TWO-LOOP RENORMALISATION OF $d=2$ σ-MODELS WITH TORSION

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A calculation of the two-loop β-function for σ-models with torsion is presented, for the bosonic and the $N=1$ supersymmetric cases (using component fields). In the bosonic case the result agrees with a recent calculation by Hull and Townsend, while in the supersymmetric case, we obtain a vanishing two-loop β-function.

Recently the calculation of the two-loop β-function, $\beta^{(2)}$, in two-dimensional non-linear σ-models with torsion has been the subject of some controversy in the literature [1-5]. The reason the calculation is not a straightforward exercise in dimensional regularisation arises from the presence of the antisymmetric tensor $\epsilon^{\mu\nu}$. One may choose to deal with this object by systematically employing the relationship

$$\epsilon^{\mu\nu}\epsilon^{\rho\sigma}=-(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\rho\nu}) , \tag{1}$$

and then continuing to d dimensions. However, as is well known, consideration of the quantity

$$\epsilon^{\mu\nu}\epsilon^{\rho\sigma}\epsilon_{\mu\nu} \tag{2}$$

then leads to the relationship

$$(d+1)(d-2)\epsilon^{\rho\sigma}=0 . \tag{3}$$

Employment of eq. (1) is therefore not in itself a well-defined prescription for extracting the simple pole in $\epsilon=2-d$ from a two-loop graph. Clearly by virtue of eq. (3), the simple pole from any graph involving three or more ϵ tensors will depend on the order in which the contractions are performed, if the graph has a double pole. These problems persist in general even if regularisation by dimensional reduction is employed. (The equivalent problem concerning $\epsilon^{\mu\nu\rho\sigma}$ and the associated γ^5 is familiar in four dimensions: for a discussion of two-loop calculations in that case see ref. [6].) For this reason, the result of ref. [1] for the bosonic case cannot be regarded as the consequence of any particular scheme. It should be noted, however, that in ref. [1] the analogous calculation is also performed for the $N=1$ supersymmetric case (using superfields) with the result that $\beta^{(2)}$ vanishes. In fact, the power of superfields is such that the ambiguity described above seems not to arise, and this result remains undisputed. It is clear, therefore, that insight into the bosonic result may be gained by repeating the supersymmetric calculation in component form, and the purpose of this paper is to present such a calculation.

While this calculation was in its final stages I received a copy of ref. [3] which treats the bosonic case. For this case my conclusions are in complete accordance with those of Hull and Townsend (HT). In view of the importance of the result, however, it seems to me legitimate to present an independent computation, differing in some details. The treatment of fermions in the torsion case presents additional subtleties, and the cancellation in the supersymmetric case provides a powerful check on the bosonic result.

The result of this investigation is that a prescription for $\epsilon^{\mu\nu}$ which avoids the ambiguities described above is to set

$$\eta_{\mu\rho}\epsilon^{\mu\nu}\epsilon^{\rho\sigma}=-\eta^{\nu\sigma} . \tag{4}$$

This is, of course, the two-dimensional relation. In ref. [3], HT consider a generalization of eq. (4) to

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$$\eta_{\mu\rho}\epsilon^{\mu\nu}\epsilon^{\rho\sigma} = -(1+c\cdot\epsilon)\eta^{\nu\sigma} \tag{5}$$

and conclude that $c=0$ is the correct choice [#1]. As we shall see, it also leads to the expected result in the supersymmetric case. This is natural in that superfield Feynman rules and manipulations automatically perform tensor contractions in two dimensions, in accordance with eq. (5). Other values of c do not give the supersymmetric result, and would also result in a lack of manifest covariance (with respect to the connection with torsion).

The lagrangian for the bosonic case is

$$L(\phi) = \tfrac{1}{2}[\eta^{\mu\nu}g_{ij}(\phi) + \epsilon^{\mu\nu}e_{ij}(\phi)]\partial_\mu\phi^i\,\partial_\nu\phi^j , \tag{6}$$

where $\eta^{\mu\nu} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $\epsilon^{\mu\nu} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ and g_{ij} and e_{ij} are symmetric and antisymmetric, respectively.

Computation of the radiative corretions to L is facilitated by means of a background field expansion in terms of a quantum field ξ^i which is a vector on the manifold M with coordinates ϕ^i and metric g_{ij}. For terms quadratic in ξ we have [1,7–9]

$$L_B^{(2)} = \tfrac{1}{2}g_{ij}(\phi)\hat\nabla_\mu\xi^i\hat\nabla^\mu\xi^i + \tfrac{1}{2}\hat R_{iklj}(\eta^{\mu\nu}+\epsilon^{\mu\nu})\partial_\mu\phi^i\partial_\nu\phi^j\xi^k\xi^l , \tag{7}$$

and for terms cubic in $\xi^{(1)}$

$$L_B^{(3)} = \tfrac{1}{2}A^\nu_{ijk}\hat\nabla_\nu\xi^i\xi^j\xi^k + \tfrac{1}{3}\epsilon^{\mu\nu}S_{ijk}\hat\nabla_\mu\xi^i\hat\nabla_\nu\xi^j\xi^k , \tag{8}$$

where

$$\hat\nabla_\mu\xi^i = \nabla_\mu\xi^i + S^i{}_{jk}\epsilon_{\mu\nu}\partial^\nu\phi^j\xi^k , \tag{9}$$

$$S_{ijk} = \tfrac{1}{2}(\partial_i e_{jk} + \partial_j e_{ki} + \partial_k e_{ij}) , \tag{10}$$

$$\hat R_{iklj} = R_{iklj} + S_{iml}S^m{}_{kj} - S_{imj}S^m{}_{kl} - \nabla_l S_{ikj} + \nabla_j S_{ikl} , \tag{11}$$

$$A^\mu_{ijk} = \tfrac{2}{3}\partial_\mu\phi_l[\eta^{\mu\nu}(\hat R_{j(li)k} + \hat R_{k(li)j}) + \epsilon^{\mu\nu}(\hat R_{j[li]k} + \hat R_{k[li]j})] . \tag{12}$$

We have omitted from $L_B^{(3)}$ a term proportional to ξ^3 which makes no contribution to two-loop divergences. There are two-loop graphs involving ξ^4 vertices, but the conclusion of ref. [1] that these do not contribute to $\beta^{(2)}$ is correct so we refrain from repeating this analysis.

[#1] Similar remarks are made in ref. [4].

For the supersymmetric case we will also require L_F, given by [7,8,10,11]

$$L_F = \tfrac{1}{2}g_{ij}\bar\psi^i\gamma^\mu\nabla_\mu\psi^j + (\psi^4 \text{ terms}) \tag{13}$$

(where ψ is a Majorana function), and the terms linear in ξ in its background field expansions, given by

$$L_F^{(1)} = \tfrac{1}{2}\bar\psi^i\gamma^\mu\psi^j(\xi^k B_{\mu ijk} - S_{ijk}\epsilon_{\mu\nu}\hat\nabla^\nu\xi^k) , \tag{14}$$

where

$$B_{\mu ijk} = \tfrac{1}{4}\partial^\nu\phi^l[(\eta_{\mu\nu}-\epsilon_{\mu\nu})(\hat R_{klij} + \hat R_{jlik} + \hat R_{lijk}) - (\eta_{\mu\nu}+\epsilon_{\mu\nu})(\hat R_{ijlk} + \hat R_{kjli} + \hat R_{kijl})] . \tag{15}$$

Terms of the form ψ^4 and $\psi^2\phi^2$ do not contribute to $\beta^{(2)}$.

The relevant graphs for the calculation of the divergent terms in the two-loop effective action for ϕ are shown in tables 1 and 2. Propagators are defined in the usual way by referring the vectors ψ^i, ξ^i to a tangent frame. We follow the procedure of ref. [12] in performing the subtractions at the level of the integrals and discarding any integral or diagram which (after subtraction) gives only double poles. While this method gives rise to fewer cross-checks that that of directly computing counter-term insertions, it is simpler and more suited to higher-loop calculations. We regulate infra-red divergences when

Table 1
Contributions to the effective action for ϕ from graphs with ξ propagators only. X, Y, Z are defined as follows:

$$\begin{pmatrix}X\\ Y\\ Z\end{pmatrix} = \frac{\pi^2}{(2\pi)^4}\frac{1}{\epsilon}(\partial^\mu\phi^i\partial_\mu\phi^j + \epsilon_{\mu\nu}\partial^\mu\phi^i\partial^\nu\phi^j)\begin{pmatrix}X_{ij}\\ Y_{ij}\\ Z_{ij}\end{pmatrix} .$$

See eq. (24) for the definitions of X_{ij}, Y_{ij}; $Z_{ij} = \hat R_{iklj}S^k{}_{mn}S^{lmn}$.

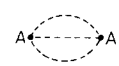
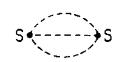
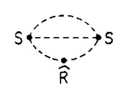
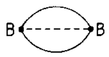
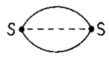
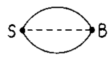
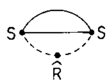
	$\frac{1}{3}(X-Y)$
	$\frac{1}{12}(2X+Y)$
	Z
TOTAL	$\frac{1}{4}(2X-Y)+Z$

Table 2
As table 1 for graphs involving ψ propagators.

	$\frac{1}{8}(-2X+3Y)$
	$-\frac{3}{8}(2X+Y)$
	$\frac{1}{4}(2X+Y)$
	$-Z$
TOTAL	$-\frac{1}{4}(2X-Y)-Z$

necessary by inserting masses on the propagators [12,13].

As an example, consider the first graph of table 2. This gives rise to the following contribution to the effective action:

$$\Delta=-(2\pi)^{-4}\tfrac{1}{4}B_{\mu ijk}B_{\nu}^{ijk}I^{\mu\nu}\,, \tag{16}$$

where

$$I^{\mu\nu}=\int\frac{d^2k\,d^2q\,d^2r}{k^2q^2r^2}\,\delta^{(2)}(k-q+r)\,\mathrm{Tr}(\gamma^\mu \not{k}\gamma^\nu \not{q}) \tag{17}$$

$$=2\int\frac{d^2k\,d^2q\,d^2r}{k^2q^2r^2}\,(2k^\mu q^\nu-\eta^{\mu\nu}k\cdot q)$$

$$\times\delta^{(2)}(k-q+r)\,. \tag{18}$$

The second term in eq. (18) gives double poles only on subtraction. We drop this and thus obtain

$$I^{\mu\nu}\Rightarrow\frac{4}{n}g^{\mu\nu}\int\frac{d^2k\,d^2q\,d^2r}{k^2q^2r^2}\,k\cdot q\,\delta^{(2)}(k-q+r)$$

$$=\frac{2}{n}I^2\eta^{\mu\nu}\,, \tag{19}$$

where

$$I=\int\frac{d^dk}{k^2}\Rightarrow\int\frac{d^dk}{k^2+\mu^2}=i\pi\left(\frac{2}{\epsilon}-\gamma\right)\mu^{-\epsilon}\,. \tag{20}$$

The subtracted result for $I^{\mu\nu}$ is hence (discarding the double pole)

$$I^{\mu\nu}=-\frac{2}{\epsilon}\pi^2\eta^{\mu\nu}\,. \tag{21}$$

In fact, of course, direct evaluation of $I^{\mu\nu}$ from eq. (18) with $r^2\to r^2+\mu^2$ reveals that $I^{\mu\nu}$ actually has no double pole, so that the various discarded terms actually cancel, and eq. (12) is obtained directly. Substituting in eq. (16)

$$\Delta=\frac{\pi^2}{(2\pi)^4}\frac{1}{2\epsilon}B^{\mu}_{ijk}B^{ijk}_{\mu} \tag{22}$$

or

$$\Delta=\frac{\pi^2}{(2\pi)^4}\frac{1}{8\epsilon}\,(\partial_\mu\phi^i\partial^\mu\phi^j+\epsilon_{\mu\nu}\partial^\mu\phi^i\partial^\nu\phi^j)$$

$$\times(-2X_{ij}+3Y_{ij})\,, \tag{23}$$

where

$$X_{ij}=\hat{R}_{iklm}\hat{R}^{klm}{}_j \quad\text{and}\quad Y_{ij}=\hat{R}_{iklm}\hat{R}^{lmk}{}_j\,. \tag{24}$$

There is a subtle point associated with the fermion part of the calculation, as follows: We could, in eqs. (13)–(15), have removed $\epsilon_{\mu\nu}$ by use of the relation

$$\epsilon_{\mu\nu}\gamma^\nu=\gamma^3\gamma_\mu\,, \tag{25}$$

where

$$\gamma^3=\gamma^0\gamma^1\,. \tag{26}$$

It is not then equivalent, however, to proceed with a totally anticommuting γ^3, because we would then have interactions whereby $\bar{\psi}\gamma^\mu(1\pm\gamma^3)\psi$ coupled differently to the background fields like a chiral Schwinger model. As is well known, the result is potential anomalies, and a totally anticommuting γ^3 in conjunction with dimensional regularization is inadequate. We would introduce an 't Hooft–Veltman style [14] γ^3 or evaluate the fermion loop using a Pauli–Villars (PV) regulator, but for ease of comparison with the bosonic sector it is easier to stick with $\epsilon_{\mu\nu}$. The effect is that terms of the form X_{ij} and Y_{ij} are produced, whereas use of an anticommuting γ^3 leads to terms like $\hat{R}_{iabc}\hat{R}_j{}^{abc}$, etc. [5].

The other graphs are calculated in similar fashion. In order to rewrite the resulting expressions in the same form as Δ above, the following relations are useful:

$$\hat{\nabla}^{\mu}S^{ijk}\hat{\nabla}_{\mu}S_{ijk}=3\epsilon^{\mu\nu}B_{\mu ijk}\hat{\nabla}_{\nu}S^{ijk}$$
$$=\tfrac{3}{4}(\partial_{\mu}\phi^{i}\partial^{\mu}\phi^{j}+\epsilon_{\mu\nu}\partial^{\mu}\phi^{i}\partial^{\nu}\phi^{j})(2X_{ij}+Y_{ij}) . \quad (27)$$

Of course the second graph in table 1, for example, actually gives rise to a term of the form $(\partial_{\mu}S_{ijk})^2$. We have assumed (as did HT) that connection insertions from $L_{\mathrm{B}}^{(2)}$, eq. (7), result in covariantisation $\partial_{\mu}\to\hat{\nabla}_{\mu}$. While this presumably occurs, it would be worthwhile to check this, particularly in view of the remarks above concerning the lurking of proportional anomalies in the fermion sector. Note also that covariantisation would presumably be disturbed by the use of an n-dependent prescription for $\epsilon^{\mu\nu}$ [for example $c\neq 0$ in eq. (5)].

Note that the null result in the supersymmetric case (evident from the totals in tables 1 and 2) is not a result of graph by graph cancellation (except in the case of the last graph). This makes it improbable that there exists a simple alternative prescription for $\epsilon^{\mu\nu}$ which gives the same supersymmetric result. The final result [3] for the two-loop contribution to the β-function is

$$\beta_{ij}^{(2)}=\frac{-1}{16\pi^2}(2\hat{R}_{iklm}\hat{R}^{klm}{}_{j}-\hat{R}_{iklm}\hat{R}^{lmk}{}_{j}$$
$$+4S^{k}{}_{mn}S^{lmn}\hat{R}_{iklj}) . \quad (28)$$

[This differs from the result in table 1 by a factor of 4: one factor of 2 because of the $(\frac{1}{2})$ in eq. (6), and 2 because of the relationship [15,16] between β_{ij} and the corresponding counterterm in ϵ^{-1}].

In the torsion-free case we obtain

$$\beta_{ij}^{(2)}=-(8\pi^2)^{-1}\,R_{iklm}R_{j}{}^{klm} \quad (29)$$

in accordance with previous calculations [15,16].

It would clearly be of considerable interest to consider whether an effective action for the g_{ij} and e_{ij} fields can be constructed whose equations of motion are related to the β function of eq. (28), and the relationship between such an action and that of ref. [17].

While this paper was in preparation I received a preliminary version of ref. [5]. The authors obtain a β-function which is neither covariant with respect to the connection with torsion nor zero in the supersymmetric case. I believe that these results are a consequence of the ambiguities associated with the use of a d-dependent prescription for $\epsilon_{\mu\nu}$.

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