

**Theorem 2.4.** Let \(T \in L(E, F)\) such that \(T^* \in \prod^\phi(F^*, E^*)\). If \(F\) has the metric approximation property, then for any continuous linear map \(\gamma: F \to L^\phi\), the map \(\gamma T\) is \(\phi\)-decomposable.

**Proof.** First, we claim that there exists an \(M > 0\) such that for all
\( x_1, x_2, \ldots, x_n \in E, \|x_i\| \leq 1 \) and for all measurable disjoint sets \( A_1, \ldots, A_n \) in \([0,1]\) we have

\[
\sum_{i=1}^{n} \int_{A_i} \phi |\gamma T(x_i)(t)| \, dt \leq M. \tag{*}
\]

By the remark preceding the theorem and the assumption that \( F \) has the metric approximation property, it is enough to prove (*) for operators \( \gamma = \sum_{i=1}^{k} y'_i \otimes 1_{B_i}, y'_i \in F^* \) and \( B_i \) measurable in \([0,1]\). One can take \( B_i \) to be disjoint of equal length and \( \bigcup_{i=1}^{k} B_i = [0,1] \).

Let \( \gamma - \sum_{i=1}^{k} y'_i \otimes 1_{B_i}, B_i \) disjoint in \([0,1]\) and \( m(B_i) = 1/k, y'_i \in F^* \), for \( i = 1, \ldots, k \). If \( x_1, \ldots, x_n \in E \), with \( \|x_i\| \leq 1 \) and if \( A_1, \ldots, A_n \) are disjoint measurable subsets in \([0,1]\), then

\[
\sum_{i=1}^{n} \int_{A_i} \phi |\gamma T(x_i)(t)| \, dm(t)
\]

\[
= \sum_{i=1}^{n} \int_{A_i} \phi \left| \sum_{j=1}^{k} \langle Tx_i, y_j \rangle 1_{B_j}(t) \right| \, dm(t)
\]

\[
\leq \sum_{i=1}^{n} \sum_{j=1}^{k} \phi |\langle Tx_i, y'_j \rangle| m(B_j \cap A_i) \tag{since \( \phi \) is subadditive}
\]

\[
\leq \sum_{j=1}^{k} \frac{1}{k} \phi \|T^* y'_j\| \cdot \sum_{i} m(B_j \cap A_i) \tag{since \( A_i \)'s are disjoint}
\]

\[\leq \lambda \sup_{\|x^*\| \leq 1, \|x\| \leq 1} \sum_{j=1}^{k} \phi |\langle y'_j, x^* \rangle| m(B_j) \tag{since \( T^* \in \prod^\Phi(F^*, E^*) \)}
\]

\[= \lambda \sup_{\|x^*\| \leq 1} \int \phi |\gamma x^*(t)| \, dm(t).
\]

Since \( \gamma \) is continuous, by the remark preceding the theorem we get \( \sup_{\|x^*\| \leq 1} \int \phi |\gamma x^*(t)| \, dm(t) \leq M \) for some \( M > 0 \), and (*) is proved.

It follows from (*) that the image of the unit ball of \( E \) under \( \gamma T \) is bounded in the lattice \( L^\Phi \). If \( g \in L^\Phi \) such that \( \gamma T(x) \leq g \) for all \( x \in E, \|x\| \leq 1 \), then the function \( \theta(t) = \gamma T x(t)/g(t) \) if \( g(t) \neq 0 \) and \( \theta(0) = 0 \), is an element of \( L^\infty \). Consequently, the linear map

\[
S: E \rightarrow L^\infty,
\]

\[
S(x) = \gamma T x \big| g
\]
is continuous and \( \| S \| \leq 1 \). Hence, by the lifting theorem, there exists \( Q: [0, 1] \to (L^\infty)^* \) such that the function \( \langle Q(t), f \rangle \) is \( m \)-measurable a.e., for all \( f \in L^\infty \), and \( f(t) = \langle Q(t), f \rangle \) a.e. Further \( \| Q(t) \| = 1 \) for all \( t \in [0, 1] \).

Now, consider the function \( \psi: [0, 1] \to E^* \) defined by \( \psi(t) = g(t) \cdot S^*(Q(t)) \). It is not difficult to see that \( \psi \) is the function needed for \( \gamma T \) to be \( \phi \)-decomposable, noting that \( g \in L^\infty \subseteq L^\phi \).

Before we prove the next result, we need the following two lemmas:

**Lemma 2.5.** Let \( T: L^\phi \to L^2 \) be a continuous linear operator. Then \( \| Tf \| \leq \lambda \int \phi(t) dm(t) \) for all \( f \in L^\phi \) for which \( \int \phi(t) dm(t) = \| f \|_\phi \leq 1 \).

**Proof:** First we prove it for \( f \in L^\phi \), \( \| f \|_\phi = 1 \). If the inequality \( \| Tf \| \leq \lambda \| f \|_\phi \) is not true, then we can find a sequence \( (f_n) \) such that \( \| f_n \|_\phi = 1 \) but \( \| Tf_n \| > \lambda \| f_n \|_\phi \). Then the sequence \( f_n/n \to 0 \) in \( L^\phi \), but \( \| T(f_n/n) \| > 1 \), which contradicts the continuity of \( T \).

Now, let \( f \in L^\phi \), \( \| f \|_\phi < 1 \). Then one can find an \( \alpha > 1 \) such that \( \| \alpha f \|_\phi = 1 \). Hence

\[
\| Tf \| = \frac{1}{\alpha} \| T\alpha f \| \\
\leq \frac{\lambda}{\alpha} \| \alpha f \|_\phi \\
\leq \lambda \frac{\alpha + 1}{\alpha} \| f \|_\phi \\
\leq 2\lambda \| f \|_\phi.
\]

Q.E.D.

It should be remarked that for every \( r > 0 \) there exists \( \lambda > 0 \) such that \( \| Tf \| \leq \lambda \| f \|_\phi \) for all \( f \in L^\phi \), \( \| f \|_\phi \leq r \).

**Lemma 2.6.** Let \( T: L^2 \to L^\phi \) be \( p \)-summing operator. Then \( ST: L^2 \to L^2 \) is \( p \)-summing for continuous operators \( S: L^\phi \to L^2 \).

**Proof:** Using Lemma 2.5 and the argument in the proof of Theorem 1.6, the result follows. Q.E.D.

Now we prove:

**Theorem 2.7.** Let \( \phi \) be any modulus function. Then \( \Pi^\phi(L^2, L^2) \subseteq \Pi^2(L^2, L^2) \).

**Proof:** Let \( T: L^2 \to L^2 \) be \( \phi \)-summing operator. By Theorem 2.4, \( \gamma T^*: L^2 \to L^2 \to L^\phi \) is \( \phi \)-decomposable for all continuous linear operators \( \gamma: L^2 \to L^\phi \). In particular, we can choose \( \gamma(f) = \int f(t) \, dx \), [2, 5], where \( (x_i) \)
is a symmetric stable process on \([0, 1], m\) with exponent 2. This makes \(\gamma\) an isomorphic embedding of \(L^2\) into \(L^\phi\) and also into \(L^0\). Hence \(\gamma T^*: L^2 \rightarrow L^0\) is zero decomposable. Using Theorem 3 in [5], we get \(T^*: L^2 \rightarrow L^2\) is zero summing. By Lemma 2.6, \(\gamma T^*: L^2 \rightarrow L^0\) is zero decomposable. Another application of Theorem 3 in [5]: we get \(T: L^2 \rightarrow L^2\) is zero-summing. However, every zero-summing map is 2-summing, [5]. Hence \(T \in \Pi^2(L^2, L^2)\).

**Theorem 2.8.** For any modulus function \(\phi\), \(\Pi^2(L^2, L^3) \subseteq \Pi^\phi(L^2, L^2)\).

**Proof.** Let \(T: L^2 \rightarrow L^2\) be 2-summing operator. If \(\gamma\) is the isomorphic embedding of \(L^2\) into \(L^\phi\) as in Theorem 2.7, then using Theorem 3 in [5], we get

\[\gamma T: L^2 \rightarrow L^2 \rightarrow L^\phi\]

is \(\phi\)-decomposable. By Theorem 2.3, \(\gamma T\) is \(\phi\)-summing. Using Lemma 2.5, we get \(T: L^2 \rightarrow L^2\) is \(\phi\)-summing. Q.E.D.

**Acknowledgment**

The first author thanks Professor J. Jacod for helpful discussions.

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