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On Modules with DICC

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INTRODUCTION

Let R be an arbitrary commutative ring with unit. R-modules with DICC, i.e., modules which have no doubly infinite chain of submodules, were first introduced in [C1] and extensively investigated in [C2].

Let us recall from those papers some results which will be referred to later on, but first let us fix some notation. If R is a commutative ring with unit, $\mathfrak{n}$ will denote its nilradical. Also, a min/max ideal of R is a prime ideal which is both minimal and maximal. If M is an R-module, then $N(M) = N =$ the sum of all submodules of M of finite length. In [C1] we proved that a non-reduced ring S with no min/max ideals has DICC if and only if S reduced is Noetherian, and $\mathfrak{n}$ has DCC, is nilpotent, and is almost divisible (i.e., $\forall y \in S - \mathfrak{n}$, $\mathfrak{n}/y\mathfrak{n}$ has finite length). In [C2] we proved a similar result for modules—that is, an R-module M has DICC if and only if N has DCC; M/N has ACC; and, for all $x \in M - N$, $N/Rx \cap N$ has finite length. Moreover, a DICC not DCC R-module M can be generated by finitely many elements outside the submodule N.

In this paper we make a further investigation of DICC modules. We call *weakly DICC* (abbreviated to WDICC a ring whose nilradical is nilpotent and DCC, and such that the reduced ring is Noetherian; we call *standardly DICC* (abbreviated to SDICC) a DICC module (resp. ring) which is not ACC, not DCC (resp. not ACC and, hence, also not DCC). We pay particular attention to the question of how the existence of a DICC R-module affects the ring itself.

It is known that if a ring admits a faithful ACC module, then the ring is Noetherian. Recently, W. Heinzer and D. Lantz proved that if a complete quasi-local ring admits a faithful DCC module, then it is Noetherian. For SDICC modules there seems to be no obvious way of proving a similar result, as the Example 1 given after Theorem 1 shows. In this subtle example we exhibit a quasi-local WDICC ring R such that R_{red} is complete local, R has a faithful SDICC module, but R is not DICC.

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Finally, among other results, we provide with Theorem 4 a new characterization of SDICC modules.

Let us mention some further notation. If $(R, \mathfrak{m})$ is a quasi-local ring, then $K = R/\mathfrak{m}$ is its residue field, $\hat{R}$ is the $\mathfrak{m}$-adic completion of R, $E_R(K)$ the injective hull of K, and $-^{\vee} = \operatorname{Hom}_R(-, E_R(K))$ the Matlis dual of $-$ [M]. The symbol $\subset$ means strict inclusion and the symbol $\subseteq$ allows equality. All other notation is standard unless stated otherwise.

Let us start out with some definitions.

DEFINITION 1. We call a non-reduced ring *weakly DICC* if its nilradical is nilpotent and has DCC, and the reduced ring is Noetherian. We shall call *standardly DICC* an R-module (resp. a ring) which is DICC, but not ACC, not DCC (resp. not ACC and, hence, not DCC).

Henceforth we will abbreviate weakly DICC to WDICC and standardly DICC to SDICC.

Our first aim is to extend to an SDICC module a fact true for ACC and DCC modules over an arbitrary ring.

Fact. It is known [N, Chap. 1, Theorem 3.6, Corollary 3.17, and Note 3.18] that a Noetherian module over an arbitrary ring R can always be thought of as a Noetherian module over a Noetherian ring, namely $R/\operatorname{Ann}_R M$. Hence, if a ring admits a faithful Noetherian module, then it is Noetherian.

Recently, W. Heinzer and D. Lantz [HL, Proposition 4.3] proved that if a complete quasi-local ring admits a faithful Artinian module, then the ring is Noetherian.

An immediate consequence is then

PROPOSITION 1. *A DCC R-module M can be thought of as a module over a Noetherian ring.*

Proof. Let $\operatorname{Supp}(M) = \{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_k \mid \mathfrak{m}_i \in \operatorname{Max} \operatorname{Spec} R\}$. Then $M = \bigoplus_{i=1}^k M_{\mathfrak{m}_i}$, where $M_{\mathfrak{m}_i}$, the localization of M at $\mathfrak{m}_i$, is a DCC $R_{\mathfrak{m}_i}$-module. Set $\mathfrak{J}_i = \operatorname{Ann}_{R_{\mathfrak{m}_i}} M_{\mathfrak{m}_i}$, $R_i = R_{\mathfrak{m}_i}/\mathfrak{J}_i$, and $\mu_i = \mathfrak{m}_i R_{\mathfrak{m}_i}/\mathfrak{J}_i$, $i = 1, 2, \ldots, k$. Now, observe that $M_{\mathfrak{m}_i}$ is a faithful DCC R_i-module and, hence, an $\hat{R}_i$-module. Since $\hat{R}_i$ is Noetherian by [HL, Proposition 4.3], $M = \bigoplus_{i=1}^k M_{\mathfrak{m}_i}$ is a module over the ring $\prod_{k=1}^k \hat{R}_i$, which is Noetherian.

For faithful SDICC modules over an arbitrary ring there is no obvious way to get a similar result, as Example 1, below, shows. The best we could obtain is

THEOREM 1. *A ring which admits a faithful SDICC module is WDICC.*

For the proof let us give a lemma.

LEMMA 1. *Let M be an SDICC R-module. Then, for all $x \in M - N$, M/Rx is a Noetherian R-module.*

Proof. Pick an element $x \in M - N$ and consider the short exact sequence $0 \to N/N \cap Rx \to M/Rx \to M/N + Rx \to 0$. Since M/N is Noetherian, so is the last term; while $N/N \cap Rx$ has finite length. Hence, M/Rx is Noetherian.

Proof of Theorem 1. Let us prove first that $R/\mathfrak{n}$ is Noetherian. For any given element $x \in M - N$, set $\mathfrak{i} = \operatorname{Ann}_R x$ and $\mathfrak{j} = \operatorname{Ann}_R M/Rx$. Then $R/\mathfrak{j}$ is Noetherian and $R/\mathfrak{i} \cong Rx$ is an SDICC ring. Moreover, $\mathfrak{j} \cdot \mathfrak{i} = (0)$. Set $\mathfrak{q} = \operatorname{rad}(\mathfrak{i})$, i.e., $\mathfrak{q}/\mathfrak{i} = \mathfrak{n}\ (R/\mathfrak{i})$. Hence $\mathfrak{q}/\mathfrak{i}$ is nilpotent, that is, $\mathfrak{q}^t \subseteq \mathfrak{i}$ for some t, which implies $\mathfrak{q}^t \cdot \mathfrak{j} = (0)$, and therefore $\mathfrak{q} \cap \mathfrak{j} \subseteq \mathfrak{n}$ since $(\mathfrak{q} \cap \mathfrak{j})^{t+1} = (\mathfrak{q} \cap \mathfrak{j})^t(\mathfrak{q} \cap \mathfrak{j}) \subseteq \mathfrak{q}^t \cdot \mathfrak{j} = (0)$. Now, in the short exact sequence

$$0 \to \mathfrak{q}/\mathfrak{q} \cap \mathfrak{j} \to R/\mathfrak{q} \cap \mathfrak{j} \to R/\mathfrak{q} \to 0,$$

the last term has ACC, and the first term has ACC because $\mathfrak{q}/\mathfrak{q} \cap \mathfrak{j} \cong \mathfrak{q} + \mathfrak{j}/\mathfrak{j} \subsetneq R/\mathfrak{j}$, so the middle term has it too. Hence, $R/\mathfrak{n}$ is Noetherian. We still need to prove that $\mathfrak{n}$ has DCC and is nilpotent. For this, pick $x_1, \dots, x_h \in M - N$ which are generators of M [C2, Corollary 2.6]. Set $\mathfrak{i}_i = \operatorname{Ann}_R x_i$, $i = 1, 2, \dots, h$. Let $\mathfrak{n}_i$ be the nilradical of $R/\mathfrak{i}_i$. The map $R \to Rx_1 \oplus \cdots \oplus Rx_h$ defined by the rule $r \mapsto (rx_1, \dots, rx_h)$ is an injective homomorphism since M is faithful. Hence $\mathfrak{n} \subsetneq \bigoplus_{i=1}^h \mathfrak{n}_i$ has DCC. Also, $\mathfrak{n}$ is nilpotent because $\mathfrak{n}^{\max_i t_i} \subseteq \mathfrak{q}_i^{\max_i t_i} \subseteq \bigcap_{i=1}^h \mathfrak{i}_i = (0)$, where $\mathfrak{q}_i$ lifts $\mathfrak{n}_i$ back to R.
Q.E.D.

Before exhibiting the example of a quasi-local WDICC ring R such that R_{red} is complete which is not DICC but which has a faithful SDICC module, we explain some background.

Let A be a complete local domain. Recall that $E_A(k)$ is the injective hull of the residue field K of A and $-^{\vee} = \operatorname{Hom}_A(-, E_A(k))$. Let E be a DCC not almost-divisible A-module (i.e., $\exists a \in A - \{0\}$ such that E/aE does not have finite length [C2, Definition 1.2]) and let $E_1, E_2 \subset E$ be such that $E_1 \cap E_2 = (0)$, E/E_i $(i = 1, 2)$ is divisible, and $E/E_i \cong_{\alpha_i} F$ as A-modules. Hence F has DCC. Consider the diagram

$$\begin{tikzcd} & E/E_1 \arrow[dr, "\alpha_1", "\cong"'] & \\ E \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & & F \\ & E/E_2 \arrow[ur, "\alpha_2"', "\cong"] & \end{tikzcd} \tag{1}$$

where π_1, π_2 are the canonical quotient surjections and set $\beta_i = \alpha_i \circ \pi_i$, $i = 1, 2$.

Consider the ring $R = A \oplus E$ obtained from A by idealizing E (see [N]). Since E is not almost divisible, R is not DICC, but WDICC. (We will show that for a suitable choice of A, R admits a faithful DICC module which will have the form $M = A \oplus A \oplus F$ as an A-module.) We first note:

LEMMA 2. *To give an R-module H is the same as to give an A-module H together with an A-linear map* $\lambda\colon E \to \operatorname{End}_A(H)$ *such that for all* $e_1, e_2 \in E$, $\lambda(e_1)\,\lambda(e_2) = 0$.

The proof is straightforward. Q.E.D.

Let $M = A \oplus A \oplus F$, with F as above. Then, by Lemma 2, the A-linear map $\lambda\colon E \to \operatorname{End}_A(M)$ defined by $e \to \lambda(e)$, where $\lambda(e)(a_1 + a_2 + f) = a_1\beta_1(e) + a_2\beta_2(e) \in F$ defines an R-module structure on M: the scalars act by the rule $(a \oplus e)(a_1 + a_2 + f) = aa_1 + aa_2 + af + a_1\beta_1(e) + a_2\beta_2(e)$. By [C2, Theorem 2.4] already quoted, M will be DICC if we can prove that for all $x \in M - F$ (Note: our N is F now), $F/Rx \cap F$ has finite length. It will suffice if for all $x \in M - F$, $F/Rx \cap F = 0$.

An arbitrary element $(a_1, a_2, f) \in M - F$ (i.e., with a_1, a_2 not both 0) has a multiple $(aa_1, aa_2, 0)$: pick $a \neq 0$ in $\operatorname{Ann} f$. Hence M has DICC provided that for all $a_1, a_2 \in A$ not both zero and for all $f \in F$, there exists $e \in E$ such that $(0, 0, f) = (0 \oplus e)(a_1, a_2, 0)$, i.e., $f = a_1\beta_1(e) + a_2\beta_2(e)$. This means that M has DICC provided that the composite map

$$E \xrightarrow{(a_1, a_2)} \begin{array}{c} E \\ \oplus \\ E \end{array} \to \begin{array}{c} E/E_1 \\ \oplus \\ E/E_2 \end{array} \to \begin{array}{c} F \\ \oplus \\ F \end{array} \leftarrow F, \tag{2}$$

$$\begin{aligned} e &\mapsto (a_1 e, a_2 e) \mapsto (a_1\pi_1(e), a_2\pi_2(e)) \mapsto (a_1\beta_1(e), a_2\beta_2(e)) \\ &\mapsto a_1\beta_1(e) + a_2\beta_2(e) \end{aligned}$$

is surjective, or, equivalently, that the dual map

$$E^\vee \leftarrow \begin{array}{c} E^\vee \\ \times \\ E^\vee \end{array} \leftarrow \begin{array}{c} (E/E_1)^\vee \\ \times \\ (E/E_2)^\vee \end{array} \leftarrow \begin{array}{c} F^\vee \\ \times \\ F^\vee \end{array} \leftarrow F^\vee \tag{3}$$

is injective.

This is not true in general but *is* true for the choices below.

EXAMPLE 1. Let us take $A = \mathbb{R}[[X, Y]]$ which is a complete local domain. Here, $\mathbb{R}$ is the real numbers, $\mathbb{C}$ the complex numbers. Consider the A-module

$$W = \mathbb{C}[[X, Y]] \oplus (\mathbb{C}[[X, Y]]/(Y)) = \mathbb{C}[[X, Y]]\,u \oplus (\mathbb{C}[[X, Y]]/(Y))\,v,$$

where $u=(1,0)$, $v=(0,1+(Y))$, and take $W_1=\mathbb{C}[[X,Y]]\,u$, $W_2=\mathbb{C}[[X,Y]](u+v)$. Then $W_1 \xrightarrow{\cong} W_0=\mathbb{C}[[X,Y]](u\mapsto 1)$ and $W_2 \xrightarrow{\cong} W_0$ $(u+v\mapsto i)$ and $W=W_1+W_2$. Let us identify the previous objects, i.e., $E=W^{\vee}$, $E_i=(W/W_i)^{\vee}$, $i=1,2$. Then E has DCC and is not almost divisible, $E_1\cap E_2=0$, E/E_i is divisible, and $E/E_i \xrightarrow{\cong}_{\alpha_i} W_0^{\vee}$, $i=1,2$. Our task is now to form diagram (3) and check that the composite map is injective. We get

$$\mathbb{C}[[X,Y]] \leftarrow \begin{matrix} \mathbb{C}[[X,Y]]\,u \\ + \\ \mathbb{C}[[X,Y]] \end{matrix} \leftarrow \begin{matrix} \mathbb{C}[[X,Y]]\,u \oplus (\mathbb{C}[[X,Y]]/(Y))\,v \\ \oplus \\ \mathbb{C}[[X,Y]]\,u \oplus (\mathbb{C}[[X,Y]]/(Y))\,v \end{matrix} \leftarrow \begin{matrix} \mathbb{C}[[X,Y]]\,u \\ \oplus \\ \mathbb{C}[[X,Y]](u+v)\,i \end{matrix}$$

$$\leftarrow \begin{matrix} \mathbb{C}[[X,Y]] \\ \oplus \\ \mathbb{C}[[X,Y]] \end{matrix} \leftarrow \mathbb{C}[[X,Y]],$$

$$\gamma[(a_1+a_2 i)\,u+a_2 iv] \leftarrow\!\shortmid \gamma(a_1 u+a_2 i(u+v))$$

$$\leftarrow\!\shortmid \begin{matrix} \gamma u \\ \oplus \\ \gamma i(u+v) \end{matrix} \leftarrow\!\shortmid \begin{matrix} \gamma u \\ \oplus \\ \gamma i(u+v) \end{matrix} \leftarrow\!\shortmid \begin{matrix} \gamma \\ \oplus \\ \gamma \end{matrix} \leftarrow\!\shortmid \gamma \neq 0.$$

The element $\gamma[(a_1+a_2)\,u+a_2 iv]\neq 0$ since $\gamma(a_1+a_2 i)\neq 0$ and this is so because $a_1+a_2 i\in\mathbb{C}[[X,Y]]$ is not zero as $a_1, a_2\in\mathbb{R}[[X,Y]]$ have been chosen not both zero.

Remark 1. If a DICC R-module M is a direct sum of two modules M_1, M_2, then either both modules are Noetherian, or both are Artinian, or one of them must be of finite length.

The next result is similar to [C1, Theorem 3] for rings. First, let us recall from [C2] that a min/max ideal for an R-module M is a maximal ideal of R which is minimal in the support of M.

THEOREM 2. *A faithful SDICC R-module with finitely many min/max ideals decomposes into a direct sum of a module of finite length and an* SDICC *module with no min/max ideals.*

Proof. Assume there is only one such min/max ideal and set $Q=\bigcup_{t\geqslant 1}\operatorname{Ann}_M \mathfrak{m}^t$. Let us observe that the number r of min/max ideals is finite and, if it were $r>1$, we would simply set $Q=\bigoplus_{i=1}^r(\bigcup_{t\geqslant 1}\operatorname{Ann}_M \mathfrak{m}_i^t)$. Consider the short exact sequence $0\to Q\to M\to M/Q\to 0$.

Since M/Q is an SDICC module with no min/max ideals, by [C2, Theorem 2.9] $\operatorname{Supp}(M/Q)$ is an irreducible closed subset of $\operatorname{Spec}(R)$ and is disjoint from $\operatorname{Supp} Q=\{\mathfrak{m}\}$, which is closed in $\operatorname{Spec}(R)$. Hence $M=M/Q\oplus Q$. As M/Q is SDICC, then Q must have finite length by Remark 1. Q.E.D.

As a consequence we have that M is an $R/\mathfrak{i}\times R/\mathfrak{j}$-module, where

$\mathfrak{i} = \operatorname{Ann}_R Q$ (hence $R/\mathfrak{i}$ is Artinian as it is 0-dimensional and Noetherian), and where $\mathfrak{j} = \operatorname{Ann}_R M/Q$ (hence $R/\mathfrak{j}$ is WDICC by Theorem 1).

THEOREM 3. *If M is a faithful* SDICC *R-module with no min/max ideals, then the following statements hold*:

1. Min Supp$(M) = \{\mathfrak{p}\}$,
2. $\mathfrak{p}$ = nilradical of R.

Proof. (1) This has been proved in [C2, Theorem 2.9].

(2) If not, there exists a prime ideal $\mathfrak{q} \subset \mathfrak{p}$. Hence $M_\mathfrak{q} = 0$. Let $M = (m_1, \ldots, m_k)$, where $m_i \in M - N$ all i, and let $\mathfrak{i}_\nu = \operatorname{Ann}_R m_\nu$. As $R/\mathfrak{i}_\nu \subsetneq M$, it follows that $(R/\mathfrak{i}_\nu)_\mathfrak{q} = 0$, all ν, and therefore there exists $x_\nu \in R - \mathfrak{q}$ such that x_ν kills m_ν. Then $x = x_1^{N_1} \cdots x_k^{N_k} \notin \mathfrak{q}$ and yet x kills M, a contradiction since M is faithful.

This shows that $\mathfrak{p}$ is minimal in Spec(R) and that $\mathfrak{p}$ equals the nilradical of R. Q.E.D.

For some purposes, the next result reduces the study of arbitrary DICC modules M to those in which M/N is a direct sum of prime cyclic modules.

THEOREM 4. *Let M be an arbitrary R-module. M is* SDICC *if and only if there exists an* SDICC *submodule M_0 of M such that $M_0 \not\subset N$, M/M_0 is Noetherian, M/N is an essential extension of $M_0/N \cap M_0$, and $N/M_0 \cap N$ has finite length. If M is* SDICC, *then M_0 above can be chosen so that $M_0/N \cap M_0 \cong \bigoplus_{\text{finite}} R/\mathfrak{p}_i$.*

Proof. First, let us assume that M is SDICC and construct M_0. Since M/N is Noetherian, we can choose $\bar{u}_1, \ldots, \bar{u}_\tau$, where $u_i \in M$, $i = 1, \ldots, \tau$, such that $\operatorname{Ann}_R \bar{u}_i = \mathfrak{p}_i$, a prime ideal of R, the sum $\sum_{i=1}^{\tau} Ru_i$ is direct, and τ is maximum. Then $\bar{M}_0 = \sum_{i=1}^{\tau} R\bar{u}_i \cong \bigoplus_{i=1}^{\tau} R/\mathfrak{p}_i$ and $\bar{M}_0 \subset M/N$ is essential. (If it is not essential, pick $\bar{u} \in \bar{M} - \{0\}$ such that $R\bar{u} \cap \bar{M}_0 = 0$. Replace $\bar{u}$ by a multiple with prime annihilator, and then $\bar{u}_1, \ldots, \bar{u}_\tau, \bar{u}$ give a larger value for τ.) Let $M_0 = \sum_{i=1}^{\tau} Ru_i + N$. Note that M_0/N is $\bar{M}_0$.

M_0 is DICC simply because it is a submodule of M. $M/M_0 \cong \bar{M}/\bar{M}_0$ is Noetherian. Since M is not Noetherian, neither is M_0. Finally if M_0 were Artinian, we would have $M_0 = N$, so that M/N is an essential extension of 0. It follows that $M = N$, and so M is Artinian.

Conversely, suppose M_0 is SDICC, N/N_0, where $N_0 = M_0 \cap N$ has finite length, $M_0/N_0 \subsetneq M/N$ is essential, and M/M_0 is ACC. We must show that M is SDICC. Since N_0 is DCC and N/N_0 has finite length, N is DCC. M/N is Noetherian because the first and last terms of the short exact sequence $0 \to M_0/N_0 \to M/N \to M/M_0 + N \to 0$ are Noetherian. Given $x \in M - N$, we need to show that $N/Rx \cap N$ has finite length. Since $\bar{M}_0 \subsetneq \bar{M}$ is essential, pick $\bar{x} \in \bar{M} - \{0\}$, hence $a\bar{x} \in \bar{M}_0 - \{0\}$ for some $a \in R - \{0\}$, i.e., $a\bar{x} = \bar{m}_0 \neq 0$ or, equivalently, $ax = m_0 + n_0$. Multiplying further, we can kill n_0 and get $y = a'x \in M_0 - N_0$. It is sufficient to show that $N/Ry \cap N$ has finite length since $Ry \subset Rx$. For this, consider the short exact sequence

$$0 \to \frac{N_0}{Ry \cap N_0} \to \frac{N}{Ry \cap N} \to \frac{N}{N_0} \to 0,$$

where $Ry \cap N = Ry \cap N_0$ because $Ry \cap N \subset N_0$ since $Ry \subset M_0$ and $N_0 = M_0 \cap N$. Since the first and last terms have finite length, the middle term has it too. Finally, M is SDICC since $M_0 \subset M$ is SDICC.

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