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# Positivity of a System of Differential Operators

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## 1. INTRODUCTION

Let $p(x, \xi) = \sum_{|\alpha+\beta| \leqslant 2} a_{\alpha\beta} x^\alpha \xi^\beta$ be a real quadratic polynomial of $(x, \xi) \in R^{2n}$. Let $p_2(x, \xi) = \sum_{|\alpha+\beta|=2} a_{\alpha\beta} x^\alpha \xi^\beta$ be the second order part of $p(x, \xi)$ and $P((x, \xi), (y, \eta))$ be the polarized form of $p_2(x, \xi)$. Let $\sigma(\cdot, \cdot)$ be the standard symplectic form on $R^{2n}$. $F$ is the Hamiltonian map of $p_2$ defined by $\sigma((x, \xi), F(y, \eta)) = P((x, \xi), (y, \eta))$ and $\operatorname{tr}^+ p_2$ is defined as the sum of the positive eigenvalues of $-i \cdot F$.

Let $p^w(x, D)$ be the Weyl operator with symbol $p(x, \xi)$, i.e., $p^w(x, D)\, u = (2\pi)^{-2n} \iint p((x+y)/2, \xi)\, e^{i(x-y)\cdot\xi}\, u(y)\, dy\, d\xi$, where $u \in \mathscr{S}(R^n)$, the space of rapidly decreasing $C^\infty$ functions.

Melin proved in [3] that $\langle p^w(x, D)\, u, u \rangle \geqslant 0$ for any $u \in \mathscr{S}(R^n)$ if and only if $\inf p(x, \xi) + \operatorname{tr}^+ p_2 \geqslant 0$. In particular, if $p(x, \xi) \geqslant 0$, then $p^w(x, D) \geqslant 0$.

It is a very different case for a system of differential operators. Hörmander gave the following example in [1].

Let

$$p(x, \xi) = \begin{bmatrix} x^2 & x\xi \\ x\xi & \xi^2 \end{bmatrix}, \qquad (x, \xi) \in R^2;$$

then $p(x, \xi) \geqslant 0$ but

$$\left\langle p^w(x, D) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\rangle = -\frac{1}{2} \int (v')^2\, dx < 0,$$

where $u_1 = v''$, $u_2 = i(v - xv')$ and $v \in \mathscr{S}(R^n)$ is real-valued and not identically equal to zero.

In this paper we will study the positivity of systems of operators with the symbol

$$p(x, \xi) = \begin{bmatrix} ax^2 + b\xi^2 & \alpha x\xi \\ \alpha x\xi & cx^2 + d\xi^2 \end{bmatrix}, \qquad (x, \xi) \in R^2 \quad a, b, c, d \geqslant 0,$$

$$ad + bc \neq 0.$$



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*Remark* 1.1 By the symplectic invariance of the Weyl operators, given any $(x_0, \xi_0)$ and $u \in \mathscr{S}(R)$, we can find $u_n \in \mathscr{S}(R)$ such that $\lim_{n \to \infty} \langle p^w(x, D) u_n, u_n \rangle = \langle p(x_0, \xi_0) u, u \rangle$. See [1] for the details. It follows that $p(x, \xi) \geqslant 0$ is a necessary condition for $p^w(x, D)$ to be positive. Hence $p^w(x, D)$ cannot be positive if one of $a, b, c, d$ is strictly negative. It is also obvious that if $a, b, c, d \geqslant 0$ and $ad + bc = 0$, then $p^w(x, D) \geqslant 0$ if and only if $\alpha = 0$. Therefore from now on we will always assume $a, b, c, d \geqslant 0$ and $ad + bc \neq 0$.

The main result of this paper is the following theorem.

THEOREM 1.1. *$P^w(x, D) \geqslant 0$ if and only if $(\lambda_1, \lambda_2)$ and $(\lambda_2, \lambda_1)$ belong to the domain $\Omega$, where*

$$\lambda_1 = \frac{(ad)^{1/2} - (bc)^{1/2} + \alpha}{(ad)^{1/2} + (bc)^{1/2}}, \qquad \lambda_2 = \frac{(ad)^{1/2} - (bc)^{1/2} - \alpha}{(ad)^{1/2} + (bc)^{1/2}}$$

and $\Omega$ is a convex closed subset of $R^2$, symmetric with respect to both the $X$ and the $Y$ axes (Fig. 1). The precise definition of $\Omega$ will be given in Section 3.

In particular, the positivity of $p^w(x, D)$ depends only on $((ad)^{1/2}, (bc)^{1/2}, \alpha)$. The following corollary is a consequence of the fact that $\{(x, y) : x^2 + y^2 \leqslant 2\}$ is a proper subset of $\Omega$.

COROLLARY 1.1. *$\alpha^2 \leqslant 4(abcd)^{1/2}$ is a sufficient but not necessary condition for $p^w(x, D)$ to be positive.*

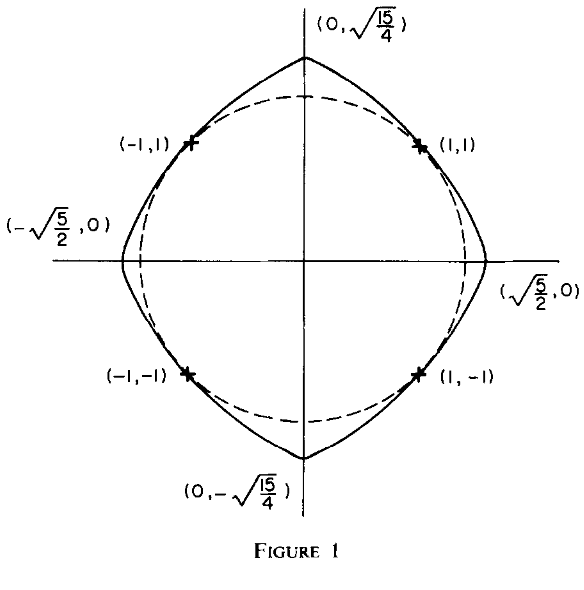


FIGURE 1

It is easy to see that $p(x, \xi) \geqslant 0$ if and only if $\alpha \leqslant (ad)^{1/2} + (bc)^{1/2}$, which corresponds to $|\lambda_1| + |\lambda_2| \leqslant 2$. Since $\Omega$ is a proper subset of $\{(x, y): |x| + |y| \leqslant 2\}$, we see that there exist positive symbols $p(x, \xi)$ such that $p^w(x, D)$ is not positive.

Moreover, if $abcd = 0$, say $b = 0$, then

$$\lambda_1 = \frac{(ad)^{1/2} + \alpha}{(ad)^{1/2}}, \qquad \lambda_2 = \frac{(ad)^{1/2} - \alpha}{(ad)^{1/2}} \qquad \text{and} \qquad \lambda_1 + \lambda_2 = 2.$$

Since $\Omega \cap \{(x, y): |x| + |y| = 2\} = \{(1,1), (1, -1), (-1, 1), (-1, -1)\}$, we obtain the following corollary.

COROLLARY 1.2. *If $abcd = 0$, then $p^w(x, D) \geqslant 0$ if and only if $\alpha = 0$.*

Hörmander's example is of course just a special case of Corollary 1.2.

The proof of Theorem 1.1 will be given in Sections 3,4 and 5. Section 2 contains some definitions and lemmas that will be needed.

## 2. HERMITE FUNCTIONS AND INFINITE HERMITIAN MATRICES

Let $H_n(x) = (-1)^n e^{x^2}(d^n/dx^n) e^{-x^2}$, $n = 0, 1, 2, \ldots$, be the Hermite polynomials. The Hermite functions $\sigma_n(x) = (2^n n! (\pi)^{1/2})^{-1/2} e^{-(1/2)x^2} H_n(x)$, $n = 0, 1, 2, \ldots$, form an orthonormal basis of $L^2(R)$.

From the well-known formulas [2]

$$2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x)$$
$$H_n'(x) = 2nH_{n-1}(x),$$

it follows that

$$x\sigma_n(x) = \left(\frac{n+1}{2}\right)^{1/2} \sigma_{n+1}(x) + \left(\frac{n}{2}\right)^{1/2} \sigma_{n-1}(x)$$
$$D\sigma_n(x) = i\left(\frac{n+1}{2}\right)^{1/2} \sigma_{n+1}(x) - i\left(\frac{n}{2}\right)^{1/2} \sigma_{n-1}(x),$$

where $D = -i(d/dx)$ and $\sigma_{-1}(x) = 0$.

The following lemma is a direct consequence of the previous formulas.

LEMMA 2.1. *Let $u$ belong to $\mathscr{S}(R)$. With respect to the basis $\{\sigma_n(x)\}_{n=0}^{\infty}$,*

(i) *the map* $u \to x^2 u$ *is represented by the infinite matrix* $(a_{ij})_{0 \leq i,j \leq \infty}$, *where*

$$a_{ij} = \begin{cases} \frac{1}{2}(2n+1), & i=j=n \\ \frac{1}{2}((n+1)(n+2))^{1/2}, & i=n, j=n+2 \text{ or } i=n+2, j=n \\ 0, & \textit{otherwise}, \end{cases}$$

*i.e.*

$$(a_{ij}) = \begin{bmatrix} 1/2 & 0 & (1/2)^{1/2} & 0 & \\ 0 & 3/2 & 0 & (3/2)^{1/2} & \\ (1/2)^{1/2} & 0 & 5/2 & & \\ 0 & (3/2)^{1/2} & & \ddots & \end{bmatrix};$$

(ii) *the map* $u \to D^2 u$ *is represented by the infinite matrix* $(b_{ij})_{0 \leq i,j \leq \infty}$, *where*

$$b_{ij} = \begin{cases} \frac{1}{2}(2n+1), & i=j=n \\ -\frac{1}{2}((n+1)(n+2))^{1/2}, & i=n, j=n+2 \text{ or } i=n+2, j=n \\ 0, & \textit{otherwise}, \end{cases}$$

*i.e.*,

$$(b_{ij}) = \begin{bmatrix} 1/2 & 0 & -(1/2)^{1/2} & 0 & \\ 0 & 3/2 & 0 & -(3/2)^{1/2} & \\ -(1/2)^{1/2} & 0 & 5/2 & & \\ 0 & -(3/2)^{1/2} & & \ddots & \end{bmatrix};$$

(iii) *the map* $u \to \frac{1}{2}(xD + Dx)\, u$ *is represented by the matrix* $(c_{ij})_{0 \leq i,j \leq \infty}$, *where*

$$(c_{ij}) = \begin{cases} -i\frac{1}{2}((n+1)(n+2))^{1/2}, & i=n, j=n+2 \\ i\frac{1}{2}((n+1)(n+2))^{1/2}, & i=n+2, j=n \\ 0 & \textit{otherwise}, \end{cases}$$

*i.e.*,

$$(c_{ij}) = \begin{bmatrix} 0 & 0 & -i(1/2)^{1/2} & 0 & \\ 0 & 0 & 0 & -i(3/2)^{1/2} & \\ i(1/2)^{1/2} & 0 & 0 & & \\ 0 & i(3/2)^{1/2} & & \ddots & \end{bmatrix}.$$

DEFINITION 2.1. Let $A$ be an infinite complex Hermitian matrix; then $A$ is positive if $\langle Av, v\rangle \geqslant 0$ for any complex vector with finitely many non-zero components.

*Remark* 2.1. If $A \geqslant 0$ and $T$ is an infinite matrix with the property that $Tv$ has finitely many non-zero components whenever $v$ has finitely many non-zero components, then $T^*AT \geqslant 0$.

We are particularly interested in band matrices of the form

$$B=\begin{bmatrix} a_1 & b_1 & 0 & 0 \\ b_1 & a_2 & b_2 & 0 \\ 0 & b_2 & a_3 & b_3 \\ 0 & 0 & b_3 & \ddots \end{bmatrix}, \qquad a_i, b_i \in R,\ a_i > 0.$$

The following lemma is obvious.

LEMMA 2.2. *B is positive if and only if*

$$\sum_{k=1}^{n} a_k x_k^2 + 2\sum_{k=1}^{n-1} b_k x_k x_{k+1} \geqslant 0 \qquad \textit{for} \quad n=1,2,\ldots \quad \textit{and} \quad x_k \in R.$$

*It is easy to see that if* $|b'_k| \leqslant |b_k|$, *then*

$$\sum_{k=1}^{n} a_k x_k^2 + 2\sum_{k=1}^{n-1} b_k x_k x_{k+1} \geqslant 0 \qquad \textit{for any real numbers } x_1, x_2, \ldots, x_n$$

*implies that* $\sum_{k=1}^{n} a_k x_k^2 + 2\sum_{k=1}^{n-1} b'_k x_k x_{k+1} \geqslant 0$ *for any real numbers* $x_1, x_2, \ldots, x_n$.

COROLLARY 2.1. *If* $B'$ *is obtained from* $B$ *by replacing* $b_k$ *by* $b'_k$, *where* $|b'_k| \leqslant |b_k|$, *then* $B \geqslant 0$ *implies* $B' \geqslant 0$.

LEMMA 2.3. *If* $b_i \neq 0$, $i = 1, 2, \ldots$, *then* $B \geqslant 0$ *if and only if* det $B_n > 0$ *for* $n = 1, 2, \ldots$, *where*

$$B_n=\begin{bmatrix} a_1 & b_1 & & 0 \\ b_1 & a_2 & & \\ & & \ddots & b_n \\ 0 & & b_n & a_{n+1} \end{bmatrix}.$$

*Proof.* The sufficiency is a well-known theorem in linear algebra. The necessity will be proved by contradiction. We know that $B \geqslant 0$ implies

$\det B_n \geqslant 0$ for $n = 1, 2, \ldots$. Let $m$ be the first integer such that $\det B_m = 0$. Then $\det B_{m+1} = a_{m+2} \det B_m - b_{m+1}^2 \det B_{m-1} < 0$, a contradiction.

Q.E.D.

LEMMA 2.4. *Let* $\tilde{B} = (\alpha_{ij})_{1 \leqslant i,j < \infty}$, *where*

$$\alpha_{ij} = \begin{cases} 1, & i = j = n, \\ b_n/(a_n a_{n+1})^{1/2}, & i = n, j = n+1 \quad \text{or} \quad j = n, i = n+1, \end{cases}$$

*i.e.*,

$$\tilde{B} = \begin{bmatrix} 1 & b_1/(a_1 a_2)^{1/2} & 0 & \\ b_1/(a_1 a_2)^{1/2} & 1 & b_2/(a_2 a_3)^{1/2} & \\ 0 & b_2/(a_2 a_3)^{1/2} & 1 & \\ & & & \ddots \end{bmatrix}.$$

*Then* $B \geqslant 0$ *if and only if* $\tilde{B} \geqslant 0$.

*Proof.* $\tilde{B} = T^* B T$ and $B = S^* \tilde{B} S$, where $T$ is the diagonal matrix

$$\begin{bmatrix} 1/(a_1)^{1/2} & & \\ & 1/(a_2)^{1/2} & \\ & & \ddots \end{bmatrix}$$

and $S$ is the diagonal matrix

$$\begin{bmatrix} (a_1)^{1/2} & & \\ & (a_2)^{1/2} & \\ & & \ddots \end{bmatrix}.$$

The lemma now follows from Remark 2.1. Q.E.D.

## 3. TRANSFORMATION OF THE PROBLEM

Let

$$p(x, \xi) = \begin{bmatrix} ax^2 + b\xi^2 & \alpha x \xi \\ \alpha x \xi & cx^2 + d\xi^2 \end{bmatrix}.$$

We will assume that $a, b, c, d > 0$ in this section.

Let $H$ be the Hilbert space $L^2(R) \oplus L^2(R)$ and $S$ be the subspace $\mathscr{S}(R) \oplus \mathscr{S}(R)$.

LEMMA 3.1. *There exists a bounded operator $T: H \to H$ such that $T: S \to S$ is an automorphism and $T^* p^w(x, D)\, T = q^w(x, D)$, where*

$$q(x, \xi) = \begin{bmatrix} \beta x^2 + \gamma \xi^2 & \alpha x \xi \\ \alpha x \xi & \gamma x^2 + \beta \xi^2 \end{bmatrix}, \qquad \beta = (ad)^{1/2}, \qquad \gamma = (bc)^{1/2}.$$

*Proof.* Let $T_1 : H \to H$ be defined by

$$T_1 \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} (cd/ab)^{1/8}\, u \\ (ab/cd)^{1/8}\, v \end{bmatrix}.$$

$T_1$ is bounded and $T_1 : S \to S$ is an automorphism. $T_1^* p^w(x, D)\, T_1 = q_1^w(x, D)$, where

$$q_1(x, \xi) = \begin{bmatrix} (a^3cd/b)^{1/4} x^2 + (b^3cd/a)^{1/4} \xi^2 & \alpha x \xi \\ \alpha x \xi & (abc^3/d)^{1/4}\, x^2 + (abd^3/c)^{1/4}\, \xi^2 \end{bmatrix}.$$

Let $h = (bd/ac)^{1/4}$. By the symplectic invariance of Weyl operators [1], there exists a unitary operator $U$ on $L^2(R)$ such that $U: \mathscr{S}(R) \to \mathscr{S}(R)$ is an automorphism, $U^* x^2 U = hx^2$, $U^* D^2 U = (1/h)\, D^2$ and $U^*(xD + Dx)/2U = (xD + Dx)/2$.

Let $T_2 : H \to H$ be defined by $T_2 \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] = \left[\begin{smallmatrix} Uu \\ Uv \end{smallmatrix}\right]$, then $T_2^* q_1^w(x, D)\, T_2 = q^w(x, D)$. The proof is completed if we let $T = T_1 T_2$. Q.E.D.

We now introduce a basis for $H$. Let

$$e_{2k} = \begin{bmatrix} \sigma_k \\ 0 \end{bmatrix} \quad \text{and} \quad e_{2k+1} = \begin{bmatrix} 0 \\ \sigma_k \end{bmatrix},$$

where $\sigma_k$, $k = 0, 1, 2, \ldots$, are the Hermite functions. $\{e_n\}_{n=0}^{\infty}$ is a basis of $H$. Let $\omega_n = ((n+1)(n+2)/4)^{1/2}$, $n = 0, 1, 2, \ldots$.

The following lemma is a direct consequence of Lemma 2.1.

LEMMA 3.2. *With respect to the basis $\{e_n\}_{n=0}^{\infty}$, $q^w(x, D)$ is represented by the infinite matrix*

$$\begin{bmatrix} A_0 & 0 & B_0 & 0 \\ 0 & A_1 & 0 & B_1 \\ B_0^* & 0 & A_2 & 0 \\ 0 & B_1^* & 0 & \ddots \end{bmatrix}.$$

*where $A_n = [(2n+1)/2](\beta + \gamma) \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $B_n = \omega_n \left[\begin{smallmatrix} \beta - \gamma & -i\alpha \\ -i\alpha & \gamma - \beta \end{smallmatrix}\right]$.*

LEMMA 3.3. *There exists a unitary operator* $U: H \to H$ *such that* $U: S \to S$ *is an automorphism and* $U^* q^w(x, D)\, U$ *is represented the infinite matrix*

$$\begin{bmatrix} A_0 & 0 & C_0 & 0 \\ 0 & A_1 & 0 & C_1 \\ C_0^* & 0 & A_2 & 0 \\ 0 & C_1^* & 0 & \ddots \end{bmatrix},$$

*where* $A_n = [(2n+1)/2](\beta+\gamma)\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ *and*

$$C_n = \begin{cases} \omega_n \begin{bmatrix} \beta-\gamma & -\alpha \\ -\alpha & \beta-\gamma \end{bmatrix}, & n \cong 0, 1 \mod 4, \\[2ex] \omega_n \begin{bmatrix} \beta-\gamma & \alpha \\ \alpha & \beta-\gamma \end{bmatrix}, & n \cong 2,3 \mod 4. \end{cases}$$

*Proof.* Let $U: H \to H$ be defined by the infinite matrix

$$\begin{bmatrix} U_0 & 0 & 0 & \\ 0 & U_1 & 0 & \\ 0 & 0 & U_2 & \\ & & & \ddots \end{bmatrix},$$

where

$$U_n = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, & n \cong 0, 1 \mod 4, \\[2ex] \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}, & n \cong 2, 3 \mod 4. \end{cases}$$

$U^* q^w(x, D)\, U$ has the desired representation. Q.E.D.

LEMMA 3.4. *There exists a unitary operator* $W: H \to H$ *such that* $W: S \to S$ *is an automorphism and* $W^* q^w(x, D)\, W$ *is represented by the infinite matrix*

$$\begin{bmatrix} A_0 & 0 & E_0 & 0 \\ 0 & A_1 & 0 & E_1 \\ E_0 & 0 & A_2 & 0 \\ 0 & E_1 & 0 & \ddots \end{bmatrix},$$

*where* $A_n = [(2n+1)/2](\beta+\gamma)\left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$ *and*

$$E_n = \begin{cases} \omega_n \begin{bmatrix} \beta-\gamma+\alpha & 0 \\ 0 & \beta-\gamma-\alpha \end{bmatrix}, & n \cong 0, 1 \mod 4, \\ \omega_n \begin{bmatrix} \beta-\gamma-\alpha & 0 \\ 0 & \beta-\gamma+\alpha \end{bmatrix}, & n \cong 2, 3 \mod 4. \end{cases}$$

*Proof.* Let $V\colon H \to H$ be defined by

$$V\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} (u+v)/(2)^{1/2} \\ (v-u)/(2)^{1/2} \end{bmatrix}.$$

If we let $W = UV$, where $U$ is the unitary operator in Lemma 3.3, then $W^* q^w(x, D)\, W$ has the desired representation. Q.E.D.

LEMMA 3.5. *There exists a bounded operator* $Q\colon H \to H$ *such that* $Q\colon S \to S$ *is an automorphism and* $Q^* q^w(x, D)\, Q$ *is represented by the infinite matrix*

$$\begin{bmatrix} F_0 & 0 & G_0 & 0 \\ 0 & F_1 & 0 & G_1 \\ G_0 & 0 & F_2 & 0 \\ 0 & G_1 & 0 & \ddots \end{bmatrix},$$

*where* $F_n = (2n+1)/2\left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$,

$$G_n = \begin{cases} \omega_n \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, & n \cong 0, 1 \mod 4, \\ \omega_n \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix}, & n \cong 2, 3 \mod 4, \end{cases}$$

$\lambda_1 = (\beta-\gamma+\alpha)/(\beta+\gamma)$ *and* $\lambda_2 = (\beta-\gamma-\alpha)/(\beta+\gamma)$.

*Proof.* Let $Q_1\colon H \to H$ be defined by $Q_1\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right] = 1/(\beta+\gamma)^{1/2}\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right]$. If we let $Q = WQ_1$, where $W$ is the operator in Lemma 3.4, then $Q^* q^w(x, D)\, Q$ has the desired representation. Q.E.D.

Let $\zeta_n = ((2n+1)(2n+2)/(4n+1)(4n+5))^{1/2}$ and $\eta_n = ((2n+2)(2n+3)/(4n+3)(4n+7))^{1/2}$ for $n = 0, 1, 2, \ldots$.

LEMMA 3.6. *$q^w(x, D) \geqslant 0$ if and only if the following infinite matrices $N_1$, $N_2$, $N_3$, $N_4$ are positive, where*

$$N_1 = \begin{bmatrix} 1 & \zeta_0\lambda_1 & & & & 0 \\ \zeta_0\lambda_1 & 1 & \zeta_1\lambda_2 & & & \\ & \zeta_1\lambda_2 & 1 & \zeta_2\lambda_1 & & \\ & & \zeta_2\lambda_1 & 1 & \zeta_3\lambda_2 & \\ 0 & & & \zeta_3\lambda_2 & \ddots & \end{bmatrix},$$

$$N_2 = \begin{bmatrix} 1 & \zeta_0\lambda_2 & & & & 0 \\ \zeta_0\lambda_2 & 1 & \zeta_1\lambda_1 & & & \\ & \zeta_1\lambda_1 & 1 & \zeta_2\lambda_2 & & \\ & & \zeta_2\lambda 2 & 1 & \zeta_3\lambda_1 & \\ 0 & & & \zeta_3\lambda_1 & \ddots & \end{bmatrix},$$

$$N_3 = \begin{bmatrix} 1 & \eta_0\lambda_1 & & & & 0 \\ \eta_0\lambda_1 & 1 & \eta_1\lambda_2 & & & \\ & \eta_1\lambda_2 & 1 & \eta_2\lambda_1 & & \\ & & \eta_2\lambda_1 & 1 & \eta_3\lambda_2 & \\ 0 & & & \eta_3\lambda_2 & \ddots & \end{bmatrix}$$

and

$$N_4 = \begin{bmatrix} 1 & \eta_0\lambda_2 & & & \\ \eta_0\lambda_2 & 1 & \eta_1\lambda_1 & & \\ & \eta_1\lambda_1 & 1 & \eta_2\lambda_2 & \\ & & \eta_2\lambda_2 & 1 & \eta_3\lambda_1 \\ & & & \eta_3\lambda_1 & \ddots \end{bmatrix}.$$

*Proof.* By Lemma 3.5, $q^w(x, D) \geqslant 0$ if and only if the matrices

$$\begin{bmatrix} F_0 & G_0 & & 0 \\ G_0 & F_2 & G_2 & \\ & G_2 & F_4 & \\ 0 & & & \ddots \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} F_1 & G_1 & 0 & \\ G_1 & F_3 & G_3 & \\ & G_3 & F_5 & \\ 0 & & & \ddots \end{bmatrix}$$

are positive.

If we write out the components of $F_0, G_0, F_2, G_2, \ldots$, we see that the first matrix is equal to the following one:

$$\begin{bmatrix} (1/2) & 0 & \omega_0\lambda_1 & & & & \\ 0 & (1/2) & 0 & \omega_0\lambda_2 & & 0 & \\ \omega_0\lambda_1 & 0 & (5/2) & 0 & \omega_2\lambda_2 & & \\ & \omega_0\lambda_2 & 0 & (5/2) & 0 & \omega_2\lambda_1 & \\ & & \omega_2\lambda_2 & 0 & (9/2) & 0 & \\ & 0 & & \omega_2\lambda_1 & 0 & (9/2) & \\ & & & & & & \ddots \end{bmatrix}.$$

This matrix is positive if and only if the matrices

$$\begin{bmatrix} (1/2) & \omega_0\lambda_1 & & 0 \\ \omega_0\lambda_1 & (5/2) & \omega_2\lambda_2 & \\ & \omega_2\lambda_2 & (9/2) & \\ 0 & & & \ddots \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} (1/2) & \omega_0\lambda_2 & & 0 \\ \omega_0\lambda_2 & (5/2) & \omega_2\lambda_1 & \\ & \omega_2\lambda_1 & (9/2) & \\ 0 & & & \ddots \end{bmatrix}$$

are positive. By Lemma 2.4, these matrices are positive if and only if $N_1$ and $N_2$ are positive. Similarly, the second matrix at the beginning of the proof is positive if and only if $N_3$ and $N_4$ are positive. Q.E.D.

Observe that $N_1, N_2 \geqslant 0$ implies $N_3, N_4 \geqslant 0$ by Corollary 2.1. We have therefore proved the following proposition.

PROPOSITION 3.1. *$p^w(x, D) \geqslant 0$ if and only if $N_1, N_2 \geqslant 0$.*

## 4. THE DOMAIN $\Omega$

As in Section 3, let $\zeta_n = ((2n+1)(2n+2)/(4n+1)(4n+5))^{1/2}$ for $n = 0, 1, 2, \ldots$. $N(x, y)$ is the following infinite matrix.

$$\begin{bmatrix} 1 & \zeta_0 x & & & & \\ \zeta_0 x & 1 & \zeta_1 y & & 0 & \\ & \zeta_1 y & 1 & \zeta_2 x & & \\ & & \zeta_2 x & 1 & \zeta_3 y & \\ & 0 & & \zeta_3 y & 1 & \\ & & & & & \ddots \end{bmatrix}.$$

DEFINITION 4.1. $\Omega = \{(x, y) \colon N(x, y) \geqslant 0\}$.

The following two propositions give some elementary properties of $\Omega$.

PROPOSITION 4.1. (i) *$\Omega$ is convex.*

(ii) *If* $(x, y) \in \Omega$, *then the rectangle* $\{(a, b): |a| \leqslant |x|$ *and* $|b| \leqslant |y|\} \subset \Omega$. *In particular,* $\Omega$ *is symmetric with respect to the X and Y axes.*

(iii) $\Omega$ *is closed.*

*Proof.* Parts (i) and (iii) are obvious. Part (ii) follows from Corollary 2.1. Q.E.D.

PROPOSITION 4.2. (i) $(0, y) \in \Omega$ *if and only if* $y^2 \leqslant 15/4$.

(ii) $(x, 0) \in \Omega$ *if and only if* $x^2 \leqslant 5/2$.

(iii) *If* $(x, y) \in \Omega$, *then* $2x^2/5 + 4y^2/15 < 1$ *unless* $xy = 0$.

*Proof.* To prove (i) oberve that $N(0, y)$ consists of blocks of the form

$$\begin{bmatrix} 1 & \zeta_{2n+1} y \\ \zeta_{2n+1} y & 1 \end{bmatrix}, \qquad n = 0, 1, 2, \ldots.$$

Since $\zeta_{2n+1}$ is decreasing in $n$, $N(0, y) \geqslant 0$ if and only if

$$\det \begin{bmatrix} 1 & (4/15)^{1/2} y \\ (4/15)^{1/2} y & 1 \end{bmatrix} \geqslant 0.$$

Part (ii) is proved similarly. Part (iii) follows from writing down the determinant of the first 3 by 3 block in $N(x, y)$ and Lemma 2.3. Q.E.D.

From now on we will assume that $0 < y^2 < 15/4$ and $0 < x^2 < 5/2$. Consequently 1 is strictly greater than $\zeta_{2n} x^2$ and $\zeta_{2n+1} y^2$, for $n = 0, 1, 2, \ldots$. Let

$$M(x, y) = \begin{bmatrix} 1 & x/2 & & & & \\ x/2 & 1 & y/2 & & 0 & \\ & y/2 & 1 & x/2 & & \\ & & x/2 & 1 & y/2 & \\ & 0 & & y/2 & \ddots & \end{bmatrix}.$$

LEMMA 4.1. $M(x, y) \geqslant 0$ *if and only if* $|x| + |y| \leqslant 2$.

*Proof.* It is obvious that $M(0, y) \geqslant 0$ iff $y^2 \leqslant 4$ and $M(x, 0) \geqslant 0$ iff $x^2 \leqslant 4$. Let $a_0 = 1$ and $a_{n+1} = 1 - x^2/4a_n$. When $x = y$, $M(x, y) \geqslant 0$ iff $a_n > 0$ for all $n$. It is elementary to show that $a_n > 0$ for all $n$ iff $x^2 \leqslant 1$. Since the set $\{(x, y): M(x, y) \geqslant 0\}$ is a convex set, we can deduce from the information above that it is exactly the set $\{(x, y): |x| + |y| \leqslant 2\}$. Q.E.D.

PROPOSITION 4.3. $\Omega \subset \{(x, y): |x| + |y| \leqslant 2\}$.

*Proof.* If $(x, y) \in \Omega$, then by considering submatrices far away we see that $M(x, y) \geqslant 0$. Therefore $\Omega \subset \{(x, y): |x| + |y| \leqslant 2\}$ by Lemma 4.1.

Q.E.D.

In order to state a sufficient condition for $(x, y)$ to be an element of $\Omega$, we introduce the following Möbius transforms. $T_n(z) = 1 - \zeta_{2n+1}^2 y^2/(1 - \zeta_{2n}^2 x^2/z)$, $n = 0, 1, 2, \ldots$.

Let $\tau_n = 1 - \zeta_{2n+1}^2 y^2$, $\delta_n = \zeta_{2n}^2 x^2$ and $\gamma_n = \zeta_{2n}^2 x^2/(1 - \zeta_{2n+1}^2 y^2)$, for $n = 0, 1, 2, \ldots$. Then $T_n(\infty) = \tau_n$, $T_n(\delta_n) = \infty$, $T_n(\gamma_n) = 0$, $\delta_n < \gamma_n$ and $1 - \zeta_{2n}^2 x^2/z > 0$ for $z > \delta_n$. $\{\tau_n\}$ is an increasing sequence with limit $1 - y^2/4$. $\{\delta_n\}$ and $\{\gamma_n\}$ are decreasing sequences with limits $x^2/4$ and $x^2/(4 - y^2)$, respectively.

Let

$$D_n = \begin{bmatrix} 1 & \zeta_0 x & & & & & \\ \zeta_0 x & 1 & \zeta_1 y & & & 0 & \\ & \zeta_1 y & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & \zeta_{2n} x & \\ & 0 & & & \zeta_{2n} x & 1 & \zeta_{2n+1} y \\ & & & & & \zeta_{2n+1} y & 1 \end{bmatrix}, \qquad \text{for } n \geqslant 0.$$

LEMMA 4.2. *If* $T_0(1) > \gamma_1$, $T_1 T_0(1) > \gamma_2, \ldots$, $T_{n-1} \cdots T_0(1) > \gamma_n$, *then* $\det D_n > 0$.

*Proof.* We can reduce $D_n$ to the following matrix by row operations.

$$\tilde{D}_n = \begin{bmatrix} 1 & \zeta_0 x & & & & & \\ & a_0 & \zeta_1 y & & 0 & & \\ & & a_1 & & & & \\ & & & \ddots & & & \\ & 0 & & & a_{2n-1} & \zeta_{2n} x & \\ & & & & & a_{2n} & \zeta_{2n+1} y \\ & & & & & & a_{2n+1} \end{bmatrix},$$

where $a_0 = 1 - \zeta_0^2 x^2$, $a_{2k} = 1 - (\zeta_{2k}^2 x^2/T_{k-1} \cdots T_0(1))$ for $k \geqslant 1$ and $a_{2k+1} = T_k \cdots T_0(1)$ for $k \geqslant 0$.

By the assumption all the diagonal entries of $\tilde{D}_n$ are positive, therefore $\det D_n = \det \tilde{D}_n > 0$.

Q.E.D.

So far we have found some bounds for $\Omega$, the following proposition gives information about the interior of $\Omega$.

PROPOSITION 4.4. $\{(x, y): x^2+y^2 \leqslant 2\} \subset \Omega$.

We need one more lemma for the proof of Proposition 4.4.

LEMMA 4.3. $(1-(4k+5)\,t/(8k+9))(1-(4k+2)\,t/(8k+5)) - (4k+3)(4k+4)(2-t)/(8k+5)(8k+9) \geqslant 0$ *for* $t \in [0, 1]$ *and* $k = 0, 1, 2, \ldots$.

*Proof.* Define $F_k(t)$ as the left side of the inequality. By direct computation, $F'_k < 0$ on $[0, 1]$ and $F_k(1) = 0$. Therefore $F_k \geqslant 0$ on $[0, 1]$.

Q.E.D.

*Proof of Proposition* 4.4. By Proposition 4.1, we only have to consider the case where $x, y \geqslant 0$. The cases $x = 0$, $0 \leqslant y \leqslant 2^{1/2}$ and $0 \leqslant x \leqslant 2^{1/2}$, $y = 0$ follow from Proposition 4.2.

If $0 < y \leqslant x$, we are going to prove by induction that $T_n \cdots T_0(1) \geqslant (4n+5)/(8n+9) \geqslant \gamma_{n+1}$, for $n = 0, 1, 2, \ldots$.

Since $1 - 2x^2/5 - 3y^2/5 = 1 - x^2/2 - y^2/2 + (x^2 - y^2)/10 \geqslant 0$, $T_0(1) = 1 - (3/5) \cdot (4y^2/9)/(1 - 2x^2/5) \geqslant 5/9$. Also, $6x^2/13 + 7 \cdot 8 \cdot y^2/13 \cdot 17 \leqslant 6(x^2+y^2)/13 \leqslant 12/13 < 1$, which implies that $\gamma_1 = (5 \cdot 6x^2/9 \cdot 13)/(1 - 7 \cdot 8y^2/13 \cdot 17) < 5/9$.

Assume that the inequalities hold for $n = 0, 1, 2, \ldots, k-1$. $T_k \cdots T_0(1) = 1 - \zeta_{2k+1}^2 y^2/(1 - \zeta_{2k}^2 x^2/T_{k-1} \cdots T_0(1)) \geqslant 1 - \zeta_{2k+1}^2 y^2/(1 - \zeta_{2k}^2 x^2/(4k+1/8k+1))$, by the induction hypothesis and the fact that $T_k$ is increasing on $[\gamma_k, \infty)$. Since $1 - (4k+2)\,x^2/(8k+5) - (4k+3)\,y^2/(8k+5) = 1 - x^2/2 - y^2/2 + (x^2 - y^2)/(2 \cdot (8k+5)) \geqslant 0$, $T_k \cdots T_0(1) \geqslant (4k+5)/(8k+9)$. Also, $(4k+6)\,x^2/(8k+13) + (4k+7)(4k+8)\,y^2/(8k+13)(8k+17) \leqslant (x^2+y^2)(4k+6)/(8k+13) \leqslant (8k+12)/(8k+13) < 1$, which implies $\gamma_{k+1} \leqslant (4k+5)/(8k+9)$. Hence the inequalities hold for all $n$ and therefore $(x, y) \in \Omega$ by Lemma 4.2.

We still have to consider the case $0 < x \leqslant y$ (hence $x \leqslant 1$). This time we are going to prove by induction that $T_n \cdots T_0(1) \geqslant (4n+5)\,x^2(8n+9) \geqslant \gamma_{n+1}$.

$T_0(1) \geqslant 5x^2/9$ if and only if $(1 - 5x^2/9)(1 - 2x^2/5) - 3 \cdot 4 \cdot y^2/(5 \cdot 9) \geqslant 0$. On the other hand, $(1 - 5x^2/9)(1 - 2x^2/5) - 3 \cdot 4 \cdot y^2/(5 \cdot 9) \geqslant (1 - 5x^2/9)(1 - 2x^2/5) - 3 \cdot 4 \cdot (2 - x^2)/(5 \cdot 9)$. Therefore $T_0(1) \geqslant 5x^2/9$ by the case $k = 0$ in Lemma 4.3. Also, $7/13 \geqslant 7 \cdot 8 \cdot y^2/(13 \cdot 17)$ implies $1 - 7 \cdot 8 \cdot y^2/(13 \cdot 17) \geqslant 6/13$, which in turn implies that $\gamma_1 = (5 \cdot 6 \cdot x^2/9 \cdot 13)/(1 - 7 \cdot 8 \cdot y^2/13 \cdot 17) \leqslant 5x^2/9$.

Assume that the inequalities hold for $n = 0, 1, \ldots, k-1$. $T_k \cdots T_0(1) = 1 - \zeta_{2k+1}^2 y^2/(1 - \zeta_{2k}^2 x^2/(T_{k-1} \cdots T_0(1))) \geqslant 1 - \zeta_{2k+1}^2 y^2/(1 - \zeta_{2k}^2 x^2/((4k+1)\,x^2/(8k+1)))$, by the induction hypothesis. By Lemma 4.3, $(1 - (4k+5)\,x^2/(8k+9))(1 - (4k+2)\,x^2/(8k+5)) - (4k+3)(4k+4)\,y^2/(8k+5)(8k+9) \geqslant (1 - (4k+5)\,x^2/(8k+9))(1 - (4k+2)\,x^2/(8k+5)) - (4k+3)(4k+4)(2 - x^2)/(8k+5)(8k+9) \geqslant 0$, which implies that $T_k \cdots T_0(1) \geqslant (4k+5)\,x^2/(8k+9)$. Also, $(4k+7)/8k+13) \geqslant (4k+7)(4k+8)\,y^2/(8k+13)(8k+17)$ implies $1-(4k+7)(4k+8)\,y^{22}/(8k+13)(8k+17) \geqslant (4k+6)/(8k+13)$, which in turn implies that $\gamma_{k+1} \geqslant (4k+5)\,x^2/(8k+9)$. Hence the inequalities hold for all $n$ and therefore $(x, y) \in \Omega$ by Lemma 4.2. Q.E.D.

From Proposition 4.4 we know that the points $(1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$ belong to $\Omega \cap \{(x, y) \colon |x| + |y| = 2\}$. We want to show that they are the only points in the intersection.

If $x+y=2$, we can write $N(x, y) = A^*A + (1-x)\,B$, where

$$A = \begin{bmatrix} \rho_0 & \xi_1 & & 0 \\ & \rho_1 & \xi_2 & \\ & 0 & \rho_2 & \\ & & & \ddots \end{bmatrix}, \qquad \rho_k = (2k+1/4k+1)^{1/2}, \quad \xi_k = (2k/4k+1)^{1/2}$$

and

$$B = \begin{bmatrix} 0 & -\zeta_0 & & & \\ -\zeta_0 & 0 & \zeta_1 & 0 & \\ & \zeta_1 & 0 & -\zeta_2 & \\ & 0 & -\zeta_2 & 0 & \zeta_3 \\ & & & \zeta_3 & \ddots \end{bmatrix}.$$

We now define an infinite vector $v$ by the following formulas, where $v_k$ is the $k$th component of $v$. The definition of $v$ depends on a large positive integer $n$.

$$v_1 = -\rho_0 = -1.$$

$$v_2 = \rho_0/\xi_1 = (5/2)^{1/2}.$$

$$v_3 = -\rho_0\rho_1/\xi_1\xi_2 = -(1 \cdot 3 \cdot 9/2 \cdot 4)^{1/2}.$$

$$\vdots \qquad\qquad \vdots$$

$$\begin{aligned} v_k &= (-1)^k(\rho_0\rho_1 \cdots \rho_{k-2}/\xi_1\xi_2 \cdots \xi_{k-1}) \\ &= (-1)^k(1 \cdot 3 \cdots (2k-3)(4k-3)/2 \cdot 4 \cdots (2k-2))^{1/2} \qquad \text{for} \quad k \leqslant n. \end{aligned}$$

$$v_j = -v_{j-1}(j-1)/j \qquad \text{for} \quad j \geqslant n+1.$$

It is obvious that $v \in l^2$ and $(Av)_k = 0$ for $1 \leqslant k \leqslant n-1$. Also, $(Av)_j = (-1)^{j-n}\,v_n \cdot (n/j)(\rho_j - \xi_j)$ for $j \geqslant n$.

Lemma 4.4. $\|Av\|^2 = v_n^2 \cdot O(1/n)$.

*Proof.*

$$\begin{aligned}(Av)_j^2 &= v_n^2\cdot(n^2/j^2)[(2j-1)^{1/2}/(4j-3)^{1/2}-(j/j+1)(2j)^{1/2}/(4j+1)^{1/2}]^2\\ &= v_n^2\cdot(n^2/j^2)((1/2)^{1/2}\cdot 5/4j+R_j)^2\\ &= v_n^2(n^2/j^2)(25/32j^2+\tilde{R}_j), \qquad \text{where} \quad |\tilde{R}_j|\leqslant M/j^3\end{aligned}$$

with $M$ independent of $j$.

$$\begin{aligned}\therefore \sum_{j=n}^{\infty}(Av)_j^2 &= v_n^2\cdot n^2\cdot\sum_{j=n}^{\infty}(25/32j^4+\tilde{R}_j/j^2)\\ &= v_n^2\cdot n^2\cdot O(1/n^3)\\ &= v_n^2\cdot O(1/n). \qquad\qquad \text{Q.E.D.}\end{aligned}$$

On the other hand, $\langle Bv, v\rangle = 2\sum_{k=1}^{\infty}(-1)^k\zeta_{k-1}v_kv_{k+1} = 2\sum_{k=1}^{n-1}(-1)^k\zeta_{k-1}v_kv_{k+1}+2\sum_{k=n}^{\infty}(-1)^k\zeta_{k-1}v_kv_{k+1}$.

LEMMA 4.5. $2\sum_{k=1}^{n-1}(-1)^k\zeta_{k-1}v_kv_{k+1}=(-1)^n v_n^2/2+(-1)^n b_n$, *where* $\{b_n\}$ *has a positive limit.*

*Proof.* Let us define $a_n=2\sum_{k=1}^{n-1}(-1)^k\zeta_{k-1}v_kv_{k+1}+(-1)^{n+1}v_n^2/2 = 2+2\sum_{k=2}^{n-1}(-1)^{k+1}\,3\cdot5\cdots(2k-1)/2\cdot4\cdots(2k-2)+(-1)^{n+1}\,1\cdot3\cdots(2n-3)\cdot(4n-3)/2\cdot4\cdots(2n-2)\cdot2$, for $n\geqslant 2$.

Hence, $a_2=0.75$, $a_3=-0.6875$, $a_4=0.71875$, $a_5=-0.6992187$, etc. It is easy to prove by induction that $a_2>a_4>a_6>\cdots$, $-a_3<-a_4<\cdots$, $a_{2n}+a_{2n+1}>0$ and $a_{2n}+a_{2n+1}$ goes to zero as $n$ goes to infinity.

Therefore $b_n=(-1)^n a_n$ has a positive limit. In fact, $\lim b_n=0.70\cdots$.

Q.E.D.

LEMMA 4.6. $2\sum_{k=n}^{\infty}(-1)^k\zeta_{k-1}v_kv_{k+1} = (-1)^{n+1}v_n^2/2+(-1)^{n+1}v_n^2\cdot O(1/n)$.

*Proof.* $2\sum_{k=n}^{\infty}(-1)^k\zeta_{k-1}v_kv_{k+1} = 2\sum_{k=n}^{\infty}(-1)^{k+1}\zeta_{k-1}v_n^2\cdot n^2/(k(k+1))=v_n^2\cdot n^2\sum_{k=n}^{\infty}(-1)^{k+1}(1/(k(k+1)+R_k)$, where $R_k=(2\zeta_{k-1}-1)/(k(k+1))=[2((2k-1)(2k)/(4k-3)(4k+1))^{1/2}-1]/(k(k+1))$.

It is easy to see that there exists a constant $C$ such that $0\leqslant R_k\leqslant C/k^4$, for $k=n, n+1,\ldots$. Therefore $\sum_{k=n}^{\infty}R_k=O(1/n^3)$.

$\sum_{k=n}^{\infty}(-1)^{k+1}/(k(k+1)) = (-1)^{n+1}\sum_{j=0}^{\infty}2/((n+2j)(n+2j)(n+2j+1)(n+2j+2)) = (-1)^{n+1}/(2n^2)+O(1/n^3)$. The proof is now complete. Q.E.D.

Combining Lemmas 4.5 and 4.6, we obtain the following corollary.

COROLLARY 4.1. $\langle Bv, v\rangle=(-1)^n\delta_n+(-1)^{n+1}v_n^2\cdot O(1/n)$.

Finally, we want to estimate $v_n^2$.

LEMMA 4.7. $v_n^2 = O(n^{1/2})$.

*Proof.*

$$v_n^2 = 1 \cdot 3 \cdots (2n-3)(4n-3)/(2 \cdot 4 \cdots (2n-2))$$
$$= (4n-3)(2n-2)!/(2^{2n-2}((n-1)!)^2).$$

The proof is completed by an application of Stirling's formula $n! \sim (2\pi n)^{1/2} \cdot n^n \cdot e^{-n}(1 + 1/12n + \cdots)$. Q.E.D.

We are now in the position to prove the next proposition.

PROPOSITION 4.5. $\Omega \cap \{(x, y): |x| + |y| = 2\} = \{(1, 1), (1, -1), (-1, -1), (-1, 1)\}$.

*Proof.* By Proposition 4.1, we may assume that $x + y = 2$. From the discussion above, we know that $\langle N(x, y) v, v \rangle = \|Av\|^2 + (1 - x)\langle Bv, v \rangle = v_n^2 \cdot O(1/n) + (1 - x)((-1)^n b_n + v_n^2 \cdot O(1/n)) = O(n^{-1/2}) + (1-x)(-1)^n b_n$. Therefore $N(x, y)$ cannot be positive unless $x = 1$, in which case we also have $y = 1$. Q.E.D.

We can also use the vector $v$ to show that $\Omega$ is separated from the boundaries of $\{(x, y): |x| + |y| \leqslant 2\}$ by smooth curves with positive curvature.
Let

$$E = \begin{bmatrix} 0 & \zeta_0 & & & & & \\ \zeta_0 & 0 & 0 & & 0 & & \\ & 0 & 0 & \zeta_2 & & & \\ & & \zeta_2 & 0 & 0 & & \\ & 0 & & 0 & 0 & \zeta_4 & \\ & & & & \zeta_4 & \cdots & \end{bmatrix}.$$

LEMMA 4.8. $\langle Ev, v \rangle = O(n^{2/3})$.

*Proof.* First we assume $n = 2m$.

$$\langle Ev, v \rangle = 2 \sum_{k=0}^{\infty} \zeta_{2k} v_{2k+1} v_{2k+2}$$
$$= 2 \sum_{k=0}^{m} \zeta_{2k} v_{2k+1} v_{2k+2} + 2 \sum_{k=m+1}^{\infty} \zeta_{2k} v_{2k+1} v_{2k+2}.$$

By the definitions of $v$ and $\zeta_n$, the first sum

$$= -2 \sum_{k=0}^{m} (4k+1)(1 \cdot 3 \cdots (4k+1))/(2 \cdot 4 \cdots (4k+2))$$

$$= O\left( \sum_{k=0}^{m} k^{1/2} \right) \qquad \text{(cf. proof of Lemma 4.7)}$$

$$= O(m^{3/2})$$

$$= O(n^{3/2}).$$

For the second sum, we have

$$2 \sum_{k=m+1}^{\infty} \zeta_{2k} v_{2k+1} v_{2k+2}$$

$$= 2n^2 v_n^2 \sum_{k=m+1}^{\infty}$$

$$\times [(4k+1)(4k+2)/(8k+1)(8k+5)]^{1/2}/[(2k+1)(2k+2)]$$

$$= 2n^2 v_n^2 \cdot O(1/m)$$

$$= O(n^{3/2}), \qquad \text{by Lemma 4.7.}$$

The proof for odd $n$ is similar. Q.E.D.

PROPOSITION 4.6. *$\Omega$ can be separated from the boundary of $\{(x, y): |x| + |y| \leqslant 2\}$ by smooth curves of positive curvature.*

*Proof.* It suffices to prove the proposition in a small neighborhood of $(1, 1)$.

The infinite matrix $N(x, y) = N(2-y, y) + (x+y-2)E$. By the proof of Proposition 4.5, $\langle N(2-y, y)v, v\rangle = (y-1)(-1)^n b_n + O(n^{-1/2})$. Combining this estimate with Lemma 4.8, we have $\langle N(x, y)v, v\rangle = (y-1)(-1)^n b_n + I_1 + (x+y-2) I_2$, where $I_1 = O(n^{-1/2})$ and $I_2 = O(n^{3/2})$.

For $y \neq 1$, there exists $C_1 > 0$ independent of $y$ such that $C_1/(1-y)^2 < n$ implies $I_1 < |y-1|/4$. There also exists $C_2 > 0$ independent of $x$ and $y$ such that $C_2(1-y)^4 I_2 < |1-y|/4$, if $n < 2C_1/(1-y)^2$.

Therefore if $x+y-2 = C_2(1-y)^4$ and $C_1/(1-y)^2 < n < 2C_1/(1-y)^2$, then $\langle N(x, y)v, v\rangle = \langle (1-y)(-1)^n b_n + |1-y|/2$. When $(1-y)^2$ is small, we can find both odd and even $n$ to satisfy $C_1/(1-y)^2 < n < 2C_1/(1-y)^2$. Since $b_n > \frac{1}{2}$ for all $n$, by choosing a proper parity for $n$, we have $\langle N(x, y)v, v\rangle < 0$.

Hence for $y$ close to 1, $(1, 1)$ is the only point on the curve $x+y-2=C_2(1-y)^4$ that also belongs to $\Omega$. By the convexity of $\Omega$, $x+y-2=C_2(1-y)^4$ separates $\Omega$ from $x+y=2$ near $(1, 1)$.

## 5. Proof of Theorem1.1

We may now prove Theorem 1.1 in two steps. In the first step we assume $a, b, c, d>0$. From Proposition 3.1, we know that $p^w(x, D)\geqslant 0$ if and only if $N(\lambda_1, \lambda_2)$ and $N(\lambda_2, \lambda_1)\geqslant 0$, which is equivalent to $(\lambda_1, \lambda_2)$ and $(\lambda_2, \lambda_1)$ belonging to $\Omega$.

In the second step we look at the general case where $a, b, c, d\geqslant 0$ and $ad+bc>0$. Let us assume first that $p^w(x, D)\geqslant 0$. Given $\varepsilon>0$, we define $a_\varepsilon=a+\varepsilon$, $b_\varepsilon=b+\varepsilon$, $c_\varepsilon=c+\varepsilon$, $d_\varepsilon=d+\varepsilon$ and

$$p_\varepsilon(x, \xi)=\begin{bmatrix} a_\varepsilon x^2+b_\varepsilon\xi^2 & \alpha x\xi \\ \alpha x\xi & c_\varepsilon x^2+d_\varepsilon\xi^2 \end{bmatrix}.$$

Also, $\lambda_{1,\varepsilon}=[(a_\varepsilon d_\varepsilon)^{1/2}-(b_\varepsilon c_\varepsilon)^{1/2}+\alpha]/[(a_\varepsilon d_\varepsilon)^{1/2}+(b_\varepsilon c_\varepsilon)^{1/2}]$ and $\lambda_{2,\varepsilon}=[(a_\varepsilon d_\varepsilon)^{1/2}-(b_\varepsilon c_\varepsilon)^{1/2}-\alpha]/[(a_\varepsilon d_\varepsilon)^{1/2}+(b_\varepsilon c_\varepsilon)^{1/2}]$. Obviously, we have $p_\varepsilon^w(x, D)\geqslant p^w(x, D)$. From the first step we know that $(\lambda_{1,\varepsilon}, \lambda_{2,\varepsilon})$ and $(\lambda_{2,\varepsilon}, \lambda_{1,\varepsilon})$ belong to $\Omega$. Since $\Omega$ is obviously a closed set, $(\lambda_1, \lambda_2)$ and $(\lambda_2, \lambda_1)$ belong to $\Omega$ because $\lambda_{1,\varepsilon}$ and $\lambda_{2,\varepsilon}$ tend to $\lambda_1$ and $\lambda_2$ as $\varepsilon$ tends to 0.

Conversely, let us assume that $(\lambda_1, \lambda_2)$ and $(\lambda_2, \lambda_1)$ belong to $\Omega$. We may also assume that at least one of $a, b, c, d$, say $b$, equals 0 (hence $ad>0$). It follows that $\lambda_1+\lambda_2=2$, which implies that $\lambda_1=\lambda_2=1$ (Proposition 4.5). Therefore $\alpha=0$ and $p^w(x, D)$ is obviously positive. Q.E.D.

## References

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