On DICC Rings

MARIA CONTESSA

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

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INTRODUCTION

In this paper we define and try to understand the DICC rings, i.e., rings with no doubly infinite chain of ideals (see Definition 1). After investigating a few properties which hold in such rings we are able to establish in Theorem 3 that a DICC ring is either Noetherian or a direct product of an Artinian ring, and a ring \( S \) with not prime both minimal and maximal, which is not Noetherian, and with the following four properties: (1) \( S_{\text{red}} \) is Noetherian; (2) \( n \) (the nilradical of \( S \)) is nilpotent; (3) \( n \) has DCC; (4) for all \( x \in S - n \), \( n/Sx \cap n \) has finite length.

All rings are commutative with unit. The symbol \( \subseteq \) means strict inclusion, and the symbol \( \leq \) allows equality. All the notation is otherwise standard.

DEFINITION 1. A module \( M \) is said to satisfy the doubly infinite chain condition if any infinite chain of submodules of \( M \)

\[ \cdots \subseteq N_2 \subseteq N_1 \subseteq N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \]

stabilizes either to the right or to the left (or to both sides).

Henceforth "DICC" will stand for doubly infinite chain condition and "dic" for doubly infinite chain.

A ring which satisfies the DICC condition as a module is called a DICC ring.

Of course, Noetherian rings satisfy the above definition; however, a DICC ring need not be Noetherian. Let us look at the following example.

EXAMPLE. Let \((V, \mathfrak{m})\) be a DVR and let \( E \) be the injective hull of the residue class field \( K = V/\mathfrak{m} \). \( E \) is a \( V \)-module. Consider the ring \( R = V \oplus E \), where addition is defined componentwise and multiplication as follows

\[ (v, e)(v', e') = (vv', ve' + v'e) \quad \text{for} \quad (v, e), (v', e') \in V \oplus E. \]
The nilradical \( n \) of \( R \) is the ideal \( 0 \oplus E \) which is not finitely generated. Hence \( R \) is not Noetherian. Let us prove that \( R \) is a DICC ring. First note that the ideals of \( R \) are of the form

\[
\text{(I) } \quad 0 \oplus E' \quad \text{where } E' \text{ is any submodule of } E;
\]
or

\[
\text{(II) } \quad a \oplus E \quad \text{where } a \text{ is any ideal of } V.
\]

**Proof.** Type (I) is straightforward. For type (II), notice that an ideal \( b \) of \( R \) not of type (I) must be of type (II), because \( b \ni (a, e), \ a \neq 0, \) implies \( b \ni (a, e)(a, -e) = (a^2, 0), \) which in turn implies \( b \ni E \) since \( a^2E = E. \) Here \( E \) is identified with its image in \( R. \)

Now, assume \( R \) is not DICC. Let

\[
\cdot \cdot \cdot \subset b_{-2} \subset b_{-1} \subset b_0 \subset b_1 \subset b_2 \subset \cdot \cdot \cdot
\]

be a dic of ideals of \( R. \) If some \( b_k \) is of type (I), then the chain cannot be strictly decreasing since \( E \) has DCC; and if some \( b_k \) is of type (II), then the chain cannot be strictly increasing because

\[
\frac{R/b_k}{a \oplus E} = \frac{V \oplus E}{a \oplus E} \cong \frac{V}{a} \oplus \frac{E}{a} = \frac{V}{a} \oplus 0 \cong \frac{V}{a}
\]

which has ACC. This is a contradiction.

Henceforth our aim is a structure theorem for DICC rings. We shall reach this goal in Theorem 3 and up to that point we shall investigate some properties holding in such rings. Let us begin with

**Remark 1.** If \( R \) is a DICC ring, then so is any homomorphic image of \( R \) and any localization of it. (The proof is straightforward.)

**Lemma 1.** A DICC domain is Noetherian.

**Proof.** Assume not and let

\[
(0) \neq a_0 \subset a_1 \subset a_2 \subset \cdot \cdot \cdot
\]

be a strictly increasing chain of ideals of the given DICC domain \( R. \) Pick a nonzero element \( x \) in \( a_0 \) and then form the following dic:

\[
\cdot \cdot \cdot \subset (x^2) \subset (x) \subset a_0 \subset a_1 \subset \cdot \cdot \cdot .
\]

This is a contradiction.

We next give a result which holds in a quite general context and will be used later on.
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PROPOSITION 1. If an ideal \( a \subseteq R \) contains an infinite direct sum of non-zero ideals, say \( \bigoplus \alpha b_{\alpha} \subseteq a \), then \( a \) contains a dic of ideals.

Proof. By passing to a countable infinite subset we may assume the index set is the nonnegative integers. Then

\[
a_{-h} = \bigoplus_{i=h}^{\infty} b_{2i+1}, \quad \text{\( h = 0, 1, 2, \ldots \)}
\]

\[
a_{h} = \bigoplus_{i=1}^{h} b_{2i} \bigoplus a_{0}, \quad \text{\( h = 1, 2, 3, \ldots \)}
\]

provide the dic of ideals inside \( a \).

COROLLARY 1. A DICC ring \( R \) cannot contain ideals isomorphic to \( R/\mathfrak{p} \) for infinitely many distinct prime ideals \( \mathfrak{p} \).

Proof. For distinct \( \mathfrak{p}_\alpha \), if \( b_\alpha \cong R/\mathfrak{p} \), the sum \( \sum_{\alpha} b_\alpha \) is automatically direct.

COROLLARY 2. If \( m \) is a maximal ideal of a DICC ring \( R \), \( \text{Ann}_R m \) is a finite dimensional vector space over \( K = R/m \).

Proof. If not, it is an infinite direct sum of copies of \( K \).

LEMMA 2. A DICC ring with a unique prime ideal is Noetherian.

Proof. Let \((R, m)\) be such a ring. By Cohen's theorem it is enough to show that the unique prime ideal \( m \) is finitely generated. Assume not, and set \( K = R/m \). Corollary 2 applied to \((R/m^2, m/m^2)\) implies that \( m/m^2 \) is a finite-dimensional vector space over \( K \). Now, choose a finite basis for \( m/m^2 \) and pick elements \( x_1, x_2, \ldots, x_r \in m \) such that \( \{x_i + m^2\}_{i=1,...,r} \) is the chosen basis for \( m/m^2 \). Kill the ideal \( a = (x_1, \ldots, x_r) \). The ring \( \overline{R} = R/a \) has DICC and has \( \mu = m/a \) as a unique prime ideal. The hypothesis on \( a \) implies \( \mu/\mu^2 = 0 \). Therefore we can assume without loss of generality that \( m = m^2 \). We have then reduced to proving the following

SUBLEMMA. If a ring \( R \) has DICC and has a unique prime ideal \( m \) with \( m = m^2 \), then it is a field.

Proof. Assume not and let \( x \in m - \{0\} \). If \( R/(x) \) has ACC, we are done, since \( m/(x) \) is finitely generated and so is \( m \). Thus \( m = 0 \) by Nakayama's lemma and this is a contradiction. Therefore we can assume that \( R/(x) \) is not Noetherian; i.e., there is an infinite strictly increasing chain of ideals ascending from \((x)\) in \( R \). Hence \( (x) \cong R/\text{Ann}_R x \) has DCC as an \( R \)-module; i.e., \( R/\text{Ann}_R x \) is an Artinian ring with a unique prime ideal \( \mu = m/\text{Ann}_R x \).
This implies \( \mu^n = 0 \) for some \( n \) or that \( m^n \subseteq \text{Ann}_R x \) for some \( n \). Since \( m = m^2 = m^3 = \cdots \), \( m^n x = 0 \) implies \( mx = 0 \). As \( x \) was an arbitrary nonzero element in \( m \), we have \( m = m^2 = 0 \), which is a contradiction.

The next result we are told was also obtained by Professor Sylvia Wiegand, but we never saw her proof.

**Lemma 3.** A DICC ring has only finitely many minimal primes.

**Proof.** Assume not. As the set \( \text{Min Spec } R \) when equipped with the Zariski topology is a Hausdorff space, there exists an infinite strictly decreasing sequence of closed subsets of \( X_0 \), where \( X_0 = \text{Min Spec } R \), say

\[ X_0 \supset X_1 \supset X_2 \supset \cdots. \]

Choose a sequence of minimal prime ideals of \( R \) such that \( \mathfrak{P}_i \subseteq X_i \subseteq X_{i+1} \), all \( i \geqslant 1 \), and then consider the ideals of \( R \) defined by the rule

\[ b_k = \bigcap_{h-k}^\infty \mathfrak{P}_{2h+1}, \quad k \geqslant 0, \]

and

\[ b_{-k} = b_0 \cap \left( \bigcap_{i=1}^k \mathfrak{P}_{2i} \right), \quad k \geqslant 1. \]

Clearly we get a dic of ideals

\[ \cdots \subseteq b_{-2} \subseteq b_{-1} \subseteq b_0 \subseteq b_1 \subseteq b_2 \subseteq \cdots. \]

We will show that the inclusion are indeed strict, and therefore we get a contradiction. Let us prove, for example, that

\[ \begin{align*}
(1) \quad & b_0 \subset b_1; \\
(2) \quad & b_{-1} \subset b_0; \\
(3) \quad & b_{-2} \subset b_{-1}
\end{align*} \]

since the proof of the remaining inclusions is similar.

**Proof.** (1) Assume not. Then

\[ b_0 = \mathfrak{P}_1 \cap b_1 = b_1 \iff \mathfrak{P}_1 \supseteq b_1 \iff \mathfrak{P}_1 \subseteq V(b_1) \subseteq X_3, \]

which is a contradiction as \( \mathfrak{P} \subseteq X_1 - X_2 \) and \( X_3 \) is closed. (Recall that \( \mathfrak{P} \subset X \) closed \( \iff \mathfrak{P} \supseteq \bigcap_{i \in I} X_i \).)

(2) \( b_{-1} = \mathfrak{P}_{-2} \cap b_0 \subset b_0 \). For if not, then

\[ b_0 = b_{-1} \iff \mathfrak{P}_2 \supseteq b_0 = \mathfrak{P}_1 \cap \mathfrak{P}_3 \cap \mathfrak{P}_5 \cap \cdots \cap \mathfrak{P}_{2k+1} \cap \cdots \]

\[ \iff \mathfrak{P}_2 \supseteq \mathfrak{P}_1 \cap (\mathfrak{P}_3 \cap \mathfrak{P}_5 \cap \cdots) = \mathfrak{P}_1 \cap b_1 \]

\[ \iff \mathfrak{P}_2 \supseteq b_1 \quad \text{as} \quad \mathfrak{P}_2 \nsubseteq \mathfrak{P}_1. \]
Therefore $\mathcal{P}_2 \in X_3$, which is a contradiction.

(3) Assume that $b_{-2} = \mathcal{P}_4 \cap b_{-1} = \mathcal{P}_4 \cap (\mathcal{P}_2 \cap b_0) = b_{-1}$. Then

$$\mathcal{P}_4 \supseteq \mathcal{P}_2 \cap b_0 \Rightarrow \mathcal{P}_4 \supseteq b_0 = \mathcal{P}_{-1} \cap \mathcal{P}_2 \cap \mathcal{P}_3 \cap \cdots$$

$$\Rightarrow \mathcal{P}_4 \supseteq \mathcal{P}_5 \cap \mathcal{P}_7 \cap \cdots$$

$$\Rightarrow \mathcal{P}_4 \in X_5,$$

a contradiction because $\mathcal{P}_4 \notin X_4 - X_5$.

We give one more lemma and then the first step towards our goal.

**Lemma 4.** If $b_1, \ldots, b_n$ are ideals of a ring $R$ such that $R/b_j$ is Noetherian for $j = 1, \ldots, n$ and $\cap_{j=1}^n b_j = (0)$, then $R$ is Noetherian.

**Proof.** It suffices to show the case $n = 2$ since the general case follows by an inductive argument. First of all, note that $b_1$ and $b_2$ with this property must be finitely generated. Let us prove this for $b_1$. $R/b_2$ Noetherian implies that $b_1/b_2$ is finitely generated, say by $\bar{x}_1, \ldots, \bar{x}_i$. Under the assumption that $b_1 \cap b_2 = (0)$, the lifting $x_i$ of $\bar{x}_i$ in $R$, $1 \leq i \leq t$, generate $b_1$. If not, set $b' = (x_1, \ldots, x_t)$ and pick $y \in b_1 - b'$. Then $y = \sum_{i=1}^t r_ix_i + b_2$, where $r_i \in R$ and $b_2 \in b_2$, implies that $b_2 = y - \sum_{i=1}^t r_ix_i \in b_1$, which means $b_2 = 0$. Hence $y = \sum_{i=1}^t r_ix_i \in b'$, a contradiction. Now, let $\mathfrak{P} \subset R$ be a prime ideal. $\mathfrak{P} \subset (0) - b_1 \cap b_2$ implies that $\mathfrak{P}$ contains either $b_1$ or $b_2$. If $\mathfrak{P} \supseteq b_1$, then $\mathfrak{P}/b_1$ is finitely generated and hence $\mathfrak{P}$ as $b_1$ is finitely generated. Similarly if $\mathfrak{P} \supseteq b_2$.

As a corollary of all the previous lemmas we have

**Theorem 1** A reduced DICC ring is Noetherian.

**Proof.** By Lemma 3 Min Spec $R$ is finite, and by Remark 1 and Lemma 1 the domain $R/\mathfrak{P}$ is Noetherian for all $\mathfrak{P} \in$ Min Spec $R$. The conclusion now follows by applying Lemma 4 to the set of minimal prime ideals as $R$ is reduced.

This result suggests that the nilradical $n$ of a DICC ring must have some special property. Let us investigate it.

**Proposition 2.** Let $R$ be a non-Noetherian DICC ring. Then the following statements hold for the nilradical $n$ of $R$:

(a) There are only finitely many associated prime ideals of $n$ and these are all maximal.

(b) $n$ is nilpotent.

(c) $n$ has DCC.
Proof: (a) First, observe that $n$ is not finitely generated because $R$ is non-Noetherian (by Cohen's theorem $R$ contains a prime ideal $\mathfrak{p}$ which is not finitely generated, but $\mathfrak{p}/n$ is finitely generated by Theorem 1). Second, the cyclic module $Rx$ has finite length (i.e., is Artinian) for all $x \in n$. For if not, by completing to the right the infinite strictly decreasing chain starting from $Rx$ with the generators $x_i$ of $n$ we obtain a dic
\[
\cdots \mathfrak{m}_{-2} \subset \mathfrak{m}_{-1} \subset Rx \subset Rx + Rx_1 \subset \cdots \subset \cdots \subset n,
\]
where infinitely many of the weak inclusions on the right are strict, since $n$ is not finitely generated, which is a contradiction. This implies that $\text{Ass}(n) \subseteq \text{Max Spec } R$ because $R/\mathfrak{p} \subseteq Rx$, $\mathfrak{p} \in \text{Ass}(n)$, is a domain with DCC and therefore a field. Moreover, $\text{Ass}(n)$ is finite by Corollary 1. Notice that this implies that
\[
\bigoplus_{i=1}^r \left( \bigcup_{t \geq 1} \text{Ann}_{m^t} \right) = \bigoplus_{i=1}^r n_i,
\]
where $\{m_1, \ldots, m_r\} = \text{Ass}(n)$ and $n_i = \bigcup_{t} \text{Ann}_{m^t} \subset R_{m^t}$. If $R_m$ is Noetherian, then $\bigcup_i \text{Ann}_{m^t} \subset R_{m^t}$ has finite length and satisfies (b) and (c) clearly. Hence, we may assume $R$ is a local, non-Noetherian, DICC ring for the proof of (b) and (c).

Proof of (b). Assume that the powers of $n$ form an infinite strictly decreasing chain. Then each $n'/n'+1$ has ACC as an $R/n'+1$-module, for otherwise by putting together the lifting of such an infinite chain and the strictly decreasing chain
\[
n' + 1 \supset n' + 2 \supset n' + 3 \supset \cdots
\]
we produce a dic in $R$, which is impossible. Look at $n/n^2$ and pick $x_1, \ldots, x_k$ generators of it. Kill the ideal $b$ generated by the $x_i$'s in $R$. In the ring $\bar{R} = R/b$, which is still local, not Noetherian, and has DICC, $v = n/b = v^2$ is still not finitely generated. Therefore we can assume that $n = n^2$, which is a contradiction. Hence, the $n^s$ do not strictly decrease and we can assume that $n' = n^{t+1}$ for some $t$. Set $a = n'$. We want to prove that $a = (0)$. Assume not and let $x$ be a nonzero element in $a$. Then $\text{Ann } x \supset m^s \supset n' = a$ for all sufficiently large $s$. Hence $ax = 0$, which implies $a^2 = a = (0)$ and this is a contradiction. Thus $a = (0)$, i.e., $n$ is nilpotent.

(c) To show that $n$ has DCC it then suffices to show that $n'/n'+1$ has DCC, $1 \leq i \leq t$, where $n'+1 = (0)$. $Q = n'/n'+1$ is a module over the Noetherian local ring $R/n$ in which each element is killed by a power of the maximal ideal $\mu = m/n$. Such a module $Q$ has DCC if and only if $\text{Ann}_Q \mu$ is finite dimensional. But $\text{Ann}_Q \mu$ is finite-dimensional because $R/n'+1$ has DICC and the second corollary to Proposition 1 (see [5, Theorem 2]).

In the next two theorems we will reach our goal.
DEFINITION 2. A min/max ideal is a prime ideal which is both minimal and maximal.

THEOREM 2. Let $S$ be a non-Noetherian ring with no min/max ideals. Let $n$ be the nilradical of $S$. Then $S$ has DICC if and only if

1. $S_{\text{red}}$ is Noetherian;
2. $n$ is nilpotent;
3. $n$ has DCC;
4. for all $x \in S - n$, $n/Sx \cap n$ has finite length.

Proof. For the "only if" part we need to prove only that (4) holds when $S$ has DICC. Pick $x \in S - n$. If $x$ is a unit, then $Sx = S$ implies $n/Sx \cap n = 0$, which has finite length. Hence we can assume that $x$ is not a unit. This implies that $0 \neq \text{Ann}_S x \subseteq m$ for some maximal ideal $m$. Note that the image $x/1$ of $x$ in $S_m$ is not zero as $\text{Ann}_S x \subseteq m$ and is not nilpotent. If $x/1$ is not a unit, then it is clear that the sequence $\{(x/1)^i\}_{i \geq 1}$ is strictly decreasing. This implies that the $\{(x/y)^i\}_{i \geq 1}$ is strictly decreasing if not, $x/1 = (x/1)^{i+1} \Rightarrow (x/1)^{i+1} = (x/1)^i$, and therefore $n/Sx \cap n$ has finite length because $n$ having DCC implies that $n/Sx \cap n$ has DCC and $S$ having DICC implies that $Sx + n/Sx$ has ACC and hence the assertion follows from the isomorphism $n/Sx \cap n \cong Sx + n/Sx$. If $x/1$ is a unit, pick any $y \in S - n$ such that $y \in m$ and $y/1 \neq 0$ in $S_m$. Then $\{(x/1)(y/1)^i\}_{i \geq 1}$ strictly decreasing in $S_m$ implies $\{(xy)^i\}_{i \geq 1}$ strictly decreasing in $S$ which in turn implies that $n/(Sx \cap n)$ has finite length. Hence $n/Sx \cap n$ has finite length as homomorphic image of $n/(Sx \cap n)$.

Conversely, assume that $S$ is not DICC. Let

$$\cdots \subset b_{-2} \subset b_{-1} \subset b_0 \subset b_1 \subset \cdots$$

be a die of ideals of $S$. The hypothesis $S_{\text{red}}$ Noetherian implies that $b_t + n = b_{t+1} + n = \cdots$ from some $t$. Since $(\ast)$ is strictly increasing, we must have $b_t \cap n \subset b_{t+1} \cap n \subset \cdots \subset n$, for otherwise $b_t = b_{t+1} = \cdots$ and this is a contradiction. Hence $n/b_t \cap n$ does not have ACC. On the other hand, $(\ast)$ strictly decreasing implies $b_k \not\subset n$ for all $k$. Pick $x \in b_t - n$. Then, by (4), $n/Sx \cap n$ has ACC, which implies that $n/b_t \cap n$ has ACC as a homomorphic image of it. Therefore we have a contradiction again.

THEOREM 3. A ring $R$ has DICC if and only if $R$ is Noetherian or $R = S \times A$, where $S$ is a ring satisfying the four conditions of Theorem 2 and $A$ is an Artinian ring.

Proof. Let us prove first the only if part. Assume $R$ is not Noetherian
and does not have DICC either. Since $R \cong S \times A$, a strictly dic of ideals of $R$ has the form

$$\cdots \subset b_{-1} \times a_{-1} \subset b_0 \times a_0 \subset b_1 \times a_1 \subset \cdots,$$

where $\{b_k\}_{k \in \mathbb{Z}} \subset S$ and $\{a_k\}_{k \in \mathbb{Z}} \subset A$. Now, $A$ being an Artinian ring implies that the chain $\{a_k\}$ stabilizes above and below; hence the strictly increasing and decreasing part comes from $\{b_k\} \subset S$. But $S$ has DICC, a contradiction. Conversely, assume $R$ has DICC and is not Noetherian. As Min Spec $R = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is finite by Lemma 3, there are at most finitely many min/max ideals, say $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$. Notice that $k < n$, for otherwise Spec $R = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, i.e., $R$ would be zero-dimensional, hence Artinian, which is a contradiction since we are assuming $R$ not Noetherian. Set $U = \{\mathfrak{p}_1\} \cup \cdots \cup \{\mathfrak{p}_k\}$. $U$ is clopen and therefore Spec $R = X \cup U$, where $X = \text{Spec } R - U$, is clopen as well. This implies that $R \cong S \times A$, where $S$ has DICC and no min/max ideals and where $A$ is Artinian since it has DICC, and it is zero-dimensional with finite spectrum: $A$ is a finite product of rings of the type of Lemma 2.

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References