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On DICC Rings

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INTRODUCTION

In this paper we define and try to understand the DICC rings, i.e., rings with no doubly infinite chain of ideals (see Definition 1). After investigating a few properties which hold in such rings we are able to establish in Theorem 3 that a DICC ring is either Noetherian or a direct product of an Artinian ring, and a ring S with not prime both minimal and maximal, which is not Noetherian, and with the following four properties: (1) S_{red} is Noetherian; (2) $\mathfrak{n}$ (the nilradical of S) is nilpotent; (3) $\mathfrak{n}$ has DCC; (4) for all $x \in S - \mathfrak{n}$, $\mathfrak{n}/Sx \cap \mathfrak{n}$ has finite length.

All rings are commutative with unit. The symbol $\subset$ means strict inclusion, and the symbol $\subseteq$ allows equality. All the notation is otherwise standard.

DEFINITION 1. A module M is said to satisfy the *doubly infinite chain condition* if any infinite chain of submodules of M

$$\cdots \subseteq N_{-2} \subseteq N_{-1} \subseteq N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots$$

stabilizes either to the right or to the left (or to both sides).

Henceforth "DICC" will stand for doubly infinite chain condition and "dic" for doubly infinite chain.

A ring which satisfies the DICC condition as a module is called a DICC *ring*.

Of course, Noetherian rings satisfy the above definition; however, a DICC ring need not be Noetherian. Let us look at the following example.

EXAMPLE. Let $(V, \mathfrak{m})$ be a DVR and let E be the injective hull of the residue class field $K = V/\mathfrak{m}$. E is a V-module. Consider the ring $R = V \oplus E$, where addition is defined componentwise and multiplication as follows

$$(v, e)(v', e') = (vv', ve' + v'e) \qquad \text{for} \quad (v, e), (v', e') \in V \oplus E.$$

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The nilradical $\mathfrak{n}$ of R is the ideal $0 \oplus E$ which is not finitely generated. Hence R is not Noetherian. Let us prove that R is a DICC ring. First note that the ideals of R are of the form

(I) $0 \oplus E'$ where E' is any submodule of E;

or

(II) $\mathfrak{a} \oplus E$ where $\mathfrak{a}$ is any ideal of V.

Proof. Type (I) is straightforward. For type (II), notice that an ideal $\mathfrak{b}$ of R not of type (I) must be of type (II), because $\mathfrak{b} \ni (a, e)$, $a \neq 0$, implies $\mathfrak{b} \ni (a, e)(a, -e) = (a^2, 0)$, which in turn implies $\mathfrak{b} \supset E$ since $a^2 E = E$. Here E is identified with its image in R.

Now, assume R is not DICC. Let

$$\cdots \subset \mathfrak{b}_{-2} \subset \mathfrak{b}_{-1} \subset \mathfrak{b}_0 \subset \mathfrak{b}_1 \subset \mathfrak{b}_2 \subset \cdots$$

be a dic of ideals of R. If some $\mathfrak{b}_k$ is of type (I), then the chain cannot be strictly decreasing since E has DCC; and if some $\mathfrak{b}_h$ is of type (II), then the chain cannot be strictly increasing because

$$R/\mathfrak{b}_h = \frac{V \oplus E}{\mathfrak{a} \oplus E} \cong \frac{V}{\mathfrak{a}} \oplus \frac{E}{E} = \frac{V}{\mathfrak{a}} \oplus 0 \cong \frac{V}{\mathfrak{a}}$$

which has ACC. This is a contradiction.

Henceforth our aim is a structure theorem for DICC rings. We shall reach this goal in Theorem 3 and up to that point we shall investigate some properties holding in such rings. Let us begin with

Remark 1. If R is a DICC ring, then so is any homomorphic image an any localization of it. (The proof is straightforward.)

LEMMA 1. *A DICC domain is Noetherian.*

Proof. Assume not and let

$$(0) \neq \mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots$$

be a strictly increasing chain of ideals of the given DICC domain R. Pick a nonzero element x in $\mathfrak{a}_0$ and then form the following dic:

$$\cdots \subset (x^2) \subset (x) \subseteq \mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \cdots.$$

This is a contradiction.

We next give a result which holds in a quite general context and will be used later on.

PROPOSITION 1. *If an ideal* $\mathfrak{a} \subseteq R$ *contains an infinite* direct *sum of nonzero ideals, say* $\bigoplus_\lambda \mathfrak{b}_\lambda \subseteq \mathfrak{a}$, *then* $\mathfrak{a}$ *contains a dic of ideals.*

Proof. By passing to a countable infinite subset we may assume the index set is the nonnegative integers. Then

$$\mathfrak{a}_{-h} = \bigoplus_{i=h}^{\infty} \mathfrak{b}_{2i+1}, \qquad h = 0, 1, 2, \ldots$$

$$\mathfrak{a}_h = \bigoplus_{i=1}^{h} \mathfrak{b}_{2i} \oplus \mathfrak{a}_0, \qquad h = 1, 2, 3, \ldots$$

provide the dic of ideals inside $\mathfrak{a}$.

COROLLARY 1. *A* DICC *ring R cannot contain ideals isomorphic to* $R/\mathfrak{P}$ *for infinitely many distinct prime ideals* $\mathfrak{P}$.

Proof. For distinct $\mathfrak{P}_\lambda$, if $\mathfrak{b}_\lambda \cong R/\mathfrak{P}$, the sum $\sum_\lambda \mathfrak{b}_\lambda$ is automatically direct.

COROLLARY 2. *If* $\mathfrak{m}$ *is a maximal ideal of a DICC ring R,* $\mathrm{Ann}_R\, \mathfrak{m}$ *is a finite dimensional vector space over* $K = R/\mathfrak{m}$.

Proof. If not, it is an infinite direct sum of copies of K.

LEMMA 2. *A DICC ring with a unique prime ideal is Noetherian.*

Proof. Let $(R, \mathfrak{m})$ be such a ring. By Cohen's theorem it is enough to show that the unique prime ideal $\mathfrak{m}$ is finitely generated. Assume not, and set $K = R/\mathfrak{m}$. Corollary 2 applied to $(R/\mathfrak{m}^2, \mathfrak{m}/\mathfrak{m}^2)$ implies that $\mathfrak{m}/\mathfrak{m}^2$ is a finite-dimensional vector space over K. Now, choose a finite basis for $\mathfrak{m}/\mathfrak{m}^2$ and pick elements $x_1, x_2, \ldots, x_r \in \mathfrak{m}$ such that $\{x_i + \mathfrak{m}^2\}_{i=1,\ldots,r}$ is the chosen basis for $\mathfrak{m}/\mathfrak{m}^2$. Kill the ideal $\mathfrak{a} = (x_1, \ldots, x_r)$. The ring $\bar{R} = R/\mathfrak{a}$ has DICC and has $\mu = \mathfrak{m}/\mathfrak{a}$ as a unique prime ideal. The hypothesis on $\mathfrak{a}$ implies $\mu/\mu^2 = 0$. Therefore we can assume without loss of generality that $\mathfrak{m} = \mathfrak{m}^2$. We have then reduced to proving the following

SUBLEMMA. *If a ring R has DICC and has a unique prime ideal* $\mathfrak{m}$ *with* $\mathfrak{m} = \mathfrak{m}^2$, *then it is a field.*

Proof. Assume not and let $x \in \mathfrak{m} - \{0\}$. If $R/(x)$ has ACC, we are done, since $\mathfrak{m}/(x)$ is finitely generated and so is $\mathfrak{m}$. Thus $\mathfrak{m} = 0$ by Nakayama's lemma and this is a contradiction. Therefore we can assume that $R/(x)$ is not Noetherian; i.e., there is an infinite stricly increasing chain of ideals ascending from (x) in R. Hence $(x) \cong R/\mathrm{Ann}_R x$ has DCC as an R-module; i.e., $R/\mathrm{Ann}_R x$ is an Artinian ring with a unique prime ideal $\mu = \mathfrak{m}/\mathrm{Ann}_R x$.

This implies $\mu^n = 0$ for some n or that $\mathfrak{m}^n \subseteq \operatorname{Ann}_R x$ for some n. Since $\mathfrak{m} = \mathfrak{m}^2 = \mathfrak{m}^3 = \cdots$, $\mathfrak{m}^n x = 0$ implies $\mathfrak{m}x = 0$. As x was an arbitrary nonzero element in $\mathfrak{m}$, we have $\mathfrak{m} = \mathfrak{m}^2 = 0$, which is a contradiction.

The next result we are told was also obtained by Professor Sylvia Wiegand, but we never saw her proof.

LEMMA 3. *A DICC ring has only finitely many minimal primes.*

Proof. Assume not. As the set Min Spec R when equipped with the Zariski topology is a Hausdorff space, there exists an infinite strictly decreasing sequence of closed subsets of X_0, where $X_0 = \operatorname{Min\,Spec} R$, say

$$X_0 \supset X_1 \supset X_2 \supset \cdots.$$

Choose a sequence of minimal prime ideals of R such that $\mathfrak{P}_i \in X_i - X_{i+1}$, all $i \geqslant 1$, and then consider the ideals of R defined by the rule

$$\mathfrak{b}_k = \bigcap_{h=k}^{\infty} \mathfrak{P}_{2h+1}, \qquad k \geqslant 0,$$

and

$$\mathfrak{b}_{-k} = \mathfrak{b}_0 \cap \left(\bigcap_{l=1}^{k} \mathfrak{P}_{2l} \right), \qquad k \geqslant 1.$$

Clearly we get a dic of ideals

$$\cdots \subseteq \mathfrak{b}_{-2} \subseteq \mathfrak{b}_{-1} \subseteq \mathfrak{b}_0 \subseteq \mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq \cdots.$$

We will show that the inclusion are indeed strict, and therefore we get a contradiction. Let us prove, for example, that

$$(1)\ \mathfrak{b}_0 \subset \mathfrak{b}_1; \qquad (2)\ \mathfrak{b}_{-1} \subset \mathfrak{b}_0; \qquad (3)\ \mathfrak{b}_{-2} \subset \mathfrak{b}_{-1}$$

since the proof of the remaining inclusions is similar.

Proof. (1) Assume not. Then

$$\mathfrak{b}_0 = \mathfrak{P}_1 \cap \mathfrak{b}_1 = \mathfrak{b}_1 \Leftrightarrow \mathfrak{P}_1 \supseteq \mathfrak{b}_1 \Leftrightarrow \mathfrak{P}_1 \in V(\mathfrak{b}_1) \subseteq X_3,$$

which is a contradiction as $\mathfrak{P}_1 \in X_1 - X_2$ and X_3 is closed. (Recall that $\mathfrak{P} \in X$ closed $\Leftrightarrow \mathfrak{P} \supseteq \bigcap_{\mathfrak{q} \in X} \mathfrak{q}$.)

(2) $\mathfrak{b}_{-1} = \mathfrak{P}_{-2} \cap \mathfrak{b}_0 \subset \mathfrak{b}_0$. For if not, then

$$\begin{aligned} \mathfrak{b}_0 = \mathfrak{b}_{-1} &\Leftrightarrow \mathfrak{P}_2 \supseteq \mathfrak{b}_0 = \mathfrak{P}_1 \cap \mathfrak{P}_3 \cap \mathfrak{P}_5 \cap \cdots \cap \mathfrak{P}_{2k+1} \cap \cdots \\ &\Leftrightarrow \mathfrak{P}_2 \supseteq \mathfrak{P}_1 \cap (\mathfrak{P}_3 \cap \mathfrak{P}_5 \cap \cdots) = \mathfrak{P}_1 \cap \mathfrak{b}_1 \\ &\Rightarrow \mathfrak{P}_2 \supseteq \mathfrak{b}_1 \qquad \text{as} \quad \mathfrak{P}_2 \not\supseteq \mathfrak{P}_1. \end{aligned}$$

Therefore $\mathfrak{P}_2 \in X_3$, which is a contradiction.

(3) Assume that $\mathfrak{b}_{-2} = \mathfrak{P}_4 \cap \mathfrak{b}_{-1} = \mathfrak{P}_4 \cap (\mathfrak{P}_2 \cap \mathfrak{b}_0) = \mathfrak{b}_{-1}$. Then

$$\begin{aligned}\mathfrak{P}_4 \supseteq \mathfrak{P}_2 \cap \mathfrak{b}_0 \Rightarrow \mathfrak{P}_4 \supseteq \mathfrak{b}_0 &= \mathfrak{P}_{-1} \cap \mathfrak{P}_3 \cap \mathfrak{P}_5 \cap \cdots \\ &\Rightarrow \mathfrak{P}_4 \supseteq \mathfrak{P}_5 \cap \mathfrak{P}_7 \cap \cdots \\ &\Rightarrow \mathfrak{P}_4 \in X_5,\end{aligned}$$

a contradiction because $\mathfrak{P}_4 \in X_4 - X_5$.

We give one more lemma and then the first step towards our goal.

LEMMA 4. *If* $\mathfrak{b}_1,\dots,\mathfrak{b}_n$ *are ideals of a ring* R *such that* $R/\mathfrak{b}_j$ *is Noetherian for* $j=1,\dots,n$ *and* $\bigcap_{j=1}^n \mathfrak{b}_j = (0)$, *then* R *is Noetherian.*

Proof. It suffices to show the case $n=2$ since the general case follows by an inductive argument. First of all, note that $\mathfrak{b}_1$ and $\mathfrak{b}_2$ with this property must be finitely generated. Let us prove this for $\mathfrak{b}_1$. $R/\mathfrak{b}_2$ Noetherian implies that $\mathfrak{b}_1/\mathfrak{b}_2$ is finitely generated, say by $\bar{x}_1,\dots,\bar{x}_t$. Under the assumption that $\mathfrak{b}_1 \cap \mathfrak{b}_2 = (0)$, the lifting x_i of $\bar{x}_i$ in R, $1 \leqslant i \leqslant t$, generate $\mathfrak{b}_1$. If not, set $\mathfrak{b}' = (x_1,\dots,x_t)$ and pick $y \in \mathfrak{b}_1 - \mathfrak{b}'$. Then $y = \sum_{i=1}^t r_i x_i + b_2$, where $r_i \in R$ and $b_2 \in \mathfrak{b}_2$, implies that $b_2 = y - \sum_{i=1}^t r_i x_i \in \mathfrak{b}_1$, which means $b_2 = 0$. Hence $y = \sum_{i=1}^t r_i x_i \in \mathfrak{b}'$, a contradiction. Now, let $\mathfrak{P} \subset R$ be a prime ideal. $\mathfrak{P} \supseteq (0) = \mathfrak{b}_1 \cap \mathfrak{b}_2$ implies that $\mathfrak{P}$ contains either $\mathfrak{b}_1$ or $\mathfrak{b}_2$. If $\mathfrak{P} \supseteq \mathfrak{b}_1$, then $\mathfrak{P}/\mathfrak{b}_1$ is finitely generated and hence $\mathfrak{P}$ as $\mathfrak{b}_1$ is finitely generated. Similarly if $\mathfrak{P} \supseteq \mathfrak{b}_2$.

As a corollary of all the previous lemmas we have

THEOREM 1. *A reduced DICC ring is Noetherian.*

Proof. By Lemma 3 $\operatorname{Min}\operatorname{Spec} R$ is finite, and by Remark 1 and Lemma 1 the domain $R/\mathfrak{P}$ is Noetherian for all $\mathfrak{P} \in \operatorname{Min}\operatorname{Spec} R$. The conclusion now follows by applying Lemma 4 to the set of minimal prime ideals as R is reduced.

This result suggests that the nilradical $\mathfrak{n}$ of a DICC ring must have some special property. Let us investigate it.

PROPOSITION 2. *Let* R *be a non-Noetherian DICC ring. Then the following statements hold for the nilradical* $\mathfrak{n}$ *of* R:

(a) *There are only finitely many associated prime ideals of* $\mathfrak{n}$ *and these are all maximal.*

(b) $\mathfrak{n}$ *is nilpotent.*

(c) $\mathfrak{n}$ *has DCC.*

Proof. (a) First, observe that $\mathfrak{n}$ is not finitely generated because R is non-Noetherian (by Cohen's theorem R contains a prime ideal $\mathfrak{P}$ which is not finitely generated, but $\mathfrak{P}/\mathfrak{n}$ is finitely generated by Theorem 1). Second, the cyclic module Rx has finite length (i.e., is Artinian) for all $x \in \mathfrak{n}$. For if not, by completing to the right the infinite strictly decreasing chain starting from Rx with the generators x_i of $\mathfrak{n}$ we obtain a dic

$$\cdots \mathfrak{M}_{-2} \subset \mathfrak{M}_{-1} \subset Rx \subseteq Rx + Rx_1 \subseteq \cdots \subseteq \cdots \subseteq \mathfrak{n},$$

where infinitely many of the weak inclusions on the right are strict, since $\mathfrak{n}$ is not finitely generated, which is a contradiction. This implies that $\operatorname{Ass}(\mathfrak{n}) \subseteq \operatorname{Max\,Spec} R$ because $R/\mathfrak{P} \subseteq Rx$, $\mathfrak{P} \in \operatorname{Ass}(\mathfrak{n})$, is a domain with DCC and therefore a field. Moreover, $\operatorname{Ass}(\mathfrak{n})$ is finite by Corollary 1. Notice that this implies that

$$\mathfrak{n} = \bigoplus_{i=1}^{r} \left(\bigcup_{t \geqslant 1} \operatorname{Ann}_{\mathfrak{n}} \mathfrak{m}_i^t \right) = \bigoplus_{i=1}^{t} \mathfrak{n}_i,$$

where $\{\mathfrak{m}_1, ..., \mathfrak{m}_r\} = \operatorname{Ass}(\mathfrak{n})$ and $\mathfrak{n}_i = \bigcup_t \operatorname{Ann}_{\mathfrak{n}} \mathfrak{m}_i^t \subset R_{\mathfrak{m}_i}$. If $R_{\mathfrak{m}_i}$ is Noetherian, then $\bigcup_t \operatorname{Ann}_{\mathfrak{n}} \mathfrak{m}_i^t$ has finite length and satisfies (b) and (c) clearly. Hence, we may assume R is a local, non-Noetherian, DICC ring for the proof of (b) and (c).

Proof of (b). Assume that the powers of $\mathfrak{n}$ form an infinite strictly decreasing chain. Then each $\mathfrak{n}^i/\mathfrak{n}^{i+1}$ has ACC as an $R/\mathfrak{n}^{i+1}$-module, for otherwise by putting together the lifting of such an infinite chain and the strictly decreasing chain

$$\mathfrak{n}^{i+1} \supset \mathfrak{n}^{i+2} \supset \mathfrak{n}^{i+3} \supset \cdots$$

we produce a dic in R, which is impossible. Look at $\mathfrak{n}/\mathfrak{n}^2$ and pick $\bar{x}_1, ..., \bar{x}_k$ generators of it. Kill the ideal $\mathfrak{b}$ generated by the x_i's in R. In the ring $\bar{R} = R/\mathfrak{b}$, which is still local, not Noetherian, and has DICC, $v = \mathfrak{n}/\mathfrak{b} = v^2$ is still not finitely generated. Therefore we can assume that $\mathfrak{n} = \mathfrak{n}^2$, which is a contradiction. Hence, the $\mathfrak{n}^i$'s do not strictly decrease and we can assume that $\mathfrak{n}^t = \mathfrak{n}^{t+1}$ for some t. Set $\mathfrak{a} = \mathfrak{n}^t$. We want to prove that $\mathfrak{a} = (0)$. Assume not and let x be a nonzero element in $\mathfrak{a}$. Then $\operatorname{Ann} x \supset \mathfrak{m}^s \supset \mathfrak{n}^s = \mathfrak{a}$ for all sufficiently large s. Hence $\mathfrak{a}x = 0$, which implies $\mathfrak{a}^2 = \mathfrak{a} = (0)$ and this is a contradiction. Thus $\mathfrak{a} = (0)$, i.e., $\mathfrak{n}$ is nilpotent.

(c) To show that $\mathfrak{n}$ has DCC it then suffices to show that $\mathfrak{n}^i/\mathfrak{n}^{i+1}$ has DCC, $1 \leqslant i \leqslant t$, where $\mathfrak{n}^{t+1} = (0)$. $Q = \mathfrak{n}^i/\mathfrak{n}^{i+1}$ is a module over the Noetherian local ring $R/\mathfrak{n}$ in which each element is killed by a power of the maximal ideal $\mu = \mathfrak{m}/\mathfrak{n}$. Such a module Q has DCC if and only if $\operatorname{Ann}_Q \mu$ is finite dimensional. But $\operatorname{Ann}_Q \mu$ is finite-dimensional because $R/\mathfrak{n}^{i+1}$ has DICC and the second corollary to Proposition 1 (see [5, Theorem 2]).

In the next two theorems we will reach our goal.

DEFINITION 2. A *min/max ideal* is a prime ideal which is both minimal and maximal.

THEOREM 2. *Let S be a non-Noetherian ring with no min/max ideals. Let $\mathfrak{n}$ be the nilradical of S. Then S has DICC if and only if*

(1) S_{red} *is Noetherian;*

(2) $\mathfrak{n}$ *is nilpotent;*

(3) $\mathfrak{n}$ *has DCC;*

(4) *for all* $x \in S - \mathfrak{n}$, $\mathfrak{n}/Sx \cap \mathfrak{n}$ *has finite length.*

Proof. For the "only if" part we need to prove only that (4) holds when S has DICC. Pick $x \in S - \mathfrak{n}$. If x is a unit, then $Sx = S$ implies $\mathfrak{n}/Sx \cap \mathfrak{n} = 0$, which has finite length. Hence we can assume that x is not a unit. This implies that $0 \neq \operatorname{Ann} x \subseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. Note that the image $x/1$ of x in $S_{\mathfrak{m}}$ is not zero as $\operatorname{Ann} x \subseteq \mathfrak{m}$ and is not nilpotent. If $x/1$ is not a unit, then it is clear that the sequence $\{(x/1)^i\}_{i \geqslant 1}$ is strictly decreasing. This implies that the $\{(x)^i\}_{i \geqslant 1}$ is strictly decreasing [if not, $(x)^i = (x)^{i+1} \Rightarrow (x/1)^i = (x/1)^{i+1}$], and therefore $\mathfrak{n}/Sx \cap \mathfrak{n}$ has finite length because $\mathfrak{n}$ having DCC implies that $\mathfrak{n}/Sx \cap \mathfrak{n}$ has DCC and S having DICC implies that $Sx + \mathfrak{n}/Sx$ has ACC and hence the assertion follows from the isomorphism $\mathfrak{n}/Sx \cap \mathfrak{n} \cong Sx + \mathfrak{n}/Sx$. If $x/1$ is a unit, pick any $y \in S - \mathfrak{n}$ such that $y \in \mathfrak{m}$ and $y/1 \neq 0$ in $S_{\mathfrak{m}}$. Then $\{((x/1)(y/1)^i\}_{i \geqslant 1}$ strictly decreasing in $S_{\mathfrak{m}}$ implies $\{(xy)^i\}_{i \geqslant 1}$ strictly decreasing in S which in turn implies that $\mathfrak{n}/S(xy) \cap \mathfrak{n}$ has finite length. Hence $\mathfrak{n}/Sx \cap \mathfrak{n}$ has finite length as homomorphic image of $\mathfrak{n}/S(xy) \cap \mathfrak{n}$.

Conversely, assume that S is not DICC. Let

$$\cdots \subset \mathfrak{b}_{-2} \subset \mathfrak{b}_{-1} \subset \mathfrak{b}_0 \subset \mathfrak{b}_1 \subset \cdots \tag{$*$}$$

be a dic of ideals of S. The hypothesis S_{red} Noetherian implies that $\mathfrak{b}_t + \mathfrak{n} = \mathfrak{b}_{t+1} + \mathfrak{n} = \cdots$ from some t. Since $(*)$ is strictly increasing, we must have $\mathfrak{b}_t \cap \mathfrak{n} \subset \mathfrak{b}_{t+1} \cap \mathfrak{n} \subset \cdots \subset \mathfrak{n}$, for otherwise $\mathfrak{b}_t = \mathfrak{b}_{t+1} = \cdots$ and this is a contradiction. Hence $\mathfrak{n}/\mathfrak{b}_t \cap \mathfrak{n}$ does not have ACC. On the other hand, $(*)$ strictly decreasing implies $\mathfrak{b}_k \not\subset \mathfrak{n}$ for all k. Pick $x \in \mathfrak{b}_t - \mathfrak{n}$. Then, by (4), $\mathfrak{n}/Sx \cap \mathfrak{n}$ has ACC, which implies that $\mathfrak{n}/\mathfrak{b}_t \cap \mathfrak{n}$ has ACC as a homomorphic image of it. Therefore we have a contradiction again.

THEOREM 3. *A ring R has DICC if and only if R is Noetherian or $R = S \times A$, where S is a ring satisfying the four conditions of Theorem 2 and A is an Artinian ring.*

Proof. Let us prove first the only if part. Assume R is not Noetherian and does not have DICC either. Since $R \cong S \times A$, a strictly dic of ideals of R has the form

$$\cdots \subset \mathfrak{b}_{-1} \times \mathfrak{a}_{-1} \subset \mathfrak{b}_0 \times \mathfrak{a}_0 \subset \mathfrak{b}_1 \times \mathfrak{a}_1 \subset \cdots,$$

where $\{\mathfrak{b}_k\}_{k \in \mathbb{Z}} \subset S$ and $\{\mathfrak{a}_k\}_{k \in \mathbb{Z}} \subset A$. Now, A being an Artinian ring implies that the chain $\{\mathfrak{a}_k\}$ stabilizes above and below; hence the strictly increasing and decreasing part comes from $\{\mathfrak{b}_k\} \subset S$. But S has DICC, a contradiction. Conversely, assume R has DICC and is not Noetherian. As Min Spec $R = \{\mathfrak{P}_1, \ldots, \mathfrak{P}_n\}$ is finite by Lemma 3, there are at most finitely many min/max ideals, say $\mathfrak{P}_1, \ldots, \mathfrak{P}_k$. Notice that $k < n$, for otherwise Spec $R = \{\mathfrak{P}_1, \ldots, \mathfrak{P}_n\}$, i.e., R would be zero-dimensional, hence Artinian, which is a contradiction since we are assuming R not Noetherian. Set $U = \{\mathfrak{P}_1\} \cup \cdots \cup \{\mathfrak{P}_k\}$. U is clopen and therefore Spec $R = X \cup U$, where $X = \text{Spec } R - U$, is clopen as well. This implies that $R \cong S \times A$, where S has DICC and no min/max ideals and where A is Artinian since it has DICC, and it is zero-dimensional with finite spectrum: A is a finite product of rings of the type of Lemma 2.

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