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Global Existence for Systems of Parabolic Conservation Laws in Several Space Variables

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We prove the global existence of solutions of the Cauchy problem for certain systems of conservation laws with artificial viscosity terms added. The system is assumed to admit a quadratic entropy which is consistent with the viscosity matrix, and the initial data is assumed to be close to a constant in $L^2 \cap L^\infty$. In particular, our result applies to the equations of compressible fluid flow in two and three space variables. © 1987 Academic Press, Inc.

1. INTRODUCTION

In this paper we prove the global existence of solutions of the Cauchy problem for certain systems of the form

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i(u) = D\Delta u, \qquad u = u(x, t), \quad x \in \mathbb{R}^n, \ t > 0, \tag{1.1}$$

with initial data

$$u(x, 0) = u_0(x). \tag{1.2}$$

Here $u = (u^1, ..., u^m)$, $f_i = (f_i^1, ..., f_i^m)$, and D is a constant, positive matrix. The f_i are assumed to be defined and smooth in a neighborhood of a point $\bar{u}$, and the initial data u_0 is assumed to be sufficiently close to $\bar{u}$ in $L^2 \cap L^\infty$. We shall show that our results apply to the equations of compressible fluid flow (conservation of mass, momentum, and energy) in which artificial viscosity terms have been added.

The results of this paper are an extension of our previous work [2] in which corresponding results were obtained for similar problems in one space available. (References to related existence theorems may also be found in [2].)

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First, the existence of local solutions in several space variables is proved in much the same way as in the one-dimensional case. We therefore give only a sketch of this local existence in Section 2. These local solutions can then be extended to all of $t>0$ if the system (1.1) admits a quadratic entropy consistent with D. This notion is defined precisely in Definition 2.3, and the global existence result is formulated in Theorem 2.6. The main point of interest in this article is the application of this global existence result to the equations of compressible fluid flow. This is achieved in Section 3, where we explicitly construct the required entropy. (A similar construction is carried out by Harten in [5] for the case of a polytropic gas. We thank the referee for bringing this reference to our attention.)

We shall denote L^q norms in the usual way:

$$\|u\|_{q,Q}=\left(\int_Q |u|^q\right)^{1/q}=\left(\int_Q \sum_j |u^j|^q\right)^{1/q},$$

and

$$\|u\|_{\infty,Q}=\operatorname*{ess\,sup}_{j,x} |u^j(x)|.$$

The Q will be suppressed when there is no ambiguity. In addition, the H^p-norm,

$$\|u\|^2_{H^p(\mathbb{R}^n)}=\sum_{|\alpha|\leqslant p} \|D^\alpha u\|^2_{2,\mathbb{R}^n}.$$

will arise for the integer p defined by

$$p=\min\left\{k\in\mathbb{Z}: k>\frac{n}{2}+1\right\}, \tag{1.3}$$

which will be fixed throughout.

2. Global Existence of Solutions

In this section we sketch the proof of local existence of solutions, and we show how these local solutions can be extended to all of $t>0$ when the system (1.1) admits an appropriate entropy.

Suppose first that D is a constant, positive, diagonal matrix,

$$D=\operatorname{diag}(d_1,\dots,d_m), \qquad d_j>0.$$

Let $K^j(x,t)$ be the fundamental solution for the heat operator $(\partial/\partial t)-d_j\Delta$,

$$K^j(x,t)=(4\pi d_j t)^{-n/2}\exp(-|x|^2/4d_j t).$$

The solution of (1.1), (1.2) then formally satisfies the representation

$$u^j(t) = K^j(t) * u_0^j - \sum_{i=1}^{n} \int_0^t K^j_{x_i}(t-s) * f_i^j(u(s))\, ds, \qquad j = 1,..., m, \qquad (2.1)$$

where $*$ denotes convolution in the spatial variable x, $K^j_{x_i} = (\partial/\partial x_i)\, K^j$, and the x-dependence has been supressed.

We assume that each vector function $f_i = (f_i^1,..., f_i^m)$ is C^p in a closed ball $\bar{B}_r(\bar{u})$ of radius r about a point $\bar{u}$, and, without loss of generality, that $f_i(\bar{u}) = 0$. (p was defined in (1.3).) We define the set

$$G_T = \{u\colon [0, T] \to L^\infty(\mathbb{R}^n)\colon \|u(\cdot, t) - \bar{u}\|_\infty \leqslant r, 0 \leqslant t \leqslant T\}$$

and the operator L on G_T by

$$L(u)^j(t) = K^j(t) * u_0^j - \sum_{i=1}^{n} \int_0^t K^j_{x_i}(t-s) * f_i(u(s))\, ds, \qquad j = 1,..., m.$$

The following elementary properties of L can then be derived just as in [2, Lemma 2.1].

LEMMA 2.1. *Let f_i, G_T, and L be as above. Given $s < r$ there is a $T > 0$ such that, if $(u_0 - \bar{u}) \in L^2 \cap L^\infty$ with $\|u_0 - \bar{u}\|_\infty \leqslant s$, then L maps G_T into itself and is a contraction in the L^∞ norm. Moreover, given times $0 = t_1 < t_2 < ... < t_p < T$, there is a constant $C = C(s, t_1,..., t_p)$, such that, if $u \in G_T$ satisfies the inequalities*

(a) $\|u(\cdot, t) - \bar{u}\|_2 \leqslant C\, \|u_0 - \bar{u}\|_2, 0 \leqslant t \leqslant T,$

(b) $\|u_{x_i}(\cdot, t)\|_\infty \leqslant C/\sqrt{t}, 1 \leqslant i \leqslant n, 0 < t \leqslant T,$

(c) $\|D_x^\alpha u(\cdot, t)\|_2 \leqslant C\, \|u_0 - \bar{u}\|_2 / \sqrt{t - t_q}, |\alpha| = q, q = 1,..., p, t_q < t \leqslant T,$

then $L(u)$ also satisfies (a), (b), *and* (c).

Applying the above result, we can then prove the local existence of solutions as follows.

LEMMA 2.2. *Assume that $f \in C^p(\bar{B}_r(\bar{u}))$ and that D is a constant, positive, diagonal matrix. Then given $s < r$ there is a $T > 0$ and a constant C_1 such that, if $(u_0 - \bar{u}) \in L^2 \cap L^\infty$ with $\|u_0 - \bar{u}\|_\infty \leqslant s < r$, then the problem* (1.1), (1.2) *has a solution defined on $\mathbb{R}^n \times [0, T]$. Moreover, u satisfies the following five properties:*

(a) $\|u(\cdot, t) - \bar{u}\|_\infty \leqslant r, 0 \leqslant t \leqslant T;$

(b) *u_t and Δu are locally Hölder continuous in $\mathbb{R}^n \times (0, T)$;*

(c) $u(\cdot, t) - u_0 \to 0$ in $L^2(\mathbb{R}^n)$ *as* $t \to 0$;

(d) $u(\cdot, t) - \bar{u} \in H^p(\mathbb{R}^n), 0 < t \leqslant T;$

(e) $\|u(\cdot, T) - \bar{u}\|_{H^p(\mathbb{R}^n)} \leqslant C_1\, \|u_0 - \bar{u}\|_2.$

Sketch of Proof. u is obtained as the L^∞ limit of the sequence $\{u^l\}$ defined by $u^0 = u_0$ and $u^{l+1} = L(u^l)$. Observe that, by Lemma 2.1, the norms

$$\|D_x^\alpha u^l(\cdot, t) - \bar{u}\|_2, \qquad |\alpha| \leqslant p, 0 < \delta \leqslant t \leqslant T,$$

are bounded independently of l for any fixed $\delta > 0$. Since $p > (n/2) + 1$, it follows that the functions $u^l_{x_i}$ are uniformly Hölder continuous in x on compact subsets of $\mathbb{R}^n \times (0, T)$. A simple argument [2, p. 219] then shows that the $u^l_{x_i}$ are also uniformly locally Hölder continuous in t. Standard results [4, Theorem 51] applied to the equation

$$u_t^{l+1} - D\Delta u^{l+1} = -\sum_i f_i(u^l)_{x_i}$$

then imply that u^l_t and Δu^l are also uniformly locally Hölder continuous. (b) then follows from the Ascoli–Arzela theorem, and the estimate (e) follows from Lemma 2.1 by taking any particular choice of $t_1, ..., t_p$. ∎

These local solutions will be extended to all of $t > 0$ when the system (1.1) admits an appropriate entropy–entropy flux pair. These are defined as follows:

DEFINITION 2.3. The functions α: $\bar{B}_r(u) \to \mathbb{R}$ and $\beta = (\beta_1, ..., \beta_n)$: $\bar{B}_r(u) \to \mathbb{R}^n$ form an entropy–entropy flux pair for the system (1.1) if, for each $i = 1, ..., n$ and $u \in \bar{B}_r(\bar{u})$,

$$\nabla\alpha(u)^t f_i(u) = \nabla\beta_i(u)^t. \tag{2.2}$$

The entropy α will always be assumed to satisfy

$$\delta\,|u - \bar{u}|^2 \leqslant \alpha(u) \leqslant \delta^{-1}\,|u - \bar{u}|^2, \qquad u \in \bar{B}_r(\bar{u}), \tag{2.3}$$

for some positive constant δ. Finally, α is said to be consistent with the diagonal matrix D if

$$w^t D\alpha''(u)\, w \geqslant 0 \tag{2.4}$$

holds for all $u \in \bar{B}_r(\bar{u})$ and $w \in \mathbb{R}^m$.

The existence of such a pair (α, β) enables us to derive the following *a priori* energy estimate.

LEMMA 2.4. *Assume that there is an entropy–entropy flux pair* (α, β) *for the system* (1.1) *satisfying* (2.2)–(2.4), *and that D is a constant, positive, diagonal matrix. Then there is a constant $C_2 \geqslant 1$ depending only on the* *properties of* α *and* f *in* $\bar{B}_r(\bar{u})$ *such that, if* u *is any solution of* (1.1), (1.2) *on* $\mathbb{R}^n \times [0, t]$ *satisfying* (a)–(d) *of Lemma* 2.2, *then*

$$\|u(\cdot, t)-\bar{u}\|_2 \leqslant c_2 \|u_0-\bar{u}\|_2, \qquad 0 \leqslant t \leqslant \bar{t}.$$

Proof. Without loss of generality, we may take $\beta_i(\bar{u})=0$, $i=1,..., n$. Multiply (1.1) on the left by $\nabla\alpha^t$ to obtain

$$\alpha(u)_t+\sum_i \beta(u)_{x_i}=\nabla\alpha^t D\Delta u$$

$$=\sum_{i=1}^n (\nabla\alpha^t D u_{x_i})_{x_i}-\sum_{i=1}^n (D\alpha'' u_{x_i})^t u_{x_i}.$$

Integrating over $\mathbb{R}^n \times [t_0, \bar{t}]$, we then have, using (2.4) and (d) of Lemma 2.2, that

$$\int \alpha(u(x, \cdot))\,|_{t_0}^{t}\, dx \leqslant 0.$$

Thus from (2.3),

$$\|u(\cdot, t)-\bar{u}\|_2 \leqslant \delta^{-2} \|u(\cdot, t_0)-\bar{u}\|_2^2.$$

The result then follows by letting $t_0 \to 0$ and using (c) of Lemma 2.2. ∎

We can now state our global existence theorem for the case that the diffusion matrix D is diagonal. The proof will require the Sobolev inequality

$$\|v\|_{\infty, \mathbb{R}^n} \leqslant C_3 \|v\|_{H^{p-1}(\mathbb{R}^n)} \leqslant C_3 \|v\|_{H^p(\mathbb{R}^n)}; \tag{2.5}$$

see [3, p. 144]. (Recall that p was defined in (1.3).)

LEMMA 2.5. *Assume that* f *is in* $C^p(\bar{B}_r(\bar{u}))$ (p *is defined in* (1.3)), *that* D *is a constant, positive, diagonal matrix, and that the system* (1.1) *admits an entropy–entropy flux pair* (α, β) *satisfying* (2.2)–(2.4). *Let* C_1, C_2, *and* C_3 *be as in Lemma* 2.2, *Lemma* 2.4, *and* (2.5), *respectively. Then if*

$$u_0 \in L^2 \cap L^\infty, \qquad \|u_0-\bar{u}\|_\infty \leqslant s < r, \qquad \text{and} \qquad C_1 C_2 C_3 \|u_0-\bar{u}\|_2 \leqslant s,$$

then the Cauchy problem (1.1), (1.2) *has a unique global solution.*

Proof. We take $\bar{u}=0$ in the proof. Let T be as in Lemma 2.1 and take $T_k=kT$, $k=1, 2,....$ We shall show by induction on k that a solution exists for $0 \leqslant t \leqslant T_k$ and satisfies

$$(\mathrm{a}_k) \qquad \|u(\cdot, t)\|_\infty \leqslant r, \qquad 0 \leqslant t \leqslant T_k$$

and

$$(\mathrm{b}_k) \qquad \|u(\cdot, T_k)\|_{H^p(\mathbb{R}^n)} \leqslant C_1 C_2 \|u_0\|_2.$$

(a_1) and (b_1) hold by Lemma 2.2(a) and (e), since $C_2 \geqslant 1$. Assuming that (a_k) and (b_k) hold, we then have from (2.5) that

$$\|u(\cdot, T_k)\|_\infty < C_3 \|u(\cdot, T_k)\|_{H^p}$$
$$\leqslant C_1 C_2 C_3 \|u_0\|_2 \leqslant s$$

by (b_k) and our hypothesis. Since $u(\cdot, T_k) \in L^2$, Lemma 2.2 then applies at the new initial time T_k to show that the solution can be extended up to time T_{k+1}. (a_{k+1}) is then satisfied by Lemma 2.2(a), and, by Lemma 2.2(e) and Lemma 2.3,

$$\|u(\cdot, T_{k+1})\|_{H^p} \leqslant C_1 \|u(\cdot, T_k)\|_2 \leqslant C_1 C_2 \|u_0\|_2$$

as required. ∎

Finally, we can dispense with the requirement that D be a diagonal matrix by making a simple change of variable.

THEOREM 2.6. *Assume that $f \in C^p(\bar{B}_r(\bar{u}))$ (p is defined in (1.3)) and that the system (1.1) admits an entropy–entropy flux pair (α, β) satisfying (2.2) and (2.3). Let D be a diagonalizable matrix with positive eigenvalues, say*

$$P^{-1}DP = \Lambda = \text{diag}(d_1, ..., d_m) > 0,$$

and assume that

$$\Lambda P^t \alpha''(u)\, P \geqslant 0, \qquad u \in \bar{B}_r(\bar{u}).$$

Then the Cauchy problem (1.1), (1.2) has a global solution provided that

$$\|u_0 - \bar{u}\|_\infty < \frac{r}{\|P\|\ \|P^{-1}\|}$$

and that $\|u_0 - \bar{u}\|_2$ is sufficiently small.

Proof. Let $v = P^{-1}u$. Then v satisfies

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} g_i(v) = \Lambda \Delta v, \tag{2.6}$$

where $g_i(v) = P^{-1} f_i(Pv)$. Observe that g_i is C^p in the set

$$\left\{ v \colon |v - P^{-1}\bar{u}| \leqslant \frac{r}{\|P\|} \right\},$$

and that $\|v_0 - P^{-1}\bar{u}\|_\infty < r/\|P\|$. Also, the functions $A(v) \equiv \alpha(Pv)$ and

$B(v) \equiv \beta(Pv)$ are easily seen to satisfy the requirements (2.2) and (2.3) for the system (2.6). Finally,

$$\Lambda A''(v) = \Lambda P' \alpha''(Pv)\, P \geqslant 0$$

by hypothesis. The result then follows by applying Lemma 2.5 to the system (2.6). ∎

3. Application to Compressible Flow in Several Space Variables

In this section we show how our main result, Theorem 2.6, can be applied to the two systems

$$\begin{aligned} \rho_t + \operatorname{div}(\rho u) &= 0 \\ (\rho u_i)_t + \operatorname{div}(\rho u_i u + P e_i) &= 0, \qquad i = 1, \ldots, n \\ (\rho S_t) + \operatorname{div}(\rho S u) &= 0 \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \rho_t + \operatorname{div}(\rho u) &= 0 \\ (\rho u_i)_t + \operatorname{div}(\rho u_i u + P e_i) &= 0, \qquad i = 1, \ldots, n \\ (\rho E)_t + \operatorname{div}(\rho E u + P u) &= 0 \end{aligned} \tag{3.2}$$

when appropriate (artificial) viscosity terms are added to the right-hand sides. (e_i denotes the ith standard basic vector.) These systems describe the flow of a compressible fluid in which ρ, $u = (u_1, \ldots, u_n)$, P, S, and E are, respectively, the density, velocity, pressure, entropy, and energy. In (3.1), P is a smooth function of ρ and S and is to satisfy

$$P_\rho > 0 \qquad \text{for} \quad \rho > 0. \tag{3.3}$$

In (3.2), $P = P(\rho, e)$, where $E = e + |u|^2/2$. System (3.2) can be derived formally from (3.1) and a fundamental thermodynamic relation involving e, S, and ρ; see [1, pp. 15–16].

We first display an entropy–entropy flux pair for (3.1) when $n = 3$. The existence of the required entropy–entropy flux pair for (3.2) will then follow by a simple change of variables. The $n = 2$ case is similar; we omit the details. The final application of Corollary 2.5 will then be stated at the end of this section.

Observe that all nonlinear functions appearing in (3.1) and (3.2) are defined and smooth in the region $\rho > 0$. We therefore fix a point $(\bar{\rho}, \bar{u}, \bar{S})$ with $\bar{\rho} > 0$, and, without loss of generality, we take $\bar{u} = 0$, $\bar{S} = 0$.

Define functions α and β_i, $i = 1, 2, 3$, by

$$\alpha(\rho, u, S) = \frac{\rho\, |u|^2}{2} + \rho \int_{\bar{\rho}}^{\rho} \frac{P(\sigma, S) - \bar{P}}{\sigma^2}\, d\sigma + C\rho S^2 \tag{3.4}$$

and

$$\beta_i(\rho, u, S) = \frac{\rho\, |u|^2\, u_i}{2} + \rho u_i \int_{\bar{\rho}}^{\rho} \frac{P(\sigma, S) - \bar{P}}{\sigma^2}\, d\sigma + u_i(P - \bar{P}) + C\rho u_i S^2, \tag{3.5}$$

where $\bar{P} = P(\bar{\rho}, \bar{S})$ and C is a positive constant to be chosen later. We have to show first that

$$\nabla\alpha' f_1' = \nabla\beta_1', \tag{3.6}$$

where from (3.1),

$$f_1 = \begin{pmatrix} \rho u_1 \\ \rho u_1^2 + P \\ \rho u_1 u_2 \\ \rho u_1 u_3 \\ \rho u_1 S \end{pmatrix}. \tag{3.7}$$

The derivatives in (3.6) are understood to be with respect to the conserved quantities $(\rho, \rho u, \rho S)$. To facilitate the computation, we let

$$w = (\rho - \bar{\rho}, u, S)$$

and

$$z = (\rho - \bar{\rho}, \rho u, \rho S).$$

(3.6) is equivalent to

$$\left(\frac{d\alpha}{dw}\right)^t \frac{dw}{dz} \frac{df_1}{dw} = \left(\frac{d\beta_1}{dw}\right)^t. \tag{3.8}$$

We verify (3.8) by direct computation:

$$\frac{d\alpha}{dw} = \begin{bmatrix} \dfrac{|u|^2}{2} + \displaystyle\int_{\bar{\rho}}^{\rho} \frac{P - \bar{P}}{\sigma^2}\, d\sigma + \frac{P - \bar{P}}{\rho} + CS^2 \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho \displaystyle\int_{\bar{\rho}}^{\rho} \frac{P_S}{\sigma^2}\, d\sigma + 2C\rho S \end{bmatrix} \tag{3.9}$$

and

$$\frac{dw}{dz} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -u_1/\rho & 1/\rho & 0 & 0 & 0 \\ -u_2/\rho & 0 & 1/\rho & 0 & 0 \\ -u_3/\rho & 0 & 0 & 1/\rho & 0 \\ -S/\rho & 0 & 0 & 0 & 1/\rho \end{pmatrix}. \tag{3.10}$$

Therefore

$$\left[\left(\frac{d\alpha}{dw}\right)^t \frac{dw}{dz}\right]^t = \begin{bmatrix} -\frac{|u|^2}{2} + \int_{\bar\rho}^{\rho} \frac{P-\bar P}{\sigma^2}\, d\sigma + \frac{P-\bar P}{\rho} - S\int_{\bar\rho}^{\rho} \frac{P_S}{\sigma^2}\, d\sigma - CS^2 \\ u_1 \\ u_2 \\ u_3 \\ \int_{\bar\rho}^{\rho} \frac{P_S}{\sigma^2}\, d\sigma + 2CS \end{bmatrix}. \tag{3.11}$$

Next, from (3.7),

$$\frac{df_1}{dw} = \begin{pmatrix} u_1 & \rho & 0 & 0 & 0 \\ P_\rho + u_1^2 & 2\rho u_1 & 0 & 0 & 0 \\ u_1 u_2 & \rho u_2 & \rho u_1 & 0 & 0 \\ u_1 u_3 & \rho u_3 & 0 & \rho u_1 & 0 \\ u_1 & \rho S & 0 & 0 & \rho u_1 \end{pmatrix}.$$

Combining this with (3.11), we therefore obtain

$$\left[\left(\frac{d\alpha}{dw}\right)^t \frac{dw}{dz}\frac{df_1}{dw}\right]^t = \begin{bmatrix} \frac{|u|^2 u_1}{2} + u_1\int_{\bar\rho}^{\rho} \frac{P-\bar P}{\sigma^2}\, d\sigma + u_1\frac{P-\bar P}{\rho} + u_1 P_\rho + C u_1 S^2 \\ \frac{\rho\, |u|^2}{2} + \rho u_1^2 + \rho\int_{\bar\rho}^{\rho} \frac{P-\bar P}{\sigma^2}\, d\sigma + (P-\bar P) + C\rho S^2 \\ \rho u_1 u_2 \\ \rho u_1 u_3 \\ \rho u_1 \int_{\bar\rho}^{\rho} \frac{P_S}{\sigma^2}\, d\sigma + 2C\rho u_1 S \end{bmatrix},$$

which is precisely $(d\beta_1/dw)'$. The computations for β_1 and β_2 are similar. This proves that (2.2) holds.

Next, in order to establish the hypothesis (2.3) and (2.4), we compute the Hessian matrix $d^2\alpha/dz^2$ at $(\bar{\rho}, 0, 0)$:

$$\begin{aligned}\frac{d^2\alpha}{dz^2} &= \left[\frac{d}{dw}\left(\frac{dw'}{dz}\frac{d\alpha}{dw}\right)\right]\frac{dw}{dz} \\ &= \left[\frac{d}{dw}\left(\frac{d\alpha'}{dw}\frac{dw}{dz}\right)\right]'\frac{dw}{dz}.\end{aligned} \tag{3.12}$$

From (3.11),

$$\frac{d}{dw}\left(\frac{d\alpha}{dw}\right)'\frac{dw}{dz}\bigg|_{(\bar{\rho},0,0)} = \begin{bmatrix} \dfrac{P_\rho(\bar{\rho},0)}{\bar{\rho}} & 0 & 0 & 0 & \dfrac{P_S(\bar{\rho},0)}{\bar{\rho}} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \dfrac{P_S(\bar{\rho},0)}{\bar{\rho}^2} & 0 & 0 & 0 & 2C \end{bmatrix}$$

so that, from (3.10) and (3.12),

$$\frac{d^2\alpha}{dz^2}\bigg|_{(\bar{\rho},0,0)} = \begin{bmatrix} \dfrac{P_\rho(\bar{\rho},0)}{\bar{\rho}} & 0 & 0 & 0 & \dfrac{P_S(\bar{\rho},0)}{\bar{\rho}^2} \\ 0 & 1/\bar{\rho} & 0 & 0 & 0 \\ 0 & 0 & 1/\bar{\rho} & 0 & 0 \\ 0 & 0 & 0 & 1/\bar{\rho} & 0 \\ \dfrac{P_S(\bar{\rho},0)}{\bar{\rho}^2} & 0 & 0 & 0 & \dfrac{2C}{\bar{\rho}} \end{bmatrix}.$$

Since $\bar{\rho}$ and $P_\rho(\bar{\rho}, 0)$ are positive (see (3.3)), we conclude that, if C is sufficiently large, $d^2\alpha/dz^2$ is positive definite at $z=0$ and therefore in a neighborhood of 0. But α and $d\alpha/dz$ vanish at $z=0$ (see (3.3) and (3.11). It therefore follows that

$$\delta\,|z|^2 \leqslant \alpha \leqslant \delta^{-1}\,|z|^2$$

for some $\delta > 0$ and for z in a neighborhood of 0. Thus (2.3) holds. Finally, since $d^2\alpha/dz^2$ is positive definite, the hypothesis (2.4) will be satisfied if D is diagonalizable and is sufficiently close to a multiple of the identity matrix.

We have thus shown that the entropy–entropy flux pair (α, β) given by (3.4), (3.5) satisfies the required conditions (2.2)–(2.4) for the system (3.1).

The corresponding result for the energy formulation of the compressible flow equations, (3.2), then follows easily. Note first that system (3.2) can be derived formally from (3.1) by making the change of variable

$$(\rho, \rho u, \rho S) = h(\rho, \rho u, \rho E)$$

(see [1, pp. 15–16]). It then follows easily that the functions $A = \alpha \circ h$ and $B = \beta \circ h$ form an entropy–entropy flux pair for (3.2), satisfying (2.2) and (2.3) in a neighborhood of $(\bar{\rho}, 0, 0)$ in $(\rho, \rho u, \rho E)$ space, and that A'' is positive definite there. (For a proof, see [2, Proposition 3.1].) Thus A and B satisfy (2.2)–(2.4) for system (3.2) when D is diagonalizable and close to a positive multiple of the identity.

We can now give a formal statement of the application of Theorem 2.6 to the compressible flow systems (3.1) and (3.2). In this statement we let the dependent variables be (ρ, z), where now

$$z = \begin{bmatrix} \rho u \\ \rho S \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} \rho u \\ \rho E \end{bmatrix}$$

in systems (3.1) and (3.2), respectively.

THEOREM 3.1. *Let $(\bar{\rho}, \bar{z})$ be a given constant state with $\bar{\rho} > 0$. Then there is a number $r > 0$ such that, if the initial data (ρ_0, z_0) satisfies $\|(\rho_0, z_0) - (\bar{\rho}, \bar{z})\|_\infty \leqslant r$ and $\|(\rho_0, z_0) - (\bar{\rho}, \bar{z})\|_2 \leqslant r$, and if D is a diagonalizable matrix with $\|D - dI\| \leqslant r$ for some $d > 0$, then the systems* (3.1) *and* (3.2), *modified by the addition of terms $D\Delta[{}^{\rho}_{z}]$ to the right-hand sides, have unique solutions defined in all of $t > 0$.*

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