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On a 2-Generated Infinite 3-Group: Subgroups and Automorphisms

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1. INTRODUCTION

This is the second paper dedicated to a detailed study of the infinite 3-group

$$\mathscr{G} = \langle \gamma, x \rangle, \qquad \gamma^3 = x^3 = 1,$$

$$\gamma = (\gamma, x, x^{-1}) \qquad \text{(a recursive definition to be explained later)},$$

which was introduced by the author and Narain Gupta in [2] as a group of automorphisms of an infinite regular tree $\mathscr{T}$. As was shown in [4], $\mathscr{G}$ has the remarkable properties of containing a copy of every finite 3-group, and having all its proper quotients finite.

The first paper [5] in this detailed study $\mathscr{G}$ dealt with the presentation problem. Here we study further aspects of its subgroup structure. Our prime objective though is to determine the group of automorphisms of $\mathscr{G}$. This task represents a large portion of our present paper.

The main results are as follows.

THEOREM 1. *Let $i \geqslant 0$, and $\mathscr{G}_i$ be the subgroup of $\mathscr{G}$ which fixes pointwise the ith level vertices of the tree $\mathscr{T}$. Then,*

$$\mathscr{G}_0 = \mathscr{G}, \qquad \mathscr{G}_1 = \langle \gamma \rangle^{\mathscr{G}}, \qquad \mathscr{G}_2 = \mathscr{G}_1 \langle (\gamma, \gamma, \gamma) \rangle,$$

and for $i \geqslant 3$,

$$\mathscr{G}_i = \mathscr{G}_{i+1} \times \mathscr{G}_{i+1} \times \mathscr{G}_{i+1}.$$

Moreover,

$$|\mathscr{G}/\mathscr{G}_1| = 3, \qquad |\mathscr{G}/\mathscr{G}_i| = 3^{2 \cdot 3^{i-2}+1} \qquad \text{for } i \geqslant 2.$$

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THEOREM 2. *For every* $g \in \mathcal{G}$, $\mathcal{C}_{\mathcal{G}}(g)$ *is finitely generated* $\Leftrightarrow$ *g is conjugate to* 1, x, *or* x^{-1}. *Furthermore, the only finite conjugacy class of* $\mathcal{G}$ *is* $\{1\}$.

THEOREM 3. *The automorphism group of* $\mathcal{G}$, $\mathrm{Aut}(\mathcal{G})$, *is a* $\{2, 3\}$*-group. Indeed,*

$$\mathrm{Aut}(\mathcal{G}) = (\mathcal{G} \rtimes \mathcal{X}) \rtimes \mathcal{V}$$

where

$\mathcal{X}$ *is an elementary abelian* 3*-group of infinite rank, and*

$\mathcal{V}$ *is an elementary abelian group of order* 4.

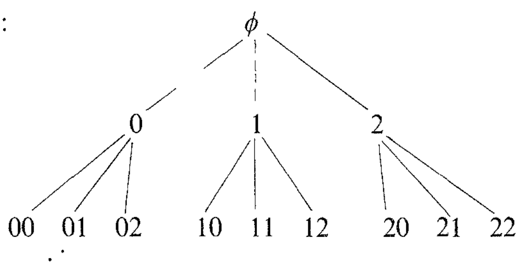
An outline of the proof of Theorem 3 is given in the introduction to Section 5.

2. GENERAL FACTS ABOUT $\mathcal{T}$ AND $\mathcal{G}$

2.1. *The Tree* $\mathcal{T}$ *and Its Automorphisms*

We adapt some of the material from [3] to the present situation. Let

$\mathcal{T}$:



be the infinite ternary tree having for its set of vertices the set $\mathcal{M}$ of finite sequences on 0, 1, 2.

Let Σ_3 be the permutation group on $\{0, 1, 2\}$ and define its generators

$$x = (0, 1, 2), \qquad t = (0, 1).$$

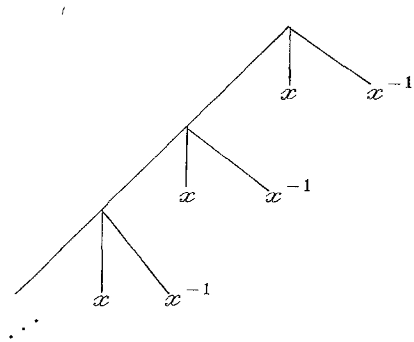
The group Σ_3 is embedded in $\mathcal{A} = \mathrm{Aut}(\mathcal{T})$ by

$$\text{for } y \in \Sigma_3, j \in \{0, 1, 2\}, s \in \mathcal{M}$$

$$y{:}\, js \to y(j)\, s.$$

Given an automorphism $\alpha \in \mathcal{A}$, and a vertex s of $\mathcal{T}$, we may define an autormorphism like α on the subtree headed by s and extend it to an automorphism of $\mathscr{T}$ by fixing the vertices outside the subtree. We denote this new automorphism by the tree $\mathscr{T}$ with α attached to the vertex s.

We introduce the automorphism γ by



For every natural number i, let $\mathscr{A}_i$ denote the subgroup of $\mathscr{A}$ which fixes pointwise the ith-level vertices (that is, sequences of length i). Then

$$\mathscr{A}_i \cong \mathop{\mathsf{X}}_{3^i} \mathscr{A} \qquad \text{(a direct product of } 3^i \text{ copies of } \mathscr{A}\text{)}.$$

On making the identification

$$\mathscr{A}_1 = \mathscr{A} \times \mathscr{A} \times \mathscr{A},$$

we get

$$\mathscr{A} = \mathscr{A} \wr \Sigma_3,$$

the wreath product of $\mathscr{A}$ by Σ_3. Define

$$\pi_i \colon \mathscr{A} \times \mathscr{A} \times \mathscr{A} \to \mathscr{A} \qquad (i = 0, 1, 2)$$

to be the projection map on the ith coordinate.

Given $\alpha \in \mathscr{A}$, it can be expressed uniquely as

$$\alpha_\phi (:= \alpha) = (\alpha_0, \alpha_1, \alpha_2).\, y_\phi,$$

where

$$y_\phi \in \Sigma_3, \qquad \alpha_i \in \mathscr{A} \qquad (i = 0, 1, 2).$$

Likewise, each α_i can be expressed in this form. Thus, for each $s \in \mathscr{M}$, we obtain

$$\alpha_s = (\alpha_{s0}, \alpha_{s1}, \alpha_{s2})\, y_s$$

with

$$y_s \in \Sigma_3, \qquad \alpha_{si} \in \mathscr{A} \qquad (i=0, 1, 2).$$

Clearly,

$$\alpha \text{ is determined by } \{ y_s | s \in \mathscr{M} \}.$$

The automorphism γ can be described recursively as

$$\gamma = (\gamma, x, x^{-1})$$

and

$$\mathscr{G} = \langle \gamma, x \rangle.$$

2.2. *Facts about $\mathscr{G}$*

2.2.1. (i) *Let Γ denote $\langle \gamma \rangle^{\mathscr{G}}$, the normal closure of $\langle \gamma \rangle$ in $\mathscr{G}$. Then,*

$$\Gamma < \mathscr{A}_1, \qquad \mathscr{G} = \Gamma \cdot \langle x \rangle, \qquad \pi_i \colon \Gamma \twoheadrightarrow \mathscr{G} \qquad (i=0, 1, 2).$$

(ii) $\Gamma' = \mathscr{G}' \times \mathscr{G}' \times \mathscr{G}'$.

(iii) $\mathscr{G}/\Gamma' \cong C_3 \wr C_3$.

(iv) *Let $\alpha \in \mathscr{G}$. Then there exists*

$$j \colon \mathscr{M} \to \{0, 1, 2\}$$

such that

$$\alpha_\phi (:= \alpha) = (\alpha_0, \alpha_1, \alpha_2)\, x^{j(\phi)},$$

$$\alpha_s = (\alpha_{s0}, \alpha_{s1}, \alpha_{s2})\, x^{j(s)},$$

where the α's are elements of $\mathscr{G}$.

Proof. The group Γ is generated by

$$\gamma = (\gamma, x, x^{-1}), \qquad \gamma^{x} = (x^{-1}, \gamma, x), \qquad \gamma^{x^{-1}} = (x, x^{-1}, \gamma).$$

Also

$$[\gamma\gamma^{x}, \gamma^{x^{-1}}\gamma] = ([x^{-1}, \gamma^{-1}], 1, 1).$$

With these observations, assertions (i)–(iv) can be obtained routinely.

2.2.2. *Let $\omega \in \mathscr{A}$. Then, $\omega \in \Gamma$ iff there exist integers $n_0, n_1, n_2 \in \{0, 1, 2\}$ and commutators $c_0, c_1, c_2 \in \mathscr{G}'$, such that*

$$w = (c_0 \gamma^{n_0} x^{n_2 - n_1}, c_1 \gamma^{n_1} x^{n_0 - n_2}, c_2 \gamma^{n_2} x^{-n_0 + n_1});$$

this representation of $w \in \Gamma$ is unique.

Proof. This follows from the fact that $w \in \Gamma$ is a word in $\gamma, \gamma^x, \gamma^{x^{-1}}$ and from item (ii) of the previous proposition.

2.2.3. (i) $w = (w_0, w_1, w_2) \in \Gamma \Rightarrow w_0 w_1 w_2 \in \mathscr{G}' \langle \gamma \rangle$,

(ii) $\gamma \gamma^x \gamma^{x^{-1}} \equiv (\gamma, \gamma, \gamma) \pmod{\Gamma'}$,

(iii) $\gamma^{(1)} = (\gamma, \gamma, \gamma) \in \mathscr{G}'$,

(iv) $w = (w_1, w_1, w_1) \in \Gamma \Leftrightarrow w_1 \in \mathscr{G}' \langle \gamma \rangle$,

(v) $w = (x^{n_0}, x^{n_1}, x^{n_2}) \in \Gamma \Rightarrow n_0 \equiv n_1 \equiv n_2 \equiv 0 \pmod 3$,

(vi) *$\mathscr{G}'$ is generated by $\gamma^{(1)}$, $[x, \gamma]$ modulo Γ'.*

Proof. These facts are derived from 2.2.2.

2.2.4. $\ker \pi_0 = (1 \times \mathscr{G}' \times \mathscr{G}') \cdot \langle (1, \gamma x^{-1}, \gamma x) \rangle$.

Proof. Note that

$$\gamma^x \gamma^{x^{-1}} = (1, \gamma x^{-1}, \gamma x) \in \ker \pi_0 .$$

Also,

$$1 \times \mathscr{G}' \times \mathscr{G}' = \Gamma' \cap \pi_0 .$$

Now apply 2.2.2.

2.2.5 (Order of elements).

(i) *Let $w \in \Gamma$. Then,*

$$0(wx) = 3 \Leftrightarrow wx \text{ is conjugate to } x \text{ by an element from } \Gamma.$$

(ii) *Let $w \in \mathscr{G}'$. Then,*

$$0(w\gamma) = 3 \Rightarrow w\gamma = \gamma, \gamma^x \text{ or } \gamma^{x^{-1}} \qquad (\text{modulo } \Gamma').$$

Proof. (i) Write

$$w(w_0, w_1, w_2),$$

where

$$w_0 = c_0 \gamma^{n_0} x^{n_2 - n_1}, \qquad w_1 = c_1 \gamma^{n_1} x^{n_0 - n_2}, \qquad w_2 = c_2 \gamma^{n_2} x^{-n_0 + n_1},$$

as in 2.2.2. Then,

$$0(wx)=3 \Rightarrow w_2=w_1^{-1}w_0^{-1}, \qquad n_2=-n_0-n_1.$$

Observe that if for some $v_0 \in \mathscr{G}$,

$$v=(v_0,\, v_0w_0,\, v_0w_0w_1) \in \Gamma,$$

then

$$[v, x^{-1}]=(w_0, w_1, w_1^{-1}w_0^{-1}), \qquad wx=x^v.$$

By direct application of 2.2.2 the element x^{n_1} is such a v_0.

(ii) By 2.2.3(vi),

$$w=(\gamma^{i-j}x^{-j}, \gamma^{i+j}x^{-j}, \gamma^i x^{-j}) \qquad (\text{modulo } \Gamma').$$

Thus,

$$w\gamma=(\gamma^{i-j+1}x^{-j}, \gamma^{i+j}x^{-j+1}, \gamma^i x^{-j-1}) \qquad (\text{modulo } \Gamma').$$

By straightforward considerations in $\mathscr{G}/\Gamma'$ $(\cong C_3 \wr C_3)$, $(w\gamma)^3=1$ (mod Γ') implies the following system of congruences hold modulo 3:

$$\begin{aligned} i-j+1 \quad &\text{or} \quad j\equiv 0 \\ i+j \quad &\text{or} \quad -j+1\equiv 0 \\ i \quad &\text{or} \quad -j-1\equiv 0. \end{aligned}$$

The conclusion follows by analyzing the different possibilities.

2.2.6 (Length functions; see [2]). *To the natural epimorphism*

$$\langle \underline{\gamma} \rangle * \langle \underline{\gamma}^{\underline{x}} \rangle * \langle \underline{\gamma}^{\underline{x}^{-1}} \rangle \langle \underline{x} \rangle \twoheadrightarrow \mathscr{G}$$

there is associated a syllable length ℓ which satisfies

(i) $g=wx$, *with* $w \in \Gamma$, $\Rightarrow \ell(g^3)<\ell(g)$,

(ii) $g=w=(w_0, w_1, w_2) \Rightarrow \ell(w_i)<\ell(w)$ $(i=0, 1, 2)$

unless possibly for

$$w \in \langle \gamma^h \rangle \cup \langle \gamma^h \rangle \langle \gamma^{h'} \rangle \cup \langle \gamma^h \rangle \langle \gamma^{h'} \rangle \langle \gamma^h \rangle$$

for some $h, h' \in \langle x \rangle$.

2.2.7. *$\mathcal{G}$ is a 3-group.*

Proof. Follows by induction on $\ell(g)$; or see [2].

2.2.8 (see [4]).

(i) *$\mathcal{G}$ contains a copy of every finite 3-group.*

(ii) *Every proper quotient of $\mathcal{G}$ is finite; in particular, $Z(\mathcal{G})=1$.*

3. Stabilizers, Centralizers, and Derived Groups

3.1. *Stabilizers.*

Let $\mathcal{G}_i$ be the pointwise stabilizer in $\mathcal{G}$ of the ith-level vertices of $\mathcal{T}$ (that is $\mathcal{G}_i = \mathcal{G} \cap \mathcal{A}_i$) for $i \geqslant 0$. Then,

$$\mathcal{G}_{i+1} \leqslant \mathop{\mathsf{X}}_{3} \mathcal{G}_i \leqslant \mathop{\mathsf{X}}_{3} \mathcal{A}_i = \mathcal{A}_{i+1},$$

and thus, if $\mathsf{X}_3\, \mathcal{G}_i$ is a subgroup of $\mathcal{G}$, then $\mathcal{G}_{i+1} = \mathsf{X}_3\, \mathcal{G}_i$.

We note also that the derived group $\mathcal{G}_i'$ is a subgroup of $\mathcal{G}_{i+1}$; so,

$$\mathcal{G}_i' \leqslant \mathcal{G}_{i+1} < \mathcal{G}_i \qquad \text{for } i \geqslant 0.$$

Clearly,

$$\mathcal{G}_0 = \mathcal{G}, \qquad \mathcal{G}_1 = \Gamma.$$

Let $w \in \mathcal{G}_2$. Then, as $\Gamma' \leqslant \mathcal{G}_2$, we find that

$$w \equiv (\gamma^{(1)})^n \qquad (\bmod \Gamma')$$

for some integer n. Hence

$$\mathcal{G}_2 = \Gamma' \cdot \langle \gamma^{(1)} \rangle.$$

By 2.2.3 item (iii), $\gamma^{(1)} \in \mathcal{G}'$; thus it follows that

$$\mathcal{G}' > \mathcal{G}_2, \qquad \Gamma' = \mathcal{G}' \times \mathcal{G}' \times \mathcal{G}' > \mathcal{G}_2 \times \mathcal{G}_2 \times \mathcal{G}_2.$$

Hence,

$$\Gamma' = \mathop{\mathsf{X}}_{3} \mathcal{G}' > \mathop{\mathsf{X}}_{3} \mathcal{G}_2 > \mathop{\mathsf{X}}_{3} \mathcal{G}_i \qquad \forall i \geqslant 3.$$

By the first paragraph,

$$\mathcal{G}_{i+1} = \mathop{\mathsf{X}}_{3} \mathcal{G}_i \qquad \forall i \geqslant 2.$$

From $\mathscr{G}/\Gamma' \cong C_3 \wr C_3$, we have

$$|\mathscr{G}/\Gamma| = 3, \qquad |\Gamma/\mathscr{G}'| = |\mathscr{G}'/\mathscr{G}_2| = |\mathscr{G}_2/\Gamma'| = 3;$$

so,

$$|\mathscr{G}/\mathscr{G}_1| = 3, \qquad |\mathscr{G}_1/\mathscr{G}_2| = 3^2.$$

Now,

$$[\mathscr{G}_2 : \mathscr{G}_3] = [\mathscr{G}_2 : \Gamma'][\Gamma' : \mathscr{G}_3],$$

and

$$[\Gamma' : \mathscr{G}_3] = [\mathscr{G}' : \mathscr{G}_2]^3.$$

Therefore,

$$[\mathscr{G}_2 : \mathscr{G}_3] = 3^4.$$

The other numerical results in Theorem 1, as announced in the introduction, follow by simple induction.

3.2. *Centralizers*

3.2.1. $\mathscr{C}_G(x) = \text{Diagonal } (\mathbf{X}_3\, \mathscr{G}'\langle\gamma\rangle) \oplus \langle x\rangle$ *and is finitely generated. Also, the conjugacy class of x in $\mathscr{G}$ is infinite.*

Proof. First, we observe that

$$w = (w_0, w_1, w_2) \in \mathscr{C}_\Gamma(x) \Leftrightarrow w_0 = w_1 = w_2,$$

and then apply 2.2.2 *to reach the desired conclusion.*

3.2.2. (i) $\mathscr{C}_{\mathscr{G}}(\gamma) = \mathscr{C}_{\Gamma'}(\gamma) \oplus \langle\gamma\rangle$, $\mathscr{C}_{\Gamma'}(\gamma) = \mathscr{C}_{\Gamma'}(\gamma) \times \mathscr{C}_{\mathscr{G}'}(x) \times \mathscr{C}_{\mathscr{G}'}(x)$.

(ii) $\mathscr{C}_{\mathscr{G}}(\gamma)$ *is not finitely generated. Also, the conjugacy class of γ in $\mathscr{G}$ is infinite.*

Proof. By considering $\mathscr{G}/\Gamma'$, we have that

$$\mathscr{C}_{\mathscr{G}}(\gamma) \leqslant \Gamma.$$

Let $w = (w_0, w_1, w_2) \in \mathscr{C}_\Gamma(\gamma)$. Then,

$$[w_0, \gamma] = [w_1, x] = [w_2, x] = 1.$$

From the representation of w,

$$w_0 = c_0\gamma^{n_0}x^{n_2-n_1}, \qquad w_1 = c_1\gamma^{n_1}x^{n_0-n_2}, \qquad w_2 = c_2\gamma^{n_2}x^{-n_0+n_1},$$

we obtain

$$n_2 = n_1, \qquad n_1 = 0, \qquad n_2 = 0,$$
$$[c_0, \gamma] = 1, \qquad [c_1, x] = [c_2, x] = 1.$$

Thus,

$$w = (c_0, c_1, c_2)\, \gamma^{n_0},$$

an element of $(\mathscr{C}_{\mathscr{G}'}(\gamma) \times \mathscr{C}_{\mathscr{G}'}(x) \times \mathscr{C}_{\mathscr{G}'}(x)) \bigoplus \langle \gamma \rangle$.

The rest of the assertions are easy consequences.

3.2.3. (i) *A subgroup K of $\mathscr{G}$ is finitely generated iff $K \cap \Gamma$ is finitely generated.*

(ii) *A subgroup K of Γ is finitely generated iff $\pi_i(K)$ is finitely generated* ($i = 0, 1, 2$).

We recall the statement of Theorem 2: $\forall g \in \mathscr{G}$, $\mathscr{C}_{\mathscr{G}}(g)$ is finitely generated $\Leftrightarrow g$ is conjugate to some power of x. Furthermore, the only finite conjugacy class of $\mathscr{G}$ is $\{1\}$.

Proof of Theorem 2. $(\Leftarrow)$ Obvious.

$(\Rightarrow)$ Clearly, as $\mathscr{C}_{\mathscr{G}}(g)$ is finitely generated, so is $\mathscr{C}_{\Gamma}(g)$.

We will proceed by induction on $\ell(g)$, and we may assume $\ell(g) > 1$ (by 3.2.1 and 3.2.2).

(i) Suppose $g = ux$, where $u = (u_0, u_1, u_2) \in \Gamma$. Then, by direct calculation,

$$w = (w_0, w_1, w_2) \in \mathscr{C}_{\Gamma}(g) \Leftrightarrow w_1 = w_0^{u_0}, \qquad w_2 = w_0^{u_0 u_1}, \qquad w_0 \in \mathscr{C}_{\mathscr{G}}(u_0 u_1 u_2).$$

Thus, $w \in \mathscr{C}_{\Gamma}(g)$ implies

$$w = (w_0, w_0^{u_0}, w_0^{u_0 u_1}), \qquad (w_0, w_0, w_0) \in \Gamma,\ w_0 \in \mathscr{G}'\langle \gamma \rangle.$$

Hence,

$$w \in \mathscr{C}_{\Gamma}(g) \Leftrightarrow w_0 \in \mathscr{G}'\langle \gamma \rangle \cap \mathscr{C}_{\mathscr{G}}(u_0 u_1 u_2);$$

that is,

$$\pi_0(\mathscr{C}_{\Gamma}(g)) = \mathscr{G}'\langle \gamma \rangle\, \mathscr{C}_{\mathscr{G}}(u_0 u_1 u_2).$$

Let us denote $\mathscr{G}'\langle \gamma \rangle$ by N and $\mathscr{C}_{\mathscr{G}}(u_0 u_1 u_2)$ by $\mathscr{C}$. Clearly,

$$[\mathscr{G} : N] < \infty, \qquad [NC : N] < \infty, \qquad [\mathscr{C} : \mathscr{C} \cap N] < \infty,$$

and as

$$\pi_0(\mathscr{C}_\Gamma(g)) \text{ is finitely generated,}$$

we have that

$$\mathscr{C} \text{ is finitely generated.}$$

Since $u_0 u_1 u_2$ is an entry of g^3, by 2.2.6 item (i), $\ell(u_0 u_1 u_2) < \ell(g)$. Hence, as $\mathscr{C}_{\mathscr{G}}(u_0 u_1 u_2)$ is finitely generated, there exists $h \in \mathscr{G}$ such that

$$u_0 u_1 u_2 \in \langle x \rangle^h.$$

However, by 2.2.3 item (i), $u_0 u_1 u_2 \in \mathscr{G}' \langle \gamma \rangle$; thus,

$$u_0 u_1 u_2 = 1, \qquad \text{and} \qquad O(g) = 3.$$

By 2.2.5. item (i),

$$g \text{ is conjugate to } x.$$

We conclude from this case that for $i \not\equiv 0 \pmod 3$, $\mathscr{C}_{\mathscr{G}}(\gamma^i x^j)$ is not finitely generated.

(ii) Suppose $g = (u_0, u_1, u_2) \in \Gamma$. Denote $\mathscr{C}_{\mathscr{G}}(u_i)$ by $\mathscr{C}_i$ $(i = 0, 1, 2)$. Since $[\mathscr{G} : \mathscr{G}'] < \infty$, we have

$$[\mathscr{C}_i : \mathscr{G}' \cap \mathscr{C}_i] < \infty, \qquad [\mathscr{C}_i : \pi_i(\mathscr{C}_\Gamma(g))] < \infty \qquad (i = 0, 1, 2).$$

Hence

$$\mathscr{C}_i \text{ is finitely generated} \qquad (i = 0, 1, 2).$$

By 2.2.6. item (ii),

$$\ell(u_i) < l(g) \qquad \text{for } i = 0, 1, 2,$$

unless possibly for

$$g \in \langle \gamma^h \rangle \cup \langle \gamma^h \rangle \langle \gamma^{h'} \rangle \cup \langle \gamma^h \rangle \langle \gamma^{h'} \rangle \langle \gamma^h \rangle$$

for some $h, h' \in \langle x \rangle$. The exceptional cases lead to $g = 1$, by looking at the entries of

$$\gamma^i \gamma^{jx} = (\gamma^i x^{-j}, x^i \gamma^j, x^{-i+j}),$$

and

$$\gamma^i \gamma^{jx} \gamma^k = (\gamma^i x^{-j} \gamma^k, x^i \gamma^j x^k, x^{-i} x^j x^{-k}).$$

Thus, we may assume $\ell(u_i) < \ell(g)$ $\forall i$. By induction,

$$u_i = (x^{j(i)})^{h_i}$$

for some integers $j(i)$, $h_i \in \mathcal{G}$ $(i = 0, 1, 2)$. Hence,

$$g \equiv (x^{j(0)}, x^{j(1)}, x^{j(2)}) \qquad (\text{mod } \Gamma'),$$

and

$$(x^{j(0)}, x^{j(1)}, x^{j(2)}) \in \Gamma.$$

We conclude from 2.2.3 item (v),

$$j(0) \equiv j(1) \equiv j(2) \equiv 0 \qquad (\text{mod } 3),$$

and thus $g = 1$.

As for the second conclusion of the theorem, let $g \in \mathcal{G}$ be a representative of a finite conjugacy class. Then,

$$[\mathcal{G} : \mathcal{C}_{\mathcal{G}}(g)] < \infty,$$

and thus

$$\mathcal{C}_{\mathcal{G}}(g) \text{ is finitely generated.}$$

Hence, we may assume $g \in \langle x \rangle$. By 3.2.1, $g = 1$.

3.3. *The Derived Group* $\mathcal{G}'$

Our aim is to prove that $\mathcal{G}'$ cannot be decomposed as a direct sum of two nontrivial subgroups; or simply, $\mathcal{G}'$ is an indecomposable group. First, we prove a Krull–Schmidt property.

3.3.1. *Let A be a finitely generated infinite periodic group having all its proper quotients finite. Then,*

(i) $1 \neq N \lhd A \Rightarrow Z(N) = 1$.

(ii) *Let*

$$\begin{aligned} A &= A_1 \oplus \cdots \oplus A_s \\ &= B_1 \oplus \cdots \oplus B_t \end{aligned}$$

where the A_i's and B_i's are indecomposable subgroups of A. Then,

$$s = t, \qquad \{A_1, ..., A_s\} = \{B_1, ..., B_s\}.$$

Proof. (i) Suppose there exists $z \in Z(N)^{\#}$. Then as $[A:N] < \infty$, it follows that $\{z^a | a \in A\}$ is finite, and $\langle z \rangle^A$ is a finite normal subgroup of A. Hence,

$$|A| = [A: \langle z \rangle^A] |\langle z \rangle^A| < \infty;$$

a contradiction.

(ii) Let p_i be the projection of G onto A_i $(1 \leqslant i \leqslant s)$. Then,

$$A_i = \langle p_i(B_j) | 1 \leqslant j \leqslant t \rangle \qquad (1 \leqslant i \leqslant s).$$

As

$$p_i(B_j) \text{ commutes with } p_i(B_k) \qquad \forall j \neq k,$$

and as $Z(A_i) = 1$, we have

$$A_i = p_i(B_1) \oplus \cdots \oplus p_i(B_t) \qquad (1 \leqslant i \leqslant s).$$

Since A_i is indecomposable, there exists a unique $f(i)$ such that

$$A_i = p_i(B_{f(i)}), \qquad p_i(B_j) = 1 \qquad \forall j \neq f(i).$$

Thus,

$$f\colon \{1, 2, ..., s\} \to \{1, 2, ..., t\}$$

is injective, and as we could have assumed $t \leqslant s$, we obtain $t = s$. By rearranging the indices we have

$$A_i = p_i(B_i), \qquad p_i(B_j) = 1 \qquad \forall i \neq j.$$

Thus,

$$A_i = B_i \qquad \forall i.$$

3.3.2. *$\mathscr{G}'$ is indecomposable.*

Proof. As $\mathscr{G}'$ is generated by

$$[x, y], [x, y]^x, [x, y]^y, [x, y]^{xy},$$

and as $Z(\mathscr{G}') = 1$, we get that

$\mathscr{G}'$ cannot have more than two nontrivial factors.

Suppose

$$\mathscr{G}' = A_1 \oplus A_2$$

where A_1, A_2 are nontrivial subgroups of $\mathscr{G}'$. On conjugating $\mathscr{G}'$ by x, we get

$$\mathscr{G} = A_1^x \oplus A_2^x.$$

Thus, by the previous proposition,

$$\{A_1, A_2\} = \{A_1^x, A_2^x\};$$

since $o(x) = 3$,

$$A_i = A_i^x \qquad (i = 1, 2).$$

Similarly,

$$\gamma \text{ normalizes } A_1 \text{ and } A_2.$$

Hence,

$$A_i \lhd \mathscr{G} \qquad (i = 1, 2).$$

Now, by the finite quotient property of $\mathscr{G}$,

$$[\mathscr{G}: A_i] < \infty \qquad \text{and so} \qquad [A_1 \oplus A_2 : A_i] < \infty \qquad (i = 1, 2).$$

Thus,

$$|A_i| < \infty \qquad (i = 1, 2);$$

a contradiction is reached.

4. Word Structure and Depth Function

4.1. *Word Structure in $\mathscr{G}$*

An element from $\mathscr{G}$ is a word

$$w(\gamma, x) = x^{i_1}\gamma^{j_1}x^{i_2} \cdots x^{i_n}\gamma^{j_n}$$

for some natural number n, i's and j's from $\{-1, 0, 1\}$. This word can be rewritten as

$$w(\gamma, x) = (\gamma^{j_1})^{x^{-i_1}}(\gamma^{j_2})^{x^{-i_1-i_2}} \cdots (\gamma^{j_k})^{x^{-i_1-i_2-\cdots-i_k}} \cdots (\gamma^{j_n})^{x^{-i_1-i_2-\cdots-i_n}} x^{i_1+i_2+\cdots+i_n}$$

and clearly,

$$w(\gamma, x) \in \Gamma \Leftrightarrow i_1 + i_2 + \cdots + i_n \equiv 0 \pmod 3.$$

Let $w(\gamma, x) \in \Gamma$. Then

$$w(\gamma, x) = (w_0(\gamma, x), w_1(\gamma, x), w_2(\gamma, x))$$

where the structure of the entries is obtained from

$$(\gamma^j)^{x^{-i}} = \begin{cases} (\gamma^j, x^j, x^{-j}) & \text{if } i = 0 \\ (x^j, x^{-j}, \gamma^j) & \text{if } i = 1 \\ (x^{-j}, \gamma^j, x^j) & \text{if } i = -1, \end{cases}$$

and from the "additive structure" of the sequence $\{i_1, i_2, \ldots, i_n\}$. As a direct application of this analysis, we will show that $\mathscr{G}$ admits an elementary abelian group of automorphisms of order 4.

4.1.1. *Additive Structure of Sequences.*

Let

$$v = \{i_1, i_2, \ldots, i_n\}$$

be a sequence with entries from $\{-1, 0, 1\}$. Let $q \in \{-1, 0, 1\}$. Then the *q-sum marks* of v is defined to be the set of indices

$$\{k_1, k_2, \ldots, k_f\}$$

with

$$1 \leqslant k_1 < k_2 < \cdots < k_f, \qquad f \text{ maximal},$$

such that

$$q \equiv \sum_{v=1}^{k_1} i_v \equiv \sum_{v=1}^{k_2} i_v \equiv \cdots \equiv \sum_{v=1}^{k_f} i_v \qquad (\text{mod } 3);$$

we will write $f(q)$ for f, and call it the *q-sum frequency* of v. If for a given q, the set of q-sum marks is empty, we will write $f(q) = 0$.

The following remarks are easy to verify:

(i) $f(-1) + f(0) + f(1) = n$,

(ii) let $(-v) = \{-i_1, -i_2, \ldots, -i_n\}$. Then, for all q,

$$q\text{-sum marks of } (-v) = (-q)\text{-sum marks of } v.$$

4.1.2. *Structure of $w_i(\gamma, x)$.* Let

$$(k_1, k_2, \ldots, k_s), (\ell_1, \ell_2, \ldots, \ell_t), (m_1, m_2, \ldots, m_u)$$

be the q-sum marks ($q=0, -1, 1$) of $\{i_1, i_2, \ldots, i_n\}$, where $s=f(0)$, $t=f(-1)$, $u=f(1)$. Then

$$w_0(y, x) = x^{v_1} y^{j_{k_1}} x^{v_2} \cdots x^{v_s} y^{j_{k_s}},$$

$$v_1 = \sum_{z=1}^{k_1-1} \left(\sum_{v=1}^{z} i_v \right) j_z, \qquad v_2 = \sum_{z=1}^{k_2-1} \left(\sum_{v=1}^{z} i_v \right) j_z, \ldots,$$

$$v_s = \sum_{z=1}^{k_s-1} \left(\sum_{v=1}^{z} i_v \right) j_z \qquad (\text{if } k_1 = 1, \text{ then } v_1 = 0), \tag{1}$$

$$w_1(y, x) = x^{\mu_1} y^{j_{l_1}} \cdots x^{\mu_t} y^{j_{l_t}} x^{\mu_{t+1}},$$

$$\mu_1 = \sum_{z=1}^{l_1-1} \left[1 + \sum_{v=1}^{z} i_v \right] j_z, \ldots, \mu_t = \sum_{z=1}^{l_t-1} \left[1 + \sum_{v=1}^{z} i_v \right] j_z,$$

$$\mu_{t+1} = \sum_{z=1}^{t} \left[1 + \sum_{v=1}^{z} i_v \right] j_z, \tag{2}$$

$$w_2(y, x) = x^{\eta_1} y^{j_{m_1}} \cdots x^{\eta_u} y^{j_{m_u}} x^{\eta_{n+1}},$$

$$\eta_1 = \sum_{z=1}^{m_1-1} \left[-1 + \sum_{v=1}^{z} i_v \right] j_z, \ldots, \eta_m = \sum_{z=1}^{m_u-1} \left[-1 + \sum_{v=1}^{z} i_v \right] j_z,$$

$$\eta_{u+1} = \sum_{z=1}^{u} \left[-1 + \sum_{v=1}^{z} i_v \right] j_z. \tag{3}$$

4.1.3. *The Length Function l'.* To the natural presentation

$$\langle y \rangle * \langle x \rangle \twoheadrightarrow \mathcal{G},$$

there is associated a syllable length l'. Let $w = (w_0, w_1, w_2) \in \Gamma$. Then by 4.1.2, we have:

(a) $0 \leqslant l'(w_i) \leqslant l'(w)/2$ for all i; unless $w = y^{\pm 1}$;

(b) $l'(w_0) + l'(w_1) + l'(w_2) \leqslant n+2$;

(c) for $w = x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n}$, $l'(w) = 2n$, $n > 0$, the following hold:

(i) $l'(w_0) = n \Rightarrow l'(w_1) < n$ or $l'(w_1) = n$ and $w_2 \in \langle x \rangle$,

(ii) $w_0 w_1^{-1}$ or $w_1 w_2^{-1}$ has l' length less than $2n$ and ends in a power of x;

(d) for $w = x^{i_1} y^{j_1} x^{i_2} \cdots y^{j_n} x^{i_{n+1}}$, $l'(w) = 2n+1$, $n > 0$, the following hold:

(i) $l'(w_i) \leqslant n$ for all i,

(ii) $w_0 w_1^{-1}$ or $w_1 w_2^{-1}$ has l' length less than $2n$, and ends in a power of x.

4.1.4. *The Maps*

$$\tau_1: \gamma \to \gamma, \qquad x \to x^{-1},$$
$$\tau_2: \gamma \to \gamma^{-1}, \qquad x \to x,$$
$$\tau_3: \gamma \to \gamma^{-1}, \qquad x \to x^{-1},$$

extend to automorphisms of $\mathscr{G}$, and generate $\mathscr{V}$, an elementary abelian group of order 4.

Proof. It suffices to show that the τ_i's extend to endomorphisms; this will be done simultaneously for all the τ_i's.

Let

$$w(\gamma, x) = (w_0(\gamma, x), w_1(\gamma, x), w_2(\gamma, x))$$

as in 4.1.2. By Remark (ii) in 4.1.1,

$$w(\gamma, x^{-1}) = (w_0(\gamma, x^{-1}), w_2(\gamma, x^{-1}), w_1(\gamma, x^{-1})),$$
$$w(\gamma^{-1}, x) = (w_0(\gamma^{-1}, x^{-1}), w_1(\gamma^{-1}, x^{-1}), w_2(\gamma^{-1}, x^{-1})),$$
$$w(\gamma^{-1}, x^{-1}) = (w_0(\gamma^{-1}, x), w_2(\gamma^{-1}, x), w_1(\gamma^{-1}, x)).$$

Suppose there exists a word $g(\underline{\gamma}, \underline{x})$ from $\langle \underline{\gamma} \rangle * \langle \underline{x} \rangle$ such that

$$g = g(\gamma, x) = 1, \qquad \text{yet } (\gamma\tau_j, x\tau_j) \neq 1,$$

for some $j \in \{1, 2, 3\}$. Choose g to be of shortest length; clearly,

$$l'(g) > 1.$$

As

$$l'(g_i(\underline{\gamma}, \underline{x})) < l'(g(\underline{\gamma}, \underline{x})) \qquad (i = 0, 1, 2),$$

we have

$$g_i(\gamma^{\tau_1}, x^{\tau_1}) = g_i(\gamma, x^{-1}) = g_i(\gamma, x)^{\tau_1},$$
$$g_i(\gamma^{\tau_3}, x^{\tau_3}) = g_i(\gamma^{-1}, x^{-1}) = g_i(\gamma, x)^{\tau_3},$$
$$g_i(\gamma^{\tau_2}, x^{\tau_2}) = g_i(\gamma^{-1}, x) = g_i(\gamma, x)^{\tau_2} \qquad (i = 0, 1, 2).$$

Thus, by the second paragraph,

$$g(\gamma^{\tau_1}, x^{\tau_1}) = (g_0(\gamma, x)^{\tau_1}, g_2(\gamma, x)^{\tau_1}, g_1(\gamma, x)^{\tau_1}),$$
$$g(\gamma^{\tau_2}, x^{\tau_2}) = (g_0(\gamma, x)^{\tau_3}, g_1(\gamma, x)^{\tau_3}, g_2(\gamma, x)^{\tau_3}),$$
$$g(\gamma^{\tau_3}, x^{\tau_3}) = (g_0(\gamma, x)^{\tau_2}, g_2(\gamma, x)^{\tau_2}, g_1(\gamma, x)^{\tau_2}).$$

However, as $g(\gamma^{\tau_j}, x^{\tau_j}) \neq 1$, we have

$$g_i(\gamma, x)^{\tau_{j'}} \neq 1$$

for some $i \in \{0, 1, 2\}$, and $j' \in \{1, 2, 3\}$; thus, $g_i(\gamma, x) \neq 1$ which contradicts $g(\gamma, x) = (1, 1, 1)$.

4.2. *The Depth Function*

By 2.2.1 item (iv), each element $g \in \mathscr{G}$ is described by

$$\alpha_\phi (:= \alpha) = (\alpha_0, \alpha_1, \alpha_2)\, x^{i(\phi)}, \qquad \alpha_s = (\alpha_{s0}, \alpha_{s1}, \alpha_{s2})\, x^{i(s)}$$

for some function $i: \mathscr{M} \to \{-1, 0, 1\}$. Also, by 4.1.3(a),

$$l'(\alpha_i) \leqslant [l'(\alpha)/2] \qquad \text{for all } i, \text{ unless } \alpha \in \langle \gamma \rangle.$$

Thus, by induction on $l'(\alpha)$, there exists $k \geqslant 0$ such that

$$\alpha_s \in \langle \gamma \rangle \cup \langle x \rangle$$

for all sequences s of length k. Call the least such k by the *depth* of α, and denote it by $d(\alpha)$.

We note: $d(x) = d(\gamma) = 0$,

$$2^m \leqslant l'(\alpha) < 2^{m+1} \Rightarrow d(\alpha) \leqslant m = \log_2 l'(\alpha).$$

5. Aut($\mathscr{G}$)

The fact that

$$\mathscr{G}' > \Gamma' = \mathscr{G}' \times \mathscr{G}' \times \mathscr{G}'$$

has strong implications for Aut($\mathscr{G}$). For instance, let $\sigma \in N_{\mathscr{A}}(\mathscr{G})$, and suppose that σ induces the trivial automorphism on $\mathscr{G}/\mathscr{G}'$. Then,

$$\sigma^{(1)} = (\sigma, \sigma, \sigma) \in \mathscr{A}$$

also normalizes $\mathscr{G}$ and induces the trivial automorphism on $\mathscr{G}/\Gamma'$; this is so, because $\sigma^{(1)}$ commutes with x and

$$[\gamma, \sigma^{(1)}] = ([\gamma, \sigma], 1, 1) \in \mathscr{G}' \times 1 \times 1.$$

Thus, on defining

$$x^{(1)} = (x, x, x), \qquad x^{(i+1)} = (x^{(i)}, x^{(i)}, x^{(i)}) \qquad \text{for } i \geqslant 1,$$

element of $\mathscr{A}$, we find that they form a basis for an elementary abelian 3-group $\mathscr{X}$ which normalizes $\mathscr{G}$.

We will show later that

$$\mathscr{G} \cap \mathscr{X} = 1, \qquad \tilde{\mathscr{G}} = \mathscr{G} \cdot \mathscr{X} \leqslant \mathrm{Aut}(\mathscr{G}),$$

and that the action of $\mathscr{V}$ on $\mathscr{X}$ is given by

$$(x^{(i)})^{\tau_1} = (x^{(i)})^{-1} \qquad \text{for } i \geqslant 1,$$
$$(x^{(2i-1)})^{\tau_2} = (x^{(2i-1)})^{-1}, \qquad (x^{(2i)})^{\tau_2} = x^{(2i)} \qquad \text{for } i \geqslant 1.$$

Our candidate for $\mathrm{Aut}(\mathscr{G})$ is

$$\tilde{\mathscr{G}} \cdot \mathscr{V}.$$

The proof requires many steps. To given an outline, we introduce further notation:

(i) $\mathscr{C} = C_{\mathrm{Aut}(\mathscr{G})}(\mathscr{G}/\mathscr{G}')$ is the kernel of the natural map from $\mathrm{Aut}(\mathscr{G})$ into $\mathrm{Aut}(\mathscr{G}/\mathscr{G}')$,

(ii) $\mathrm{Aut}_{\mathscr{T}}(\mathscr{G})$ is the group of automorphisms of $\mathscr{G}$ induced from autormorphisms of $\mathscr{T}$, and

$$\mathscr{C}_{\mathscr{T}} = \mathscr{C} \cap \mathrm{Aut}_{\mathscr{T}}(\mathscr{G}),$$

(iii) $\mathscr{G}^{(0)} = \mathscr{G}$, $\mathscr{G}^{(i)} = \mathscr{G}^{(i-1)} \wr \langle x \rangle$ for $i \geqslant 1$, $\mathscr{G}^* = \bigcup_{i=0}^{\infty} \mathscr{G}^{(i)}$.

In 5.1, we prove

$$\mathrm{Aut}(\mathscr{G}) = \mathscr{C} \cdot \langle \tau_1, \tau_2 \rangle;$$

here, $\mathscr{G}$ imitates $C_3 \wr C_3$.

In 5.2, we show that the centralizer of $\mathscr{G}$ in $\mathrm{Aut}(\mathscr{T})$ is trivial, and so

$$N_{\mathscr{A}}(\mathscr{G}) = \mathrm{Aut}_{\mathscr{T}}(\mathscr{G}).$$

Furthermore,

$$N_{\mathscr{G}^*}(\mathscr{G}) = \mathscr{G} \cdot \mathscr{X} = \tilde{\mathscr{G}}.$$

Section 5.3 is a proof of

$$\mathscr{C}_{\mathscr{T}} = \mathscr{C}$$

which depends upon the analysis of automorphisms of the type

$$\alpha\colon \gamma \to c\gamma, \qquad x \to x,$$

where $c \in \mathscr{G}'$.

In Section 5.4, we give the last step of the proof of Theorem 3, which is

$$\mathscr{C} = \tilde{\mathscr{G}}.$$

The following notation will be necessary: for $\alpha \in \mathscr{A}$, where $\alpha = (\alpha_1, \alpha_1, \alpha_1)$, write $\alpha = \alpha_1^{(1)}$, inductively, if $\alpha_1 = \alpha_{i+1}^{(i)}$, write $\alpha = \alpha_{i+1}^{(i+1)}$.

5.1.

The following lemmas will describe the induced action of $\operatorname{Aut}(\mathscr{G})$ on $\mathscr{G}/\mathscr{G}'$.

5.1.1. *Let* $\sigma \in \operatorname{Aut}(\mathscr{G})$. *Then, there exist*

$$c \in \mathscr{G}', g \in \mathscr{G}, \text{ and } i, j \in \{-1, 1\}$$

such that

$$\sigma(\gamma) = c\gamma^i, \qquad \sigma(x) = (x^j)^g.$$

Proof. We have

$$\sigma(\gamma) = c\gamma^i x^k$$

for some $c \in \mathscr{G}'$, and $i, k \in \{-1, 0, 1\}$. Suppose $k \neq 0$. Then, as $\sigma(\gamma)$ is of order three, by 2.2.5, $\sigma(\gamma)$ is conjugate to $x^{\pm 1}$. However, this is impossible, since $\mathscr{C}_{\mathscr{G}}(\gamma)$ is not finitely generated, while $\mathscr{C}_{\mathscr{G}}(x)$ is finitely generated. Thus,

$$\sigma(\gamma) = c\gamma^i, \qquad \text{and clearly,} \qquad \sigma(x) = (x^j)^g$$

for some $g \in \mathscr{G}$, and $j \in \{-1, 1\}$.

5.1.2. *Γ is a characteristic subgroup of $\mathscr{G}$.*

Proof. Since

$$\Gamma = \mathscr{G}' \cdot \langle \gamma \rangle,$$

by 5.1.1,

$$\sigma(\gamma) \in \Gamma \text{ for all } \sigma \in \operatorname{Aut}(\mathscr{G}).$$

5.1.3. $\operatorname{Aut}(C_3 \wr C_3)$. *Let* $G = \langle y \rangle \wr \langle z \rangle$ *where* $o(y) = o(z) = 3$. *Easily,*

$$|C_G(y)| = 3^3, \qquad |C_G(z)| = 3^2.$$

This fact is used to show that to every $\sigma \in \operatorname{Aut}(G)$, there corresponds a quadruple (c_1, c_2, i, j) where $c_1, c_2 \in G'$ and $i, j \in \{-1, 1\}$ such that

$$\sigma(y) = c_1 y^i, \qquad \sigma(z) = c_2 z^j.$$

Also, all such choices are realizable. Thus,

$$|\mathrm{Aut}(G)| = 3^4 \cdot 2^2,$$
$$\mathrm{Aut}(G) = C_{\mathrm{Aut}(G)}(G/G') \cdot \langle t_1, t_2 \rangle,$$

where

$$t_1\colon \quad y \to y, \qquad z \to z^{-1}$$
$$t_2\colon \quad y \to y^{-1}, \qquad z \to z.$$

5.1.4. $\mathrm{Aut}(\mathscr{G}) = C_{\mathrm{Aut}(\mathscr{G})}(\mathscr{G}/\mathscr{G}') \cdot \langle \tau_1, \tau_2 \rangle$.

Proof. As Γ is a characteristic subgroup of $\mathscr{G}$, then so is Γ'. Thus, every automorphism of $\mathscr{G}$ induces an autormorphism of $\mathscr{G}/\Gamma'$ ($\simeq C_3 \wr C_3$).

Since $\mathscr{G}$ admits the automorphisms τ_1, τ_2 corresponding to t_1, t_2, the proof is completed by applying 5.1.1.

5.1.5. *Let G be a finitely generated, residually "a finite p-group" where p is a prime number, and G/G' a finite group. Suppose $\sigma \in \mathrm{Aut}(G)$ is of finite order and induces the trivial automorphism on G/G'. Then,*

σ is a p-automorphism.

Proof. Let $\zeta^{(i)}(G)$ $(i \geqslant 1)$ be the lower descending central series of G. Then,

$$G/\zeta^{(i)}(G) \text{ is a finite } p\text{-group}, \qquad \text{and} \qquad \bigcap_{i=1}^{\infty} \zeta^{(i)}(G) = 1.$$

Suppose σ is a p'-automorphism. Then it is well known from finite p-group theory that

$$\sigma|_{G/G'} = 1 \Rightarrow \sigma|_{G/\zeta^{(i)}(G)} = 1 \qquad \forall i \geqslant 1.$$

Thus,

$$[\sigma, G] \leqslant \zeta^{(i)}(G) \qquad \forall i \geqslant 1,$$

and

$$\sigma = 1.$$

5.1.6. *Let $\sigma \in \mathrm{Aut}(\mathscr{G})$ be of finite order. Then,*

$$o(\sigma) = 3^i 2^j$$

for some $i \geqslant 0$, $j \in \{0, 1\}$.

Proof. This is a direct consequence of the previous two lemmas.

5.2. $\mathrm{Aut}_{\mathscr{T}}(\mathscr{G})$

The main purpose here is to show that

$$N_{\mathscr{A}}(\mathscr{G}) = \mathrm{Aut}_{\mathscr{T}}(\mathscr{G}), \qquad N_{\mathscr{G}^*}(\mathscr{G}) = \mathscr{G} \cdot \mathscr{X}.$$

5.2.1. $C_{\mathscr{A}}(\mathscr{G}) = 1$ (*i.e.*, $N_{\mathscr{A}}(\mathscr{G}) = \mathrm{Aut}_{\mathscr{T}}(\mathscr{G})$).

Proof. Recall that a general element from $\mathscr{A}$ has the form

$$\alpha = (\alpha_0, \alpha_1, \alpha_2)\, x^i t^j,$$

where

$\langle x, t \rangle \cong \Sigma_3$. Let $\alpha \in C_{\mathscr{A}}(\mathscr{G})$. Then, α commutes with x, and thus with x modulo $\mathscr{A}_1$. Hence, $j = 0$. As $(\alpha_0, \alpha_1, \alpha_2)$ commutes with x, we have

$$\alpha_0 = \alpha_1 = \alpha_2.$$

On conjugating γ by α, and equating the result to γ, we obtain three equations which lead to $i = 0$; that is,

$$\alpha = (\alpha_0, \alpha_0, \alpha_0) = \alpha_0^{(1)} \in \mathscr{A}_1.$$

Clearly,

$$[\alpha, \gamma] = 1 \Rightarrow [\alpha_0, \gamma] = [\alpha_0, x] = 1,$$

and,

$$\alpha_0 \in C_{\mathscr{A}}(\mathscr{G}).$$

By repeating the above argument, we conclude that

$$\alpha \in \bigcap_{i \geq 1} \mathscr{A}_i = 1.$$

Recall that

$$\mathscr{C} = \mathscr{C}_{\mathrm{Aut}(\mathscr{G})}(\mathscr{G}/\mathscr{G}'),$$

$$\mathscr{C}_{\mathscr{T}} = \mathscr{C} \cap \mathrm{Aut}_{\mathscr{T}}(\mathscr{G}).$$

5.2.2. *Let* $\sigma \in \mathscr{C}_{\mathscr{T}}$. *Then,*

$$\sigma^{(1)} = (\sigma, \sigma, \sigma) \in \mathscr{C}_{\mathscr{T}};$$

indeed,

$$\sigma^{(1)} \textit{ is trivial on } \mathscr{G}/\Gamma'.$$

Proof. This was already shown in the introduction to Section 5.

5.2.3. *Let* $a_0, a_1, a_2 \in \mathscr{A}$, *and define* $\hat{a} = (a_0, a_1, a_2) \in \mathscr{A}$. *Then,*

$$\hat{a} \in \mathrm{Aut}_{\mathscr{T}}(\mathscr{G}) \Rightarrow a_0, a_1, a_2 \in \mathrm{Aut}_{\mathscr{T}}(\mathscr{G}).$$

Proof. This follows from the fact that Γ is a characteristic subgroup of $\mathscr{G}$, and that each a_i induces an autormorphism on $\pi_i(\Gamma) = \mathscr{G}$.

5.2.4. *Let* $a = a_0^{(1)} \in \mathscr{C}_{\mathscr{T}}$. *Then,* $a_0 \in \mathscr{C}_{\mathscr{T}}$.

Proof. By 5.2.3, $a_0 \in \mathrm{Aut}_{\mathscr{T}}(\mathscr{G})$. We know from 5.1.1,

$$a_0(\gamma) = c\gamma^i, \qquad a_0(x) = (x^j)^g$$

and need to show that $i = j = 1$. This is obtained simply by analyzing

$$[\gamma, a] = ([\gamma, a_0], [x, a_0], [x^{-1}, a_0]) \qquad (\mathrm{mod}\ \Gamma').$$

Recall the definition of $\mathscr{X}$, $\mathscr{V}$.

5.2.5. $\mathscr{X}$ *is an elementary abelian* 3-*group, generated freely by* $\{x^{(i)} | i \geqslant 1\}$, *is a subgroup of* $\mathscr{C}_{\mathscr{T}}$, *and* $\mathscr{G} \cap \mathscr{X} = 1$. *Furthermore,* $\mathscr{V}$ *normalizes* $\mathscr{X}$ *as follows*:

$$(x^{(i)})^{\tau_1} = (x^{(i)})^{-1} \qquad (i \geqslant 1),$$
$$(x^{(2i-1)})^{\tau_2} = (x^{(2i-1)})^{-1}, \qquad (x^{(2i)})^{\tau_2} = x^{(2i)} \qquad (i \geqslant 1).$$

Proof. The frist two assertions are easily checked. The third, $\mathscr{G} \cap \mathscr{X} = 1$, follows from 2.2.3 item (iv).

To determine the action of τ_1 on $x^{(1)}$ we consider:

$$\begin{aligned}[(x^{(1)})^{-1}, \gamma^{-1}]^{\tau_1} &= [\gamma\gamma^{x}, \gamma^{x^{-1}}\gamma]^{\tau_1} = [\gamma\gamma^{x^{-1}}, \gamma^{x}\gamma] \\ &= [x^{(1)}, \gamma^{-1}] = [(x^{(1)})^{-\tau_1}, \gamma^{-1}].\end{aligned}$$

Now, the fact

$$(x^{(1)})^{\tau_1} = (x^{(1)})^{-1},$$

follows from $\mathscr{C}_{\mathscr{A}}(\mathscr{G}) = 1$. The rest of the actions are determined in a similar fashion.

5.2.6. *The elements of* $\Gamma\langle x^{(1)}\rangle$ *have the form*

$$(c_0\gamma^{n_0}x^{n_2-n_1+m},\ c_1\gamma^{n_1}x^{n_0-n_2+m},\ c_2\gamma^{n_2}x^{-n_0+n_1+m})$$

for some

$$c_i \in \mathscr{G}',\ n_i \text{ and } m \in \{-1, 0, 1\} \quad (0 \leqslant i \leqslant 2);$$

in particular,

$$(1, \gamma, \gamma^{-1}) \in \Gamma\langle x^{(1)}\rangle \backslash \Gamma.$$

Also,

$$\mathscr{C}_{\mathscr{G}\langle x^{(1)}\rangle}(\gamma) = C_{\mathscr{G}}(\gamma).$$

Proof. These facts follow directly from 2.2.2.

5.2.7. $\langle \gamma, x^{(1)}\rangle \cong \mathscr{G}$ *via the map* $\gamma \to \gamma$, $x^{(1)} \to x$.

Proof. This is a direct application of 5.2.6.

Recall the definitions of $\mathscr{G}^{(i)}$ and $\mathscr{G}^*$.

5.2.8. $N_{\mathscr{G}^*}(\mathscr{G}) = \mathscr{G} \cdot \mathscr{X}$.

Proof. The following assertions clearly hold:

$$\mathscr{G} < \mathscr{G}^{(1)}, \qquad \mathscr{G} \cdot \mathscr{X} \leqslant N_{\mathscr{G}^*}(\mathscr{G}) \leqslant \mathscr{C}_{\mathscr{T}}.$$

As $\gamma^{(1)} = (\gamma, \gamma, \gamma) \in \Gamma$, and $x^{(1)} = (x, x, x) \in \mathscr{G}\langle x^{(1)}\rangle$, it follows that

$$g \in \mathscr{G} \Rightarrow g^{(1)} \in \mathscr{G}\langle x^{(1)}\rangle.$$

Also, for $k \geqslant 1$,

$$g \in \mathscr{G}\langle x^{(1)},..., x^{(n)}\rangle \Rightarrow g^{(1)} \in \mathscr{G}\langle x^{(1)},..., x^{(n+1)}\rangle.$$

We will prove by induction on n that

$$N_{\mathscr{G}^{(n)}}(\mathscr{G}) = \mathscr{G}\langle x^{(1)},..., x^{(n)}\rangle.$$

Let $n = 1$. We may work modulo Γ'. Let

$$w = (\gamma^{j_0}x^{i_0}, \gamma^{j_1}x^{i_1}, \gamma^{j_2}x^{i_2}) \in N_{\mathscr{G}^{(1)}}(\mathscr{G}),$$

for some integers $i_0, i_1, i_2, j_0, j_1, j_2$; by the first paragraph, we may assume $i_0 = j_0 = 0$. Since

$$\gamma^{(1)}, x^{(1)} \in \mathscr{G}\langle x^{(1)}\rangle, \qquad \text{and} \qquad \gamma^x\gamma^{x^{-1}} = (1, \gamma x^{-1}, x\gamma),$$

w may be reduced to an element of the form

$$w' = (1, \gamma^{k_0}, x^{k_1}\gamma^{k_2})$$

for some integers k_0, k_1, k_2. Now, as $w' \in \mathscr{C}_{\mathscr{T}}$,

$$[w', x] = (x^{k_1}\gamma^{k_2}, \gamma^{-k_0}, x^{-k_1}\gamma^{-k_2+k_0}) \in \mathscr{G}',$$

and thus

$$k_1 = 0, \qquad k_2 = -k_0.$$

But then by 5.2.6,

$$w' = (1, \gamma^{k_0}, \gamma^{-k_0}) \in \Gamma\langle x_1 \rangle.$$

Suppose our assertion is true for some $n \geqslant 1$, and let $w = (w_0, w_1, w_2) \in \mathscr{G}^{(n+1)}$ normalize $\mathscr{G}$. Then, w normalizes Γ, and consequently,

$$w_i \in \mathscr{G}^{(n)} \text{ normalizes } \pi_i(\Gamma) \ (= \mathscr{G})$$

for $i = 0, 1, 2$.

Thus, by the inductive hypothesis,

$$w_i \in \mathscr{G}\langle x^{(1)}, \dots, x^{(n)} \rangle \qquad (i = 0, 1, 2).$$

Now, as

$$w_0^{(1)} \in \mathscr{G}\langle x^{(1)}, \dots, x^{(n+1)} \rangle,$$

it follows that

$$u = (1, w_0^{-1} w_1, w_0^{-1} w_2) \text{ normalizes } \mathscr{G},$$
$$u \in \mathscr{C}_{\mathscr{T}}.$$

Hence,

$$[u, x] = (w_0^{-1} w_2, w_1^{-1} w_0, w_2^{-1} w_1) \in \mathscr{G}',$$
$$w_0^{-1} w_2, w_1^{-1} w_0, w_2^{-1} w_1 \in \mathscr{G},$$
$$u \in N_{\mathscr{G}^{(1)}}(\mathscr{G}) = \mathscr{G} \cdot \langle x^{(1)} \rangle.$$

Thence,

$$w = w_0^{(1)} u \in \mathscr{G}\langle x^{(1)}, \dots, x^{(n+1)} \rangle.$$

5.3.

Our aim is to prove

$$\mathscr{C}_{\mathscr{T}} = \mathscr{C}.$$

5.3.1. *Let $\alpha \in \mathscr{C}$. Then there exist $g \in \mathscr{G}$, $v \in \Gamma'$ such that*

$$\alpha g\colon \gamma \to v\gamma, \qquad x \to x.$$

Furthermore, if $\alpha(x) = x$, then g may be chosen from $\langle x \rangle$.

Proof. By 5.1.1, there exist $v \in \mathscr{G}'$, $h \in \mathscr{G}$, such that

$$\alpha\colon \gamma \to v\gamma, \qquad x \to x^h.$$

Hence,

$$\alpha h^{-1}\colon \gamma \to v'\gamma, \qquad x \to x$$

where,

$$v' \in \mathscr{G}'.$$

By 2.2.5 item (ii),

$$v'\gamma = \gamma, \gamma^x, \gamma^{x^{-1}} \qquad (\text{mod } \Gamma').$$

Hence,

$$\alpha h^{-1} x^i\colon \gamma \to v''\gamma, \qquad x \to x$$

for some $i \in \{-1, 0, 1\}$, and $v'' \in \Gamma'$.

5.3.2. *Let $\alpha \in \mathscr{C}$ and suppose*

$$\alpha\colon \gamma \to (c_0\gamma, c_1 x, c_2 x^{-1}), \qquad x \to x$$

for some $c_0, c_1, c_2 \in \mathscr{G}'$. Then, there exists $g \in \mathscr{G}$ such that

$$\alpha g^{(1)}\colon \gamma \to (c_0'\gamma, c_1' x, x^{-1}), \qquad x \to x$$

for some $c_0', c_1' \in \mathscr{G}'$.

Proof. As $c_2 x^{-1} = (x^{-1})^h$ for some $h \in \mathscr{G}$, and as $h^{(1)} \in \mathscr{G}\langle x^{(1)} \rangle$, we have

$$\alpha(h^{(1)})^{-1}\colon \gamma \to (c_0'\gamma, c_1' x, x^{-1})$$

for some $c_0', c_1' \in \mathscr{G}'$.

5.3.3. *Let $\alpha \in \mathscr{C}$ and suppose*

$$\alpha : \gamma \to (c_0\gamma, c_1 x, x^{-1}), \qquad x \to x$$

for some $c_0, c_1 \in \mathscr{G}'$. Then, $c_1 = 1$.

Proof. By 3.3.1, and as $\mathscr{G}'$ is indecomposable, α permutes the indecomposable factors

$$\mathscr{G}' \times 1 \times 1, \qquad 1 \times \mathscr{G}' \times 1, \qquad 1 \times 1 \times \mathscr{G}'$$

of Γ'. Now,

$$\alpha : [\gamma\gamma', \gamma'^{-1}\gamma] \quad (= ([x^{-1}, \gamma^{-1}], 1, 1))$$
$$\to ([c_0\gamma x^{-1}, c_1 x c_0 \gamma], [c_1 x c_0\gamma, x^{-1} c_1 x], [x^{-1} c_1 x, c_0\gamma x^{-1}]).$$

This image is equal to

$$([x^{-1}, \gamma^{-1}], 1, 1) \qquad \text{modulo } \zeta^{(3)}(\mathscr{G}) \times \zeta^{(3)}(\mathscr{G}) \times \zeta^{3}(\mathscr{G}).$$

Thus,

$$[\gamma\gamma', \gamma'^{-1}\gamma]^{\alpha} \in \mathscr{G}' \times 1 \times 1.$$

In other words,

$$c_1' \text{ centralizes } \langle c_1 x, c_0\gamma, c_0\gamma x^{-1} \rangle.$$

Thus, c_1' commutes with

$$(c_0\gamma x^{-1})(c_1 x c_0 \gamma)\, c_1' = (c_0\gamma \cdot c_1')^2,$$

and consequently, with

$$c_0\gamma c_1', \; c_0\gamma, \; (c_0\gamma)^{-1} \cdot c_0\gamma x^{-1} = x^{-1}.$$

Hence,

$$c_1' \text{ centralizes } \langle c_0\gamma, x \rangle.$$

Finally, since

$$\mathscr{G} = \langle \gamma'', x \rangle,$$

and

$$\pi_0 : \Gamma \twoheadrightarrow \langle c_0\gamma, c_1 x, x^{-1} \rangle = \mathscr{G},$$

we have

$$c_1^r \in Z(\mathscr{G}) = 1.$$

5.3.4. *Let $a \in \mathscr{C}$ and suppose*

$$a: \gamma \to (c\gamma, x, x^{-1}), \qquad x \to x$$

for some $c \in \mathscr{G}'$. Then,

$$a': \gamma \to c\gamma, \qquad x \to x$$

is an automorphism of $\mathscr{G}$.

Proof. First we show that a commutes with $x^{(1)}$. Let

$$w(\gamma, x) = (w_0(\gamma, x), w_1(\gamma, x), w_2(\gamma, x))$$

be some element of Γ. Then,

$$w(\gamma, x)^a = w(\gamma^a, x) = (w_0(c\gamma, x), w_1(c\gamma, x), w_2(c\gamma, x)).$$

Now, considering the fact

$$(x^{(1)})^{-1} a x^{(1)}: \gamma \to \gamma^a,$$

we have that

$$[a, x^{(1)}] \text{ centralizes } \gamma \text{ and } x.$$

Thus,

$$a \text{ centralizes } x^{(1)}, \text{ and } a \text{ normalizes } \langle \gamma, x, x^{(1)} \rangle.$$

Let $w(\gamma, x^{(1)}) \in \Gamma\langle x^{(1)} \rangle$. Then,

$$w(\gamma, x^{(1)}) = (w(\gamma, x), w(x, x), w(x^{-1}, x))$$

and

$$w(\gamma, x^{(1)})^a = w(\gamma^a, x^{(1)}) = (w(c\gamma, x), w(x, x), w(x^{-1}, x)).$$

Now, we assert that $a': \gamma \to c\gamma$, $x \to x$ extends to an automorphism of $\mathscr{G}$. Let $w(z_1, z_2)$ be a word such that $w(\gamma, x) = 1$. Then, by the isomorphism

$$\gamma \to \gamma, x \to x^{(1)}$$

between $\mathscr{G}$ and $\langle \gamma, x^{(1)} \rangle$, see 5.2.7, we have

$$w(\gamma, x^{(1)}) = 1, \qquad w(\gamma, x^{(1)})^{\alpha} = w(\gamma^{\alpha}, x^{(1)}) = 1,$$
$$w(c\gamma, x) = 1;$$

that is,

$$w(\gamma^{\alpha'}, x^{\alpha'}) = 1.$$

Hence, α' extends to an endomorphism of $\mathscr{G}$. Since

$$\pi_0 \colon \Gamma^{\alpha}(=\Gamma) \twoheadrightarrow \langle c\gamma, x \rangle = \mathscr{G},$$

α' is an endomorphism.

Finally, let $w(z_1, z_2)$ be a word such that $w(\gamma^{\alpha'}, x^{\alpha'}) = w(c\gamma, x) = 1$. Then,

$$w(\gamma, x^{(1)})^{\alpha} = (1, w(x, x), w(x^{-1}x)) \in \Gamma\langle x^{(1)} \rangle.$$

By 5.2.6,

$$w(x, x) = 1 = w(x^{-1}, x).$$

Thus,

$$w(\gamma, x^{(1)})^{\alpha} = 1 = w(\gamma, x^{(1)}),$$

and

$$w(\gamma, x) = 1.$$

Hence,

$$\alpha' \text{ is an automorphism of } \mathscr{G}.$$

5.3.5. $\mathscr{C} = \mathscr{C}_{\mathscr{T}}$.

Proof. Let $\alpha \in \mathscr{C}$. By 5.3.1, we may assume

$$\alpha \colon \gamma \to (c_0\gamma, c_1 x, c_2 x^{-1})^{\gamma^{i_1}}, \qquad x \to x$$

for some integer i_1. By 5.3.2 and 5.3.3,

$$\alpha \colon \gamma \to (c_1'\gamma, x, x^{-1})^{g_1^{(1)} \gamma^{i_1}}, \qquad x \to x$$

for some $c_1' \in \mathscr{G}'$, $g_1 \in \mathscr{G}$.

By 5.3.4,

$$\alpha' : \gamma \to c_1'\gamma, \qquad x \to x$$

is an automorphism of $\mathscr{G}$. Thus, by the first paragraph, there exist $g_2 \in \mathscr{G}$, $c_2' \in \mathscr{G}'$, i_2 an integer such that

$$\alpha' : \gamma \to (c_2'\gamma, x, x^{-1})^{g_2^{(1)} x^{i_2}}, \qquad x \to x.$$

Hence

$$\alpha : \gamma \to ((c_2'\gamma, x, x^{-1}), x, x^{-1})^{g_2^{(2)} (x^{(1)})^{i_2} g_1^{(1)} x^{i_1}}$$

$$x \to x.$$

The above procedure produces the sequences

$$\{i_j \mid j \geqslant 1\} \quad \text{from} \quad \{-1, 0, 1\},$$

$$\{g_j \mid j \geqslant 1\} \quad \text{from} \quad \mathscr{G},$$

$$\{c_j' \, j \geqslant 1\} \quad \text{from} \quad \mathscr{G}',$$

such that

$$\alpha : \gamma \to ((\cdots(c_j'\gamma, x, x^{-1}), \ldots), x, x^{-1})^{g_j^{(j)} (x^{(j-1)})^{i_j} g_{j-1}^{(j-1)} \cdots (x^{(1)})^{i_2} g_1^{(1)} x^{i_1}}$$

$$x \to x.$$

We observe that the infinite product

$$\zeta = \cdots g_j^{(j)} \cdot (x^{(j-1)})^{i_j} g_{j-1}^{(j-1)} \cdots (x^{(1)})^{i_2} g_1^{(1)} \cdot x^{i_1}$$

is a well-defined element from A, and

$$\gamma^{\alpha\zeta^{-1}} = \gamma, \qquad x^{\alpha\zeta^{-1}} = x.$$

Thus,

$$\alpha = \zeta.$$

5.4.

In this section, we will prove

$$\mathscr{C} = \tilde{\mathscr{G}} = \mathscr{G} \cdot \mathscr{X}.$$

5.4.1. *Let* $\alpha \in \mathscr{C}$ such that

$$\alpha : \gamma \to \gamma^g, \qquad x \to x$$

for some $g \in \mathscr{G}$. *Then,* α *is induced by conjugation by an element from* $\mathscr{C}_{\mathscr{G}}(x)$.

Proof. We proceed by induction on the length $l'(g)$. We may make the following assumptions: $l'(g) > 1$, $g \in \Gamma$,

$$g = w = w(y, x) = x^{i_1} y^{j_1} \cdots x^{i_k} y^{i_k} \text{ of length } 2k,$$

or

$$= x^{i_1} y^{j_1} \cdots x^{i_k} \text{ of length } 2k - 1;$$

also, $w = (w_0, w_1, w_2)$.

By 5.3, $a \in \mathscr{C}_{\mathscr{T}}$, and as $a(x) = x$, it follows that

$$a = a_0^{(1)} x^k$$

where $a_0 \in \mathscr{C}_{\mathscr{T}}$, and $k \in \{0, \pm 1\}$. Since $\langle y \rangle$ and $\langle x \rangle$ are not conjugate in $\mathrm{Aut}(\mathscr{G})$, we have

$$k = 0, \qquad y^{w_0} = y^{a_0}, \qquad x^{w_1} = x^{a_0}, \qquad x^{w_2} = x^{a_0},$$

and

$$w_1 w_2^{-1} \in C_{\mathscr{G}}(x).$$

We also have

$$a_0' : y \to y^{w_0 w_1^{-1}}, \qquad x \to x$$
$$a_1' : y \to y^{w_0 w_2^{-1}}, \qquad x \to x$$

are autormorphisms of $\mathscr{G}$.

We are now in a position to apply induction. By 4.1.3 part (d), one of $w_0 w_1^{-1}$, $w_0 w_2^{-1}$ has length less than $l'(w)$ and ends in a power of x; say it is $w_0 w_1^{-1}$. Clearly, we may assume $w_0 w_1^{-1} \in \Gamma$. By the inductive hypothesis, there exists $g_1 \in \mathscr{C}_{\mathscr{G}}(x)$ such that

$$a_0' = a_0 w_0^{-1} = g_1,$$
$$a_0 = g_1 w_0,$$
$$a = (g_1 w_0, g_1 w_0, g_1 w_0) \in \Gamma \langle x^{(1)} \rangle.$$

Now, as

$$ag^{-1} \text{ centralizes } y,$$

by 5.2.6,

$$ag^{-1} \in \Gamma, \qquad a \in \Gamma.$$

5.4.2. *Let $\alpha \in \mathscr{C}$. Then, there exist*

$$\alpha_1 \in \mathscr{C}, \qquad g_0, h_0 \in \mathscr{G}, \qquad k_0 \geqslant 0$$

such that

$$\alpha = (\alpha_1, \alpha_1, \alpha_1)(1, g_0, h_0)\, x^{k_0};$$

this representation is unique.

Proof. There exists $g = (g_0, g_1, g_2) \in \Gamma$ such that $x^\alpha = x^g$. Hence, $\alpha g^{-1} \in C_{\mathscr{C}}(x)$, and so

$$\alpha g^{-1} = (\alpha_1, \alpha_1, \alpha_1)\, x^k$$

for some $\alpha_1 \in \mathscr{C}$, $k \geqslant 0$. Let $w = g^{x^{-k}}$. Since we have the factorization

$$w = (w_0, w_1, w_2) = (w_0, w_0, w_0)(1, w_0^{-1} w_1, w_0^{-1} w_2)$$

in $\mathscr{G}^{(1)}$,

$$\alpha = (\alpha_1 w_0, \alpha_1 w_0, \alpha_1 w_0)(1, w_0^{-1} w_1, w_0^{-1} w_2)\, x^k.$$

Uniqueness is easy to prove.

We note that $(1, g_0, h_0) \in \mathscr{G}^{(1)}$.

5.4.3. $\mathscr{C} = \tilde{\mathscr{G}}$.

Proof. Given $\alpha \in \mathscr{C}$, we find for all $i \geqslant 1$, $\alpha_i \in \mathscr{C}$, $g_i, h_i \in \mathscr{G}$, $k_i \geqslant 0$, such that

$$\begin{aligned}
\alpha &= \alpha_0, \\
\alpha_0 &= (\alpha_1, \alpha_1, \alpha_1)(1, g_0, h_0)\, x^{k_0}, \\
&\ \vdots \\
\alpha_i &= (\alpha_{i+1}, \alpha_{i+1}, \alpha_{i+1})(1, g_i, h_i)\, x^{k_i}. \\
&\ \vdots
\end{aligned}$$

Since

$$\gamma^\alpha = (\gamma^{\alpha_1}, x^{\alpha_1 g_0}, x^{-\alpha_1 h_0})\, x^{k_0}$$

has finite depth $d \geqslant 0$,

$$\gamma^{\alpha_d} = (\gamma^{\alpha_{d+1}}, x^{\alpha_{d+1} g_d}, x^{-\alpha_{d+1} h_d})\, x^{k_d} = (\gamma, x, x^{-1}).$$

Thus,

$$k_d = 0, \qquad \gamma^{\alpha_{d+1}} = \gamma, \qquad x^{\alpha_{d+1} g_d} = x, \qquad x^{\alpha_{d+1} h_d} = x.$$

We have

$$a_{d+1}\, g_d\colon \gamma \to \gamma^{g_d}, \qquad x \to x$$

an automorphism of $\mathscr{G}$. Thus, by 5.4.1,

$$\gamma^{g_d} = \gamma^h$$

for some $h \in C_{\mathscr{G}}(x)$. Therefore,

$$a_{d+1}\, g h^{-1} = 1, \qquad a_{d+1} = h g_d^{-1};$$

hence

$$a \in \tilde{\mathscr{G}}.$$

With this, Theorem 3 is fully established.

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References

1. R. I. Grigorchuk, On the Burnside problem for perdiodic groups, *Funktsional Anal. i Prilozhen.* **14** (1980), 53–54. [Russian]. Engl. transl.: *Functional Anal. Appl.* **14** (1980), 41–43.
2. N. Gupta and S. Sidki, On the Burnside problem for periodic groups, *Math. Z.* **182** (1983), 385–388.
3. N. Gupta and S. Sidki, Extensions of groups by tree automorphisms, *Contemp. Math.* **33** (1984), 232–246.
4. N. Gupta and S. Sidki, Some infinite p-groups, *Algebra i Logika* **22** (1983), 584–589.
5. S. Sidki, On a 2-generated infinite 3-group: The presentation problem, *J. Algebra* **110** (1987), 13–23.