JOURNAL OF FUNCTIONAL ANALYSIS 73, 152–178 (1987)

Nonlinear Superposition and Absorption of Delta Waves in One Space Dimension

JEFFREY RAUCH*

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

AND

MICHAEL C. REED†

Department of Mathematics, Duke University, Durham, North Carolina 27706

Communicated by the Editors

Received November 26, 1985; revised June 1986

1. INTRODUCTION

In the study of the initial value problem for linear hyperbolic partial differential equations it is usually possible to extend the solution operator from classical to distribution initial data. This extension is useful for several reasons. First, the existence of fundamental solutions allows one to construct other solutions by the superposition principle. Second, distribution data are idealizations of certain kinds of classical initial data: The delta functon is an idealization of a high localized peak, the derivative of a delta function is an idealization of a single sharp localized oscillation, and so forth. If one can understand how the idealized data are propagated, then, with suitable continuity, one has understood some of the essential features of the classical solutions with data which are close to the distribution data. In the linear theory there are many theorems which give precise analytical content to this idea.

In the nonlinear case the first reason for studying distribution data is not operative since there is no general superposition principal for nonlinear equations. The second reason is certainly appropriate, however, since one is very interested in the propagation of high peaks, sharp jumps, and

* Research partially supported by NSF Grant MCS-8301061.

† Research partially supported by NSF Grant DMS-8401590.

0022-1236/87 $3.00

oscillations. And, because of the second reason, the natural question to ask is how the nonlinear solutions behave for classical data which get closer and closer to certain kinds of distribution data. In this paper we study the behavior of solutions, u^ε, to the strictly hyperbolic semilinear system

$$(\partial_t + A(x, t)\,\partial_x + B(x, t))\,u^\varepsilon = f(x, t, u^\varepsilon) \qquad (1.1)$$
$$u^\varepsilon|_{t=0} = g + h^\varepsilon,$$

where g is "classical" and h^ε is smooth and converges to a distribution as $\varepsilon \to 0$. Surprisingly, in some quite general circumstances one can prove the convergence of u^ε and several interesting phenomena emerge.

In Section 2 we study this problem when f is Lipschitz and sublinear. We take $g \in L^1$ and let h^ε be a sequence of C^∞ functions which approach μ_s, a singular measure, in the sense of distributions. We suppose that $\{h^\varepsilon\}$ is uniformly bounded in L^1 and converges to zero in measure (imagine a sequence of approximations to the delta function). For simplicity we will state the result in the case where $A = \Lambda$ is diagonal and $B \equiv 0$. Let $\bar{u}$, σ^ε, and σ be the solutions of the following problems:

$$(\partial_t + \Lambda(x, t)\,\partial_x)\,\bar{u} = f(x, t, \bar{u})$$
$$\bar{u}|_{t=0} = g \qquad (1.2)$$

$$(\partial_t + \Lambda(x, t)\,\partial_x)\,\sigma^\varepsilon = 0$$
$$\sigma^\varepsilon|_{t=0} = h^\varepsilon \qquad (1.3)$$

$$(\partial_t + \Lambda(x, t)\,\partial_x)\,\sigma = 0 \qquad (1.4)$$
$$\sigma|_{t=0} = \mu_s.$$

Then the first Theorem in Section 2 states

THEOREM. *If f is sublinear, $u^\varepsilon - \bar{u} - \sigma^\varepsilon$ converges to zero in $C([0, T] : L^1)$, Further, since σ^ε goes to zero in measure and $\sigma^\varepsilon \to \sigma$ in the sense of distributions, we have that u^ε converges to $\bar{u} + \sigma$ in measure and in the sense of distributions.*

It is natural to call $\bar{u} + \sigma$ the "solution" of $(\partial_t + \Lambda\partial_x)\,u = f(x, t, u)$ with initial data $g + \mu_s$. We call these solutions delta waves. This theorem expresses a striking *superposition* principle. The singular part of the solution propagates linearly. The classical part propagates by the nonlinear equation. And, the limit of the nonlinear solution u^ε as the data becomes more singular is the sum of the two parts. The intuitive reason for this result is that the peaking part of the solution makes less and less difference in the nonlinear term since f is sublinear.

If $B \not\equiv 0$, a similar splitting takes place. Again the singular part of the limit satisfies a linear equation. The L^1 part satisfies a nonlinear equation where the off-diagonal part of Bu_{sing} appears linearly as a forcing term. As above the limit solution is the superposition of the two.

It is natural to ask whether the limit of u^ε and the superposition result hold for even more singular sequences $\{h^\varepsilon\}$. We investigate this question in Section 3 where we assume that f is bounded. Suppose that $\{h^\varepsilon\}$ is a sequence of C^∞ functions of smaller and smaller support which converge to a distribution v which has support on a set, S, of Lebesgue measure zero. There will exist such a sequence $\{h^\varepsilon\}$, for example, if v is a finite sum of derivatives of the delta function at finitely many points. For simplicity we state the result in the case where $B=0$ and S is closed. Let $\mathscr{S}$ denote the flow out of the initial singular points under the vector fields $\partial_t + \lambda_i \partial_x$.

THEOREM. *Suppose that f is bounded. Then $u^\varepsilon - \bar{u} - \sigma^\varepsilon$ converges to zero in $C([0, T]: L^1)$ and uniformly on compact subsets of the complement of $\mathscr{S}$. σ^ε converges to zero in measure and converges to σ (the solution of (1.4) with v replacing μ_s) in the sense of distributions, so, $u^\varepsilon \to \bar{u} + \sigma$ in the sense of distributions.*

There is a similar generalization of the results of Section 2 in the case $B \not\equiv 0$.

There is a relation between the growth of f and the permissible singularities in $\lim\{h^\varepsilon\}$. In Section 2, f is only required to be sublinear, but the $\{h^\varepsilon\}$ are uniformly bounded in L^1 so the most singular limits possible were measures. In Section 3, f is bounded, and v of any order can be permitted. It is clear that there is a family of theorems between these extreme cases which assume some growth in f and corresponding restrictions on v and conclude that linear superposition of the singular part occurs.

The next phenomenon we investigate is best illustrated by an example. The solution of the problem

$$u_t^\varepsilon = -(u^\varepsilon)^3$$
$$u^\varepsilon|_{t=0} = j_\varepsilon(x), \tag{1.5}$$

where $j_\varepsilon(x) = j(x/\varepsilon)/\varepsilon$, the usual mollifier, is given by

$$u^\varepsilon(t, x) = j_\varepsilon(x)(1 + 2tj_\varepsilon(x)^2)^{-1/2}.$$

Thus, for each $t > 0$, $u^\varepsilon(t, x) \to 0$ in L^1 as $\varepsilon \to 0$. More generally, if u^ε satisfies (1.5) and $u^\varepsilon|_{t=0} = g + j_\varepsilon(x)$ with $g \in L^\infty$, then $u^\varepsilon(t) \to^{L^1} \bar{u}(t)$ for $t > 0$, where $\bar{u}$ satisfies (1.5) with $\bar{u}|_{t=0} = g$. This illustrates the *disappearance* of singular data and the reason is not hard to understand. $j_\varepsilon(x)$ perturbs the data only on a set of small measure and on this set the dissipation in the nonlinear term damps any large disturbances to moderate size. The net effect is a small change as measured in L^1. This is the phenomenon investigated in Section 4. Let $g \in L^\infty$ and $\{h^\varepsilon\}$ be uniformly bounded in L^1 and converge to zero in measure. Suppose that the components of f satisfy the two conditions

$$\operatorname{sgn}(u_j) f_j(t, x, u) \leqslant c\left(1 + \sum_{i=1}^{k} |u_i|\right) \tag{1.6}$$

and

$$\lim_{|u_j| \to \infty} f_j(t, x, u)/u_j = -\infty, \tag{1.7}$$

where the limit is uniform for $(t, x, u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_k)$ in compact subsets of $\mathbb{R}^2 \times \mathbb{R}^{k-1}$. Condition (1.6) guarantees that solutions do not blow up in finite time and the condition (1.7) expresses superlinear dissipation.

THEOREM. *Assume that $A = \Lambda$ is a diagonal,* (1.6) *and* (1.7) *hold, and suppose u^ε satisfies* (1.1). *Then for each $t > 0$, $u^\varepsilon(t) \to^{L^1} \bar{u}(t)$, where $\bar{u}(t)$ satisfies*

$$(\partial_t + A\partial_x + B)\,\bar{u} = f(t, x, \bar{u}), \qquad \bar{u}|_{t=0} = g.$$

We have seen that delta waves exist but behave linearly in the sublinear case and that they are immediately absorbed in the superlinear dissipative case. It is natural to ask whether delta waves exist and whether they interact non-linearly in some superlinear non-dissipative situations? To see that the answer is yes we need just return to the canonical example [3]:

$$\begin{aligned} (\partial_t + \partial_x)\, v &= 0 \qquad & v(0, x) &= \delta(x+1) \\ (\partial_t - \partial_x)\, w &= 0 \qquad & w(0, x) &= \delta(x-1) \\ \partial_t z &= vw \qquad & z(0, x) &\equiv 0. \end{aligned} \tag{1.8}$$

The solution of this initial value problem is

$$\begin{aligned} v(x, t) &= \delta(x - t + 1) \\ w(x, t) &= \delta(x + t - 1) \\ z(x, t) &= \tfrac{1}{2} H(t-1)\, \delta(x), \end{aligned} \tag{1.9}$$

where H is the Heavyside function. Each term in the equations is meaningful in the sense of distributions and the three equalities hold. Taking the point of view of this paper, one can define v^ε, w^ε, z^ε as the solutions of (1.8) with $v(0, x) = j_\varepsilon(x+1)$, $w(0, x) = j_\varepsilon(x-1)$. Then v^ε, w^ε, z^ε converge

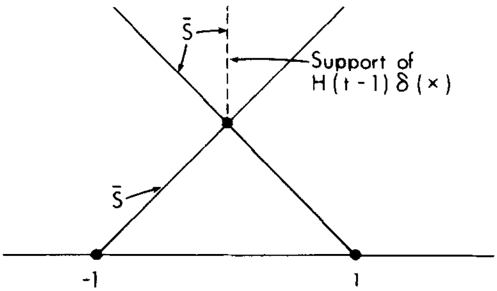

FIGURE 1

in the sense of distributions to the v, w, z in (1.9) and this convergence is uniform on compact subsets of the complement of the singular set $\bar{S}$. (See Fig. 1.)

Thus we have the standard picture of two singularities traveling on characteristics producing a new singularity at the point of intersection which travels on the other forward characteristic(s) from the point of intersection. In this case, both the incoming singularities and the outgoing anomalous singularity are delta waves. This is completely consistent with the formulas derived in [4]. There we showed that if a singularity of order n_1 interacts with a singularity of order n_2, then the anomalous singularities will (in general) have order n_1+n_2+2. (Saying that a singularity has order n_i means that n_i derivatives are continuous across the singularity bearing characteristic but the (n_i+1)th derivative jumps.) In [4] we treated the cases $n_i \geqslant -1$, where $n_i=-1$ means a jump discontinuity since one transverse integral will produce continuity. For (1.8), the incoming singularities are delta waves, i.e., $n_1=-2=n_2$, so the formula $n_1+n_2+2=-2$ predicts that the outgoing anomalous singularity should also be a delta wave which is exactly what we observed above. If one sends in more singular distributions in (1.8), say δ' singularities where $n_1=-3=n_2$, then the formula $n_1+n_2+2=-4$ predicts that the anomalous singularity will be even worse! If one does the approximation procedure described above and takes the limit then one can check directly that, indeed, z is a scalar multiple of $H(t-1)\,\delta''(x)$.

It is clear that in the superlinear non-dissipative case the existence of delta wave solutions requires restrictions on the nonlinear terms in $f(u)$. Here we do not pursue the existence and interaction of these waves.

It is a pleasure to thank Russ Caflisch, Cathleen Morawetz, and John Sylvester whose questions about previous work stimulated the author's interest in highly singular solutions. The authors are grateful for the hospitality of the Mittag-Leffler Institute where parts of this work were completed. Some results in the same direction as this paper have been obtained independently by Michael Oberguggenberger [2].

2. Nonlinear Superposition: Delta Waves

This section is devoted to the study of delta-function-like solutions in the presence of *sublinear* nonlinear terms. The governing equation is the $k \times k$ strictly hyperbolic system

$$[\partial_t + A(t, x)\, \partial_x + B(t, x)]u = f(t, x, u). \tag{2.1}$$

Because of the strict hyperbolicity we can, without loss, investigate this problem locally in x. So, given a time interval $[0, T]$ and a space interval $[-N, N]$ we denote by R the domain of determinacy of $\{t = T, -N \leqslant x \leqslant N\}$ and by R_t the set of x such that $\langle x, t \rangle \in R$. We will often make assertions like $v \in C([0, T]: L^1(R_t))$. By a change of variables which leaves each line $t = \text{constant}$ fixed, we can make R rectangular and, in these variables, it is clear what the assertion means. We will always assume that the initial data has compact support in the interval R_0 and restrict our attention to the behavior of the solution in R. For $p \in R$, the backward j characteristic from p to $\{t = 0\}$ will be denoted by $\mathscr{C}_j(p)$ or sometimes by $\mathscr{C}_j(p, q)$ or $\mathscr{C}_j(q)$, where q is the intersection point with the x-axis. Constants which depend on A, B, R, and the hypotheses on f (but not on ε) will all be denoted by c. Dependence on ε will always be explicit.

Hypotheses on A, B, f

We assume that A is in $C^1(R)$ and B is in $C(R)$. Since we are assuming strict hyperbolicity, A has k distinct real eigenvalues $\lambda_1 < \cdots < \lambda_k$ with the same smoothness properties as A. Concerning f we suppose that $f \in L^\infty_{\text{loc}}(R \times \mathbb{R}^k)$, that f is Lipschitz in u, i.e., $\nabla_u f \in L^\infty(R \times \mathbb{R}^k)$. In addition, we assume that f is *sublinear*

$$\lim_{|s| \to \infty} \frac{f(t, x, s)}{|s|} = 0, \tag{2.2}$$

the limit being uniform for $(t, x) \in R$.

Because of the sublinearity there cannot be blowup and one has the following L^p existence theorem:

Proposition. *For any $g \in L^p(R_0)$, $p < \infty$, there is a unique solution $u \in C([0, T]: L^p(R_t))$ to (2.1) such that $u(0, \cdot) = g$. In addition, there is a constant c_p such that if u_1 and u_2 are the solutions corresponding to g_1 and g_2, then*

$$\sup_{0 \leqslant t \leqslant T} \int_{R_t} |u_1 - u_2|^p \, dx \leqslant c_p \int_{R_0} |g_1 - g_2|^p \, dx.$$

Hypotheses on the data

We are interested in studying solutions u^ε of (2.1) with Cauchy data,

$$u^\varepsilon(0, \cdot) = g + h^\varepsilon, \tag{2.3}$$

with support in R_0, which becomes singular as $\varepsilon \to 0$. We assume

$$g \in L^1(R_0) \tag{2.4}$$

$$\{h^\varepsilon\}_{0<\varepsilon<1} \text{ is bounded in } L^1(R_0) \tag{2.5}$$

$\{h^\varepsilon\}$ converges to zero in measure, that is, if $S^{\varepsilon,\eta} = \{x \in R_0 : |h^\varepsilon(x)| > \eta\}$ then $\lim_{\varepsilon \to 0} m(S^{\varepsilon,\eta}) = 0$ for each $\eta > 0$, where m is Lebesgue measure. (2.6)

Sometimes we will assume the following additional hypothesis.

There is a nested family of closed measurable sets $\{T^\varepsilon\}$, $T^{\varepsilon_1} \subseteq T^{\varepsilon_2}$ if $\varepsilon_1 \leqslant \varepsilon_2$, with $\lim_{\varepsilon \to 0} m(T^\varepsilon) = 0$, which satisfy: For each fixed $\varepsilon_2 > 0$ and $\eta > 0$, there exists an ε_1 so that $S^{\varepsilon,\eta} \subset T^{\varepsilon_2}$ if $\varepsilon \leqslant \varepsilon_1$. (2.7)

When we assume (2.7), we will refer to $S \equiv \bigcap_\varepsilon T^\varepsilon$ as the singular set. For any of these sets, for example, S, we will denote the outflow in R under the vector field $\partial_t + \lambda_j \partial_x$ by $\mathscr{S}_j$, the time slice by $\mathscr{S}_j(t)$, and $\mathscr{S} \equiv \bigcup_j \mathscr{S}_j$, $\mathscr{S}(t) = \bigcup_j \mathscr{S}_j(t)$.

EXAMPLE 1. Let μ_s be a finite signed Borel measure with support in the interior of R_0 which is singular with respect to Lebesgue measure. Let $j \in C_0^\infty(\mathbb{R}), j \geqslant 0, \int j\,dx = 1$ and $j_\varepsilon(x) = \varepsilon^{-1} j(x/\varepsilon)$. Then for ε small $h^\varepsilon \equiv j_\varepsilon * \mu_s$ has support in R_0 and $\{h^\varepsilon\}$ is bounded in $L^1(R_0)$. Furthermore, Lebesgue's density theorem [6] implies that h^ε converges to zero almost everywhere with respect to Lebesgue measure. Since R_0 is compact this implies $h^\varepsilon \to 0$ in measure. In fact, that stronger hypothesis (2.7) holds too.

EXAMPLE 2. Let μ be any finite signed Borel measure with support in the interior of R_0 and let $\mu = \mu_{ac} + \mu_s = g\,dm + \mu_s$ be its Lebesgue decomposition. Then $j_\varepsilon * \mu = g + h^\varepsilon$, where $h^\varepsilon \equiv j_\varepsilon * \mu_s + (j_\varepsilon * g - g)$, satisfies the above hypotheses since $j_\varepsilon * g - g$ converges to zero in $L^1(R_0)$, pointwise a.e. and therefore almost uniformly since $m(R_0) < \infty$.

The above hypotheses and the proposition imply that for each ε there is a unique solution, u^ε, of (2.1) with initial data (2.3) which is in $C([0, T]: L^1(R_t))$. Furthermore, the family u^ε, $0 < \varepsilon < 1$, is bounded in $C([0, T]: L^1(R_t))$. We will study the limit of u^ε as $\varepsilon \to 0$. Note that a

Lipschitz change of dependent variable brings the system to characteristic form; that is, A is replaced by the diagonal matrix

$$\Lambda = \begin{bmatrix} \lambda_1(t,x) & & 0 \\ & \ddots & \\ 0 & & \lambda_k(t,x) \end{bmatrix}$$

and B is replaced by a possibly different element of $C(R)$. For our first result we assume that A is in characteristic form and that there are no lower order linear terms, i.e., $B \equiv 0$.

THEOREM 2.1. *Suppose that f is sublinear. Define u^ε, σ^ε, $\bar{u}$ by*

$$(\partial_t + \Lambda\partial_x)\, u^\varepsilon + f(t, x, u^\varepsilon) = 0 \qquad u^\varepsilon(0, \cdot) = g + h^\varepsilon \tag{2.8}$$

$$(\partial_t + \Lambda\partial_x)\, \sigma^\varepsilon = 0 \qquad \sigma^\varepsilon(0, \cdot) = h^\varepsilon \tag{2.9}$$

$$(\partial_t + \Lambda\partial_x)\bar{u} + f(t, x, \bar{u}) = 0 \qquad \bar{u}^\varepsilon(0, \cdot) = g. \tag{2.10}$$

Suppose $g \in L^1(R_0)$ and that $\{h^\varepsilon\}$ satisfies (2.5) *and* (2.6). *Then $u^\varepsilon - \bar{u} - \sigma^\varepsilon$ converges to zero in $C([0, T]: L^1(R_t))$ and*

$$\int_{R \setminus \mathscr{S}^{\varepsilon,\eta}} |u^\varepsilon - \bar{u}| < \eta m(R) + o(1). \tag{2.11}$$

Furthermore, if h^ε converges in the weak star topology on $\mathscr{M}(R_0)$ to a singular measure μ_s, then $u^\varepsilon(t) \rightharpoonup \bar{u}(t) + \sigma(t)$ weak star in $\mathscr{M}(R_t)$ uniformly for $0 \leqslant t \leqslant T$ where $\sigma(t)$ solves

$$(\partial_t + \Lambda\partial_x)\sigma = 0 \qquad \sigma(0, \cdot) = \mu_s. \tag{2.12}$$

If, in addition, $\{h^\varepsilon\}$ satisfies (2.7), *then $u^\varepsilon \to \bar{u}$ in L^1 on $R \setminus \mathscr{T}^\varepsilon$ for each ε.*

The proof of the theorem depends on the following:

LEMMA. *Suppose that X, μ is a finite measure space and that $F: X \times \mathbb{R}^k \to \mathbb{R}$ is uniformly Lipschitz with respect to the second variable:*

$$|F(x, w) - F(x, w')| \leqslant l\,|w - w'| \qquad \forall x, w, w' \in X \times \mathbb{R}^k \times \mathbb{R}^k.$$

In addition, suppose that F is measurable in X for each w and is sublinear:

$$\lim_{|w| \to \infty} F(x, w)/|w| = 0 \qquad \textit{almost uniformly on } X.$$

Suppose v^ε is a bounded family in $L^1(X)$, $v \in L^1(X)$, and $v^\varepsilon \to v$ in measure. Then $F(x, v^\varepsilon)$ converges to $F(x, v)$ in $L^1(X)$.

Proof. Given an $\eta > 0$ we will show that

$$\overline{\lim_{\varepsilon \to 0}} \int_X |F(x, v^\varepsilon(x)) - F(x, v(x))| \, d\mu(x) \leqslant \eta.$$

Let

$$G^\varepsilon = \{x \in X : |v^\varepsilon - v| \geqslant \eta/2l\mu(X)\}.$$

Then, from the Lipschitz bound on F,

$$\int_{X \setminus G^\varepsilon} |F(x, v^\varepsilon) - F(x, v)| \, d\mu \leqslant l \int_{X \setminus G^\varepsilon} |v^\varepsilon - v| \, d\mu$$
$$\leqslant \eta/2.$$

Since $v^\varepsilon \to v$ in measure, $\mu(G^\varepsilon) \to 0$ as $\varepsilon \to 0$. And, since F is sublinear, $F(x, v) \in L^1$, so $\int_{G^\varepsilon} |F(x, v)| \, d\mu = o(1)$ as $\varepsilon \to 0$. Thus, it suffices to show that

$$\overline{\lim_{\varepsilon \to 0}} \int_{G^\varepsilon} |F(x, v^\varepsilon)| \, d\mu < \eta/2.$$

Choose $M > \sup \int |v^\varepsilon| \, d\mu$ and $r > 0$ so that

$$|F(x, w)|/|w| \leqslant \eta/2M \qquad \text{for} \quad |w| \geqslant r \text{ and a.e. } x.$$

Let $E^\varepsilon = \{x : |v^\varepsilon(x)| > r\}$. Then

$$\int_{G^\varepsilon \cap E^\varepsilon} |F(x, v^\varepsilon)| \, d\mu \leqslant (\eta/2M) \int_{G^\varepsilon \cap E^\varepsilon} |v^\varepsilon| \, d\mu \leqslant \eta/2.$$

On the other hand,

$$\int_{G^\varepsilon \cap (X \setminus E^\varepsilon)} |F(x, v^\varepsilon)| \, dM \leqslant \Big(\operatorname*{ess\,sup}_{X \times \{|w| < r\}} |F| \Big) \, \mu(G^\varepsilon) = o(1),$$

so the proof is complete. ∎

Proof of Theorem 2.1. Let $v^\varepsilon = u^\varepsilon - \bar{u} - \sigma^\varepsilon$ and $L = \partial_t + \Lambda \partial_x$. Then, suppressing the t, x dependence of f, we have

$$-Lv^\varepsilon = f(u^\varepsilon) - f(\bar{u}) = f(v^\varepsilon + \bar{u} + \sigma^\varepsilon) - f(\bar{u})$$
$$= (f(v^\varepsilon + \bar{u} + \sigma^\varepsilon) - f(\bar{u} + \sigma^\varepsilon)) + (f(\bar{u} + \sigma^\varepsilon) - f(\bar{u})).$$

Thus $v^\varepsilon \in C([0, T] : L^1(R_t))$ and satisfies

$$\begin{gathered} |Lv^\varepsilon| \leqslant c\,|v^\varepsilon| + |f(\bar{u} + \sigma^\varepsilon) - f(\bar{u})| \\ v^\varepsilon(0, \cdot) \equiv 0. \end{gathered} \tag{2.13}$$

Now, σ^ε solves $L\sigma^\varepsilon = 0$ so the components of σ^ε are constant on the corresponding characteristics. Thus, since $h^\varepsilon \to 0$ in measure we conclude that $\sigma^\varepsilon \to 0$ in measure in R. Rurther, since $\{h^\varepsilon\}$ is bounded in $L^1(R_0)$, $\{\bar{u} + \sigma^\varepsilon\}$ is bounded in $L^1(R_t)$, uniformly for $t \in [0, T]$. Thus, $\bar{u}$ and $\bar{u} + \sigma^\varepsilon$ satisfy the hypotheses of the lemma and we conclude that

$$\int_{R_t} |f(\bar{u} + \sigma^\varepsilon) - f(\bar{u})| \, dx = o(1) \tag{2.14}$$

uniformly for $t \in [0, T]$. (2.13), (2.14), and Gronwall's inequality imply that v^ε is $o(1)$ in $C([0, T]: L^1(R_t))$. Since $|\sigma^\varepsilon| \leqslant \eta$ on the complement of $S^{\varepsilon, \eta}$, the estimate (2.11) holds.

Suppose now that h^ε converges in the weak star sense to a singular measure μ_s. The characteristic form of $(\partial_t + A\partial_x)$ makes it easy to define the singular measure-valued solution $\sigma(t)$ of (2.12) and to check that $\sigma^\varepsilon(t) \to \sigma(t)$ in the weak star sense. Finally, if (2.7) holds and $\varepsilon_2 > 0$, then σ^ε is small in sup norm outside of $\mathscr{T}^{\varepsilon_2}$ for ε small enough. Since $\{T^\varepsilon\}$ is nested, so is $\{\mathscr{T}^\varepsilon\}$ so $u^\varepsilon \to^{L^1} \bar{u}$ on the complement of each $\mathscr{T}^\varepsilon$. ∎

We now turn to the case where $B \neq 0$. Here it is clear that a singularity traveling in one component u_i will affect all the other components u_j in whose equation u_i appears as a linear lower order term. Nevertheless, there is a natural linear splitting of the solution into a singular part and an L^1 part. Write $B = D + E$, where D and E are the diagonal and off-diagonal part, respectively. Then if $h^\varepsilon \to \mu_s$ we will see that $u^\varepsilon \to \bar{u} + \sigma$, where $\bar{u}$ and σ satisfy

$$(\partial_t + A\partial_x + D)\sigma = 0, \qquad \sigma(0, \cdot) = \mu_s \tag{2.15}$$

$$(\partial_t + A\partial_x + B)\bar{u} + f(t, x, \bar{u}) + E\sigma = 0, \qquad \bar{u}(0, \cdot) = g. \tag{2.16}$$

Note that the first linear system determines σ which is then fed into the second system. The second is nonlinear with a singular forcing term $E\sigma$. That the solution, $\bar{u}$, is absolutely continuous (with respect to Lebesgue measure on R) depends on the fact that the jth component of $E\sigma$ is a sum of terms moving with velocities λ_i, $i \neq j$, since E has vanishing diagonal components. To see that $\bar{u}$ is absolutely continuous write the equation for $\bar{u}$ as

$$(\partial_t + A\partial_x + D)\bar{u} + E\sigma = -f(t, x, \bar{u}) - E\bar{u}$$

and let w be the solution of the linear system

$$(\partial_t + A\partial_x + D)w + E\sigma = 0, \qquad w(0, \cdot) = g.$$

Then $w \in C([0, T]: L^1(\mathbb{R}))$, since in each component one integrates transverse to the singularities in the forcing term. Setting $z \equiv \bar{u} - w$, we have

$$(\partial_t + A\partial_x + B)z + f(t, x, w + z) + Ew = 0, \qquad z(0, \cdot) = 0,$$

hence $z \in C([0, T]: L^1_{\text{loc}}(\mathbb{R}))$, too. Thus, $\bar{u} \in C([0, T]: L^1_{\text{loc}}(\mathbb{R}))$.

The next theorem asserts that (2.15), (2.16) give the correct limiting behavior.

THEOREM 2.2. *Suppose that f is sublinear. Define u^ε, σ^ε, and α^ε by*

$$(\partial_t + A\partial_x + B)\, u^\varepsilon + f(t, x, u^\varepsilon) = 0, \qquad u^\varepsilon(0, \cdot) = g + h^\varepsilon \quad (2.17)$$

$$(\partial_t + A\partial_x + D)\, \sigma^\varepsilon = 0, \qquad \sigma^\varepsilon(0, \cdot) = h^\varepsilon \quad (2.18)$$

$$(\partial_t + A\partial_x + B)\, \alpha^\varepsilon + f(t, x, \alpha^\varepsilon) + E\sigma^\varepsilon = 0, \qquad \alpha^\varepsilon(0, \cdot) = g. \quad (2.19)$$

Suppose $g \in L^1(R_0)$ and $\{h^\varepsilon\}$ satisfies (2.5) *and* (2.6). *Then $u^\varepsilon - \alpha^\varepsilon - \sigma^\varepsilon$ converges to zero in $C([0, T]: L^1(T_t))$ as $\varepsilon \to 0$. Furthermore, if $h^\varepsilon \to \mu_s$, a singular measure in the weak star sense, then $\alpha^\varepsilon \to \bar{u}$ in $C([0, T]: L^1(R_t))$ and $u^\varepsilon(t) \to \bar{u}(t) + \sigma(t)$ is weak star in $\mathcal{M}$, uniformly on $[0, T]$, where σ and $\bar{u}$ are the solutions of* (2.15) *and* (2.16).

Proof. Let $L \equiv \partial_t + A\partial_x + B$ and $v^\varepsilon = u^\varepsilon - \alpha^\varepsilon - \sigma^\varepsilon$. Then, suppressing the t, x dependence in f, we have

$$Lv^\varepsilon = f(\alpha^\varepsilon) - f(u^\varepsilon) + E\sigma^\varepsilon - L\sigma^\varepsilon.$$

Since $L\sigma^\varepsilon = E\sigma^\varepsilon$,

$$\begin{aligned} Lv^\varepsilon &= f(\alpha^\varepsilon) - f(v^\varepsilon + \alpha^\varepsilon + \sigma^\varepsilon) \\ &= \{f(\alpha^\varepsilon + \sigma^\varepsilon) - f(\alpha^\varepsilon + \sigma^\varepsilon + v^\varepsilon)\} + \{f(\alpha^\varepsilon) - f(\alpha^\varepsilon + \sigma^\varepsilon)\}, \end{aligned}$$

hence

$$\begin{gathered} |Lv^\varepsilon| \leqslant C|v^\varepsilon| + |f(\alpha^\varepsilon + \sigma^\varepsilon) - f(\alpha^\varepsilon)| \\ v^\varepsilon(0, \cdot) \equiv 0. \end{gathered} \quad (2.20)$$

As in the proof of Theorem 2.1, (2.20) and Gronwall's inequality suffice to prove that $v^\varepsilon \to 0$ in $C([0, T]: L^1(R_t))$ once we know that $|f(\alpha^\varepsilon + \sigma^\varepsilon) - f(\alpha^\varepsilon)|$ is $o(1)$ in $L^1(R)$. In that proof this was shown with α^ε replaced by a fixed element of $L^1(R)$ by using the lemma and the sublinearity of f. Using this and the fact that f is uniformly Lipschitz we see that it suffices to show that the family $\{\alpha^\varepsilon\}$ is totally bounded in $L^1(R)$. In fact, we will prove the stronger assertion that $\{\alpha^\varepsilon\}$ is precompact in $C([0, T]: L^1(R_t))$.

Define w^ε and ζ as solutions of the linear initial value problems

$$\begin{aligned}(\partial_t + A\partial_x + D)w^\varepsilon + E\sigma^\varepsilon &= 0 \qquad w^\varepsilon(0,\cdot) = g \\ (\partial_t + A\partial_x + D)\zeta &= 0 \qquad \zeta(0,\cdot) = g.\end{aligned} \tag{2.21}$$

Then $\zeta \in C([0, T]: L^1(R_t))$ and the difference satisfies

$$(\partial_t + A\partial_x + D)(w^\varepsilon - \zeta) + E\sigma^\varepsilon = 0$$

with vanishing initial data. Thus, the ith component $w_i^\varepsilon - \zeta_i$ is expressed as an integral of $\sum_{j\neq i} e_{ij}\sigma_j^\varepsilon$ over a backward i characteristic. At each point p of this characteristic, $\sigma_j^\varepsilon(p) = a_j(p)\, h_j^\varepsilon(\eta_j(p))$, where for each p and j, $\eta_j(p)$ denotes the intersection of the backward j characteristic from p with $\{t=0\}$, and a_j satisfies

$$(\partial_t + \lambda_j\, \partial_j + d_{jj})\, a_j = 0, \qquad a(0,\cdot) = 1.$$

Since A and B are continuous, $\{\lambda_i\}$, $\{e_{ij}\}$, $\{\eta_j\}$, and $\{a_j\}$ are continuous too. Thus, for each i, $w_i^\varepsilon - \zeta$ can be written

$$(w_i^\varepsilon - \zeta)(t, x) = \sum_{j\neq i} \int_{\eta_i(t,x)}^{\eta_j(t,x)} G_{ij}(t, x, y)\, h_j^\varepsilon(y)\, dy,$$

where the kernels, G_{ij}, are continuous on $R \times R_0$. Using the continuity and Fubini's theorem it is easy to check that, for each t, $\{w_i^\varepsilon - \zeta\}$ satisfy the condition of Riesz for precompactness in $L^1(R_t)$ [1]. The same calculation shows that $(w^\varepsilon - \zeta)(t)$ are uniformly bounded and equicontinuous on $[0, T]$ to $\{L^1(R_t)\}$. Thus the set $\{w^\varepsilon - \zeta\}$ is precompact in $C([0, T]$: $L^1(R_t))$. Since ζ is fixed, $\{w^\varepsilon\}$ is precompact in $C([0, T]: L^1(R_t))$, too.

Set $z^\varepsilon \equiv \alpha^\varepsilon - w^\varepsilon$. Then

$$Lz^\varepsilon + f(w^\varepsilon + z^\varepsilon) + Ew^\varepsilon = 0, \qquad z^\varepsilon(0,\cdot) = 0.$$

If we denote by $\mathscr{P}$ the map which takes w^ε to z^ε by solving this equation then $\mathscr{P}$ is continuous on $C([0, T]: L^1(R_t))$. So, $\{z^\varepsilon\} = \{\mathscr{P}(w^\varepsilon)\}$ is precompact in $C([0, T]: L^1(R_t))$. Thus $\{\alpha^\varepsilon\}$ is precompact which, as indicated above, completes the proof that $v^\varepsilon \to 0$ in $C([0, T]: L^1(R_t))$.

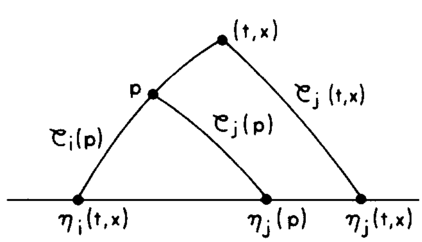

FIGURE 2

Now suppose that $h^\varepsilon \rightharpoonup \mu_s$ weak star. Since D is diagonal and continuous, the method of characteristics yields a simple proof that $\sigma^\varepsilon \rightharpoonup \sigma$ in the weak star sense. To complete the proof we must show that $\alpha^\varepsilon \to \bar{u}$ in $C([0, T]: L^1(R_t))$. We define w^ε, z^ε as above and w by (2.21) with σ replacing σ^ε. Thus

$$(\partial_t + A\partial_x + D)(w^\varepsilon - w) + E(\sigma^\varepsilon - \sigma) = 0, \qquad (w^\varepsilon - w)(0, \cdot) = 0.$$

Since $\sigma_\varepsilon \rightharpoonup \sigma$ in $\mathscr{E}'(R)$, it follows that $w^\varepsilon \rightharpoonup w$ in $\mathscr{E}'(R)$. The precompactness of $\{w^\varepsilon\}$ in $C([0, T]: L^1(R_t))$ implies that $w^\varepsilon \to w$ in $C([0, T]: L^1(R_t))$. Since $\mathscr{P}$ is continuous, $z^\varepsilon = \mathscr{P}(w^\varepsilon)$ converges to a $z \in C([0, T]: L^1(R_t))$. Thus $\alpha^\varepsilon = z^\varepsilon + w^\varepsilon$ converges to $z + w$ in $C([0, T]: L^1(R_t))$.

Finally,

$$L\alpha^\varepsilon = -f(z^\varepsilon + w^\varepsilon) - E\sigma^\varepsilon, \qquad \alpha^\varepsilon(0, \cdot) = g.$$

The three terms converge respectively to $L(z + w)$, $-f(z + w)$, and $-E\sigma$ in the sense of distributions. By uniqueness of solutions of (2.16) in $C([0, T]: L^1(R_t))$ we must have $z + w = \bar{u}$. ∎

For a strictly hyperbolic system

$$[\partial_t + A(t, x)\, \partial_x + B(t, x)]\, u + f(t, x, u) = 0, \tag{2.22}$$

where A is not in diagonal form we can choose a C^1 linear change of dependent variables $u' = V(t, x)u$ so that $V(t, x)\, A\, V(t, x)^{-1} \equiv \Lambda$ is diagonal. The matrix for the linear terms in the equation for u' is $B' \equiv VBV^{-1} + V(\partial_x V^{-1})$. We can now apply Theorem 2.2, splitting B' into diagonal and off-diagonal parts D' and E'. Transforming back we get the same statement for the general case (2.22) as we did in the diagonal case (Theorem 2.2) except that E is given by $V^{-1}E'V$ rather than as the off-diagonal part of B. The hypothesis that $A \in C^1(R)$ is used here to guarantee that B' is continuous.

3. Nonlinear Superposition: Ultrasingularities

In this section we allow the approximations h^ε to converge to distributions more singular than measures, so $\|h^\varepsilon\|_{L^1} \to \infty$ as $\varepsilon \to 0$. However, we still require $h^\varepsilon \in L^1$ for each ε to ensure the existence of solutions. We shall show that if f is uniformly bounded and h^ε tends to zero in measure, then nonlinear superposition holds as in Section 2. The following hypotheses are in force throughout this section:

Hypotheses on A, B, f, g, h^ε

We suppose that $A \in C^\infty(R)$, $B \in C(R)$, f is measurable on $R \times \mathbb{R}^k$, uniformly Lipschitzian in u and uniformly bounded, that is,

$$\sup_{R \times \mathbb{R}^k} |f| < \infty$$

$$\sup \frac{f(t, x, u) - f(t, x, u')}{|u - u'|} = l < \infty,$$

the second sup being taken over all t, x, u, u' in $R \times \mathbb{R}^k \times \mathbb{R}^k$ with $u \neq u'$. We suppose that g and h^ε are in $L^1(R_0)$ for each ε and $h^\varepsilon \to 0$ in measure, i.e., (2.6) is satisfied.

For diagonal systems we have the following analog of Theorem 2.1.

THEOREM 3.1. *If u^ε, σ^ε, $\bar{u}$ are defined by* (2.8), (2.9), *and* (2.10), *respectively, then $u^\varepsilon - \bar{u} - \sigma^\varepsilon \to 0$ in $C([0, T]: L^1(R_t))$. In addition, for any $\eta > 0$,*

$$\sup_{R \backslash \mathcal{S}^{\varepsilon,\eta}} |u^\varepsilon - \bar{u} - \sigma^\varepsilon| \leqslant c\eta \tag{3.1}$$

for ε sufficiently small. Two special cases are worthy of note:

(i) *If $h^\varepsilon \rightharpoonup v$ in $\mathscr{E}'(R_0)$, then $u^\varepsilon \rightharpoonup \bar{u} + \sigma$ in $\mathscr{E}'(R)$, where σ satisfies*

$$(\partial_t + A\partial_x)\sigma = 0 \qquad \sigma(0) = v.$$

(ii) *If h^ε satisfies the nested hypothesis* (2.7) *then $u^\varepsilon - \bar{u} - \sigma^\varepsilon$ converges uniformly to zero on $R \backslash \mathscr{T}^{\varepsilon_1}$ for any $\varepsilon_1 > 0$.*

EXAMPLE. If v is a distribution supported at finitely many points interior to R_0, then $h^\varepsilon = j_\varepsilon * v$ satisfies all the hypotheses.

The following simple lemma is the analog of the lemma in Section 2.

LEMMA. *Let X, μ be a finite measure space. Suppose that $f(x, v)$ is measurable on $X \times \mathbb{R}^k$, uniformly bounded, and uniformly Lipschitzian in v for $x \in X$. Let σ^ε and α^ε be $\mathbb{R}^k$ valued measurable functions on X such that σ^ε goes to zero in measure as $\varepsilon \to 0$. Then*

$$f(x, \alpha^\varepsilon + \sigma^\varepsilon) - f(x, \alpha^\varepsilon) \to 0 \qquad \textit{in} \quad L^1(X, \mu).$$

Proof. Let l be the Lipschitz constant for f and let $\eta > 0$ be given. Choose ε_0, so that for $\varepsilon < \varepsilon_0$, we have $\mu(G_\varepsilon) \leqslant \eta/4 \|f\|_\infty$, where

$$G_\varepsilon \equiv \{x \colon |\sigma^\varepsilon(x)| > \eta/2l\mu(X)\}.$$

Then

$$\int_{G_\varepsilon} |f(x, \alpha^\varepsilon + \sigma^\varepsilon) - f(x, \alpha^\varepsilon)| \, dM \leqslant 2 \|f\|_\infty \mu(G_\varepsilon) \leqslant \eta/2$$

$$\int_{X \setminus G_\varepsilon} |f(x, \alpha^\varepsilon + \sigma^\varepsilon) - f(x, \alpha^\varepsilon)| \, d\mu \leqslant \int_{X \setminus G_\varepsilon} l |\sigma^\varepsilon(x)| \, d\mu \leqslant \eta/2,$$

which proves the lemma. ∎

Proof of Theorem 3.1. Let $v^\varepsilon = u^\varepsilon - \bar{u} - \sigma^\varepsilon$. Then, as in the proof of Theorem 2.1,

$$\begin{gathered} |Lv^\varepsilon| \leqslant l |v^\varepsilon| + |f(t, x, \bar{u} + \sigma^\varepsilon) - f(t, x, \bar{u})| \\ v^\varepsilon(0, \cdot) = 0. \end{gathered} \tag{3.2}$$

The lemma shows that the second term on the right tends to zero in $L^1(R)$ so $v^\varepsilon \to 0$ in $C([0, T]: L^1(R_t))$. If $h^\varepsilon \rightharpoonup v$ in $\mathscr{E}'(R_0)$ then $\sigma^\varepsilon \rightharpoonup \sigma$ in $\mathscr{E}'(R)$ and therefore $u^\varepsilon \rightharpoonup \bar{u} + \sigma$ in $\mathscr{E}'(R)$.

To prove (3.1) observe that $|\sigma^\varepsilon| \leqslant c\eta$ on $R \setminus \mathscr{S}^{\varepsilon,\eta}$ so the second term on the right of (3.2) is dominated by

$$cl\eta + 2 \|f\|_\infty \chi_{\mathscr{S}^{\varepsilon,\eta}}.$$

For each $t \in [0, T]$, define

$$\gamma(t) = \sum_{i=1}^{k} \sup_{R_t \setminus \mathscr{S}_i^{\varepsilon,\eta}} |v_i|. \tag{3.3}$$

If $p \notin \mathscr{S}_j^{\varepsilon,\eta}$, then $\mathscr{C}_j(p) \cap \mathscr{S}_j^{\varepsilon,\eta} = \varnothing$ and, by strict hyperbolicity, $\mathscr{C}_j(p)$ intersects $\mathscr{S}_i^{\varepsilon,\eta}$ in a set of length $\leqslant cm\{|h^\varepsilon| > \eta\}$. For such p we have

$$\begin{aligned} |v_j^\varepsilon(p)| &\leqslant \left\{ \int_{\mathscr{C}_j(p) \cap \mathscr{S}^{\varepsilon,\eta}} + \int_{\mathscr{C}_j(p) \setminus \mathscr{S}^{\varepsilon,\eta}} \right\} |f_j(v^\varepsilon + \bar{u} + \sigma^\varepsilon) - f_j(\bar{u})| \\ &\leqslant cm\{|h^\varepsilon| > \eta\} + l \int_{\mathscr{C}_j \setminus \mathscr{S}^{\varepsilon,\eta}} |v^\varepsilon| \\ &\quad + \int_{\mathscr{C}_j \setminus \mathscr{S}^{\varepsilon,\eta}} |f_j(\bar{u} + \sigma^\varepsilon) - f_j(\bar{u})|. \end{aligned}$$

The last integrand is $\leqslant c\eta$ and the other integrand is dominated by γ. Thus, taking the sum on j we find

$$\gamma(t) \leqslant cm\{|h^\varepsilon| > \eta\} + \int_0^t \gamma(s) \, ds + c\eta.$$

As $\varepsilon \to 0$ the first term goes to zero so Gronwall's inequality yields (3.1). Case (ii) follows immediately from (2.7) and (3.1). ∎

When $B \neq 0$ we divide the analog of Theorem 2.2 into two parts.

THEOREM 3.2a. *Suppose that* u^ε, σ^ε, *and* α^ε *satisfy* (2.17), (2.18), *and* (2.19), *respectively. Then* $u^\varepsilon - \alpha^\varepsilon - \sigma^\varepsilon$ *converges to zero in* $C([0, T]: L^1(R_t))$ *and for* ε *sufficiently small*

$$\sup_{(t,x) \notin \mathscr{S}^{\varepsilon,\eta}} |u^\varepsilon - \alpha^\varepsilon - \sigma^\varepsilon| \leqslant c\eta. \tag{3.4}$$

The proof of this result is a straightforward combination of the proofs of Theorems 2.2 and 3.1 and is therefore omitted.

To obtain a more precise description of u^ε we must study α^ε more closely. The difficulty is that α^ε will, in general, be quite singular because of the forcing term $E\sigma^\varepsilon$ in the defining equation. We separate out a singular term β^ε, defined by

$$(\partial_t + \Lambda \partial_x + B)\, \beta^\varepsilon + E\sigma^\varepsilon = 0, \qquad \beta^\varepsilon(0) = 0$$

This leaves a good term $\gamma^\varepsilon \equiv \alpha^\varepsilon - \beta^\varepsilon$ which satisfies

$$(\partial_t + \Lambda \partial_x + B)\, \gamma^\varepsilon + f(t, x, \beta^\varepsilon + \gamma^\varepsilon) = 0, \qquad \gamma^\varepsilon(0) = g.$$

In addition to the hypotheses at the beginning of the section, we assume

(a) $B \in C^\infty(R)$

(b) $h^\varepsilon \to v$ in $H^{s_0}(R_0)$ for some $s_0 \in (-\infty, \infty)$.

(c) There is a closed set $T_0 \subset R_0$ of Lebesgue measure zero and $s_1 > 0$ so that:

$h^\varepsilon \to 0$ in $H^{s_1}_{\text{loc}}(R_0 \backslash T_0)$, uniformly on compact subsets of $R_0 \backslash T_0$.

We will use the notation $L \equiv \partial_t + \Lambda \partial_x$, $\partial_i \equiv \partial_t + \lambda_i \partial_x$, and denote by $\mathscr{T}$ the union of the flowouts of T_0 under the vector fields ∂_i.

THEOREM 3.2b. *Under the above assumptions*:

(i) $\sigma^\varepsilon \to \sigma$ *and* $\beta^\varepsilon \to \beta$ *in* $H^{s_0}(R)$, *where* σ *and* β *are defined by*

$$(L + D)\sigma = 0 \qquad \sigma(0) = v$$

$$(L + B)\beta + E\sigma = 0 \qquad \beta(0) = 0$$

(ii) $\sigma^\varepsilon \to \sigma$ *in* $H^{s_1}_{\text{loc}}(R \backslash \mathscr{T})$, *and uniformly on compact subsets of* $R \backslash \mathscr{T}$, *and* $\beta^\varepsilon \to \beta$ *in* $H^{s_1+1}_{\text{loc}}(R \backslash \mathscr{T})$

(iii) $\gamma^\varepsilon \to \gamma$ *in* $C([0, T]: L^1(R_t))$, *where* γ *satisfies*

$$(L+B)\gamma + f(t, x, \gamma + \chi_{R\backslash\mathscr{T}}\beta) = 0, \qquad \gamma(0) = g,$$

where $\chi\beta$ *means the function equal to* β *on* $R\backslash\mathscr{T}$ *and zero elsewhere.*

(iv) $\gamma^\varepsilon \to \gamma$ *uniformly on compact subsets of* $R\backslash\mathscr{T}$.

In particular, if $\bar{u} \equiv \beta + \gamma$, *then* $u^\varepsilon \to \bar{u} + \sigma$ *in* $H^{s_0}(R)$, $u^\varepsilon \to \bar{u} + \sigma$ *uniformly on compact subsets of* $R\backslash\mathscr{T}$, $\bar{u}$ *is continuous on* $R\backslash\mathscr{T}$ *and satisfies*

$$(L+B)\bar{u} + f(t, x, \chi_{R\backslash\mathscr{T}}\bar{u}) + E\sigma = 0, \qquad \bar{u}(0) = g.$$

Remark. Note that s_0 may be very negative and $\bar{u}$ may be very singular on $\mathscr{T}$.

Proof. (i) and the σ part of (ii) are elementary. The study of β^ε is more delicate and, for this purpose, we construct approximations $\sum_{n=0}^N w^{n,\varepsilon}$, where

$$(L+D)\, w^{0,\varepsilon} + E\sigma^\varepsilon = 0, \qquad w^{0,\varepsilon}(0) = 0$$

$$(L+D)\, w^{n,\varepsilon} + Ew^{n-1,\varepsilon} = 0, \qquad w^{n,\varepsilon}(0) = 0, \quad n \geqslant 1$$

with analogous definitions when $\varepsilon = 0$. Then

$$\begin{aligned} (L+B)\left(\sum_{n=0}^N w^{n,\varepsilon}\right) + E\sigma^\varepsilon &= Ew^{N,\varepsilon} \\ \beta^\varepsilon = \sum_{n=0}^N w^{n,\varepsilon} &- (L+B)^{-1}Ew^{N,\varepsilon}. \end{aligned} \tag{3.5}$$

By iterative arguments we shall show that for N large enough, $w^{N,\varepsilon}$ converges uniformly on R and that each $w^{n,\varepsilon}$ converges uniformly on compact subsets of $R\backslash\mathscr{T}$.

Since $\sigma_i^\varepsilon - \sigma_i \to 0$ in H^{s_0}, the microlocal elliptic regularity theorem applied to the equation $(\partial_i + d_{ii})(\sigma_i^\varepsilon - \sigma_i) = 0$ implies that

$$\sigma_i^\varepsilon - \sigma \to 0 \qquad \text{in } H^s(T^*R\backslash\text{char } \partial_i), \quad \forall s \in \mathbb{R}. \tag{3.6}$$

Now, $w_i^{0,\varepsilon} - w_i^0$ satisfies

$$\begin{aligned} (\partial_i + d_{ii})(w_i^{0,\varepsilon} - w_i^0) &= -\sum_{j\neq i} E_{ij}(\sigma_j^\varepsilon - \sigma_j) \\ w_i^{0,\varepsilon}(0) - w_i^0(0) &= 0. \end{aligned} \tag{3.7}$$

Since $j \neq i$ on the right-hand side and our system is strictly hyperbolic, (3.6) implies that the forcing term in (3.7) goes to zero in $H^s(T^*R \cap \text{char } \partial_i)$ for all $s \in \mathbb{R}$. Thus, Hörmander's theorem implies

$$w_i^{0,\varepsilon} - w_i^0 \to 0 \qquad \text{in} \quad H^s(T^*R \cap \text{char } \partial_i) \tag{3.8}$$

for all $s \in \mathbb{R}$. Since $\sigma^\varepsilon \to \sigma$ in H^{s_0},

$$w^{0,\varepsilon} - w^0 \to 0 \qquad \text{in} \quad H^{s_0+1}(R).$$

Now, the ith component of $w^{1,\varepsilon} - w^1$ satisfies

$$(\partial_i + d_{ii})(w_i^{1,\varepsilon} - w_i^1) = -\sum_{j \neq i} E_{ij}(w_j^{0,\varepsilon} - w_j^0)$$
$$w_i^{1,\varepsilon}(0) - w_i^1(0) = 0;$$

so by elliptic regularity and Hörmander's theorem

$$w_i^{1,\varepsilon} - w_i^1 \to 0 \qquad \text{in} \quad H^{s_0+1}(T^*R \cap \text{char}\, \partial_i)$$
$$w_i^{1,\varepsilon} - w_i^1 \to 0 \qquad \text{in} \quad H^{s_0+2}(T^*R \backslash \text{char}\, \partial_i)$$

for each i. From these statements and the equation for $w^{2,\varepsilon} - w^2$, it follows that

$$w^{2,\varepsilon} - w^2 \to 0 \qquad \text{in} \quad H^{s_0+2}(R).$$

Continuing inductively we find that for N even,

$$w^{N,\varepsilon} - w^N \to 0 \qquad \text{in} \quad H^{s_0+1+(N/2)}(R). \tag{3.9}$$

Now, suppose that $p \notin \mathscr{T}_i$; then $\mathscr{C}_i(p) \cap \mathscr{T}_i = \phi$ and $\mathscr{C}_i(p)$ intersects $\{t = 0\}$ at a point p_0 such that $h^\varepsilon - v \to 0$ in $H^{s_1}_{\text{loc}}(p_0)$. Thus, by Hörmander's theorem,

$$\sigma_i^\varepsilon - \sigma_i \to 0 \qquad \text{in} \quad H^{s_1}(r, \text{char}\, \partial_i),\ r \in \mathscr{C}_i(p).$$

Therefore, by elliptic regularity,

$$w_j^{0,\varepsilon} - w_j^0 \to 0 \qquad \text{in } H^{s_1+1}(r, \text{char}\, \partial_i),\ r \in \mathscr{C}_i(p),\ j \neq i.$$

Combined with (3.8), this implies that

$$w^{0,\varepsilon} - w^0 \to 0 \qquad \text{in } H^{s_1+1}(r, \text{char}\, \partial_i),\ r \in \mathscr{C}_i(p). \tag{3.10}$$

If $p \notin \mathscr{T} = \bigcup \mathscr{T}_i$ then (3.10) holds for each i. The elliptic directions are handled easily from the equation for $w^{0,\varepsilon} - w^0$, so we conclude

$$w^{0,\varepsilon} - w^0 \to 0 \qquad \text{in } H^{s_1+1}(R \backslash \mathscr{T}).$$

Suppose again that $p \notin \mathscr{T}_i$. Then the equations for the components of $w^{1,\varepsilon} - w^1$ and (3.10) imply

$$w^{1,\varepsilon} - w^1 \to 0 \qquad \text{in} \quad H^{s_1+2}(r, \text{char}\, \partial_i),\ r \in \mathscr{C}_i(p)$$

by elliptic regularity and Hörmander's theorem. For $p \notin \bigcup \mathscr{T}_i$, this implies, as above,

$$w^{1,\varepsilon} - w^1 \to 0 \qquad \text{in } H^{s_1+2}(R\backslash\mathscr{T}).$$

Continuing by induction, we find

$$w^{n,\varepsilon} - w^n \to 0 \qquad \text{in } H^{s_1+n+1}(R\backslash\mathscr{T}). \tag{3.11}$$

If we choose N even so that $s_0 + 1 + N/2 > s_1 + 1$, then (3.9) and (3.11) give the conclusion of part (ii) using formula (3.5).

Proof of Part (iii). The existence of a unique $\gamma \in C([0, T]: L^1(R_t))$ satisfying the initial value problem in (iii) is not hard to prove. Then

$$\begin{aligned} |(L+B)(\gamma^\varepsilon - \gamma)| &= |f(\gamma^\varepsilon + \beta^\varepsilon) - f(\gamma + \chi_{R\backslash\mathscr{T}}\beta)| \\ &\leqslant c|\gamma^\varepsilon - \gamma| + |f(\gamma + \beta^\varepsilon) - f(\gamma + \chi_{R\backslash\mathscr{T}}\beta)|. \end{aligned} \tag{3.12}$$

By the Lebesgue dominated convergence theorem the second term tends to zero in $L^1(R)$, since $f \in L^\infty$ and $\mathscr{T}$ has measure zero. (iii) follows by the usual application of Gronwall's inequality.

Proof of Part (iv). Let U be an open set satisfying $T \subset U \subset R_0$ and let $\mathscr{U}$ denote the union of the flowouts of U under the vector fields ∂_i. The second term in (3.12) is estimated by $c|\beta^\varepsilon - \beta|$ on $R\backslash\mathscr{U}$ and by $2\|f\|_\infty$ on $\mathscr{U}$. Thus

$$|(L+B)(\gamma^\varepsilon - \gamma)| \leqslant c|\gamma^\varepsilon - \gamma| + 2\|f\|_\infty \chi_{\mathscr{U}} + c|\beta^\varepsilon - \beta| \chi_{R\backslash\mathscr{U}}.$$

The first and last terms converge uniformly to zero. If $p \in \mathscr{U}$, then, for each i, $\mathscr{C}_i(p)$ intersects $\mathscr{U}$ in a set of length dominated by $cm(U)$. If we define

$$M(t) = \sum_i \sup_{R_t\backslash\mathscr{U}} |\gamma_i^\varepsilon(t) - \gamma_i(t)|$$

then the method of characteristics yields

$$M(t) \leqslant c\int_0^t M(s)\,ds + cm(U) + o(1);$$

hence,

$$\overline{\lim_{\varepsilon\to 0}}\, \sup_{R\backslash\mathscr{U}} |\gamma^\varepsilon - \gamma| \leqslant cm(U).$$

Given any compact $K \subset R \backslash \mathscr{T}$ we can choose an open $U \supset T$ with measure as small as we like so that $K \subset R \backslash \mathscr{U}$. Thus (iv) is proved.

4. Superlinear Dissipation

Before studying general systems we note that the lemma of Section 2 yields results for the scalar equation

$$u_t = f(u). \tag{4.1}$$

Let $\varphi(t, s)$ be the flow of the corresponding ordinary differential equation,

$$\frac{\partial \varphi}{\partial t} = f(\varphi), \qquad \varphi(0, s) = s, \tag{4.2}$$

where, as usual, f is assumed Lipschitz. The hypothesis of superlinear dissipation in the scalar case is

$$\lim_{|u| \to \infty} f(u)/u = -\infty. \tag{4.3}$$

It follows that the initial value problem for (4.2) is globally solvable in $t > 0$. In addition φ is sublinear in s, that is,

$$\lim_{|s| \to \infty} \varphi(t, s)/s = 0 \tag{4.4}$$

uniformly in $t \geqslant \bar{t}$ for any $\bar{t} > 0$. As we prove more general results later, the proof of (4.4) is omitted. Thus the initial value problem (4.1) with L^1 data has a global solution in $t > 0$ and we have an easy proof of the following proposition which shows the effect of superlinear dissipation in the scalar case.

Proposition. *Suppose that f is Lipschitz and satisfies* (4.3). *Let $g \in L^1(\mathbb{R})$ and $\{h^\varepsilon\}$ satisfy hypotheses* (2.5) *and* (2.6). *If u^ε is the solution of* (4.1) *with data $u^\varepsilon(0, \cdot) = g + h^\varepsilon$ and $\bar{u}$ is the solution of* (4.1) *with $\bar{u}(0, \cdot) = g$, then for any compact subset K of the open half plane $\{t > 0\}$ we have*

$$u^\varepsilon \to \bar{u} \qquad \text{in } L^1(K) \text{ as } \varepsilon \to 0.$$

Proof. The difference $u^\varepsilon - \bar{u}$ is equal to $\varphi(t, g(x) + h^\varepsilon(x)) - \varphi(t, g(x))$. Since φ is sublinear in the second variable the lemma in Section 2 applies and gives the result. ∎

We turn now to systems which we assume are in the characteristic form

$$(\partial_t + A\partial_x)u = f(t, x, u). \tag{4.5}$$

We have lumped the linear terms with f because the hypotheses on f (see below) are insensitive to their presence. We assume A, $D_{x,t}A \in L^\infty(R)$.

Hypotheses on f

We assume that $f(t, x, u)$ is measurable on $R \times \mathbb{R}^k$ and that

$$\frac{\partial f}{\partial u} \in L^\infty(K) \qquad \text{for any compact } K \subset R \times \mathbb{R}^k. \tag{4.6}$$

In order to prevent blowup we assume that there is a constant c so that

$$\operatorname{sgn}(u_j)\, f_j(t, x, u) \leqslant c\left(1 + \sum_{i=1}^{k} |u_i|\right) \tag{4.7}$$

holds for all t, x, u, j. One then has the differential inequality

$$(\partial_t + \lambda_j\, \partial_x)(|u_j|) \leqslant c(1 + \Sigma |u_i|). \tag{4.8}$$

It follows that

$$\sum_{i=1}^{k} \sup_{x \in R_t} |u_i(t, x)| \leqslant c'\left(1 + \sum_{i=1}^{k} \sup_{x \in R_0} |u_i(0, x)|\right) \tag{4.9}$$

for all $0 \leqslant t \leqslant T$, so if the data are in $L^\infty(R_0)$ the solution is in $L^\infty(R)$. Actually, slightly more is true. The solution is in $C([0, T]\colon L^\infty(R_t))$, continuity being in the weak star sense on L^∞. We refer to (4.7) as the non-explosive hypothesis.

The hypothesis of superlinear dissipation is

$$\lim_{|u_j| \to \infty} f_j(t, x, u)/u_j = -\infty \tag{4.10}$$

the limit holding uniformly for $(t, x, u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_k)$ in each set $R \times K$ where K is compact in $\mathbb{R}^{k-1}$.

Notice that the Lipschitz hypothesis (4.6) is only required to hold on compact subsets of u and the dissipative hypothesis for each j is only required to hold for compact subsets of u_i, $i \neq j$. A sample f satisfying these hypotheses is

$$f_j(u) = \sum_{i=1}^{k} a_{ij} u_i - u_j\left(1 + \sum_{i=1}^{k} u_i^2\right).$$

THEOREM 4.1. *Suppose that Λ and f satisfy the above hypotheses, that $g \in L^\infty(R_0)$, and that h^ε satisfies* (2.5), (2.6), *and* (2.7). *Define u^ε and $\bar{u}$ by*

$$(\partial_t + \Lambda\partial_x) u_\varepsilon = f(t, x, u^\varepsilon), \qquad u^\varepsilon(0, x) = g + h^\varepsilon$$

$$(\partial_t + \Lambda\partial_x)\bar{u} = f(t, x, \bar{u}), \qquad \bar{u}(0, x) = g.$$

Then $u^\varepsilon(t) \to \bar{u}(t)$ in $L^1(R_t)$ for each $0 < t \leqslant T$, uniformly for $t \in [\bar{t}, T]$ for any $\bar{t} > 0$, and if K is a compact set in $R \backslash \mathscr{T}$ and $\eta > 0$ is given, then for ε small enough:

$$|u^\varepsilon(t, x) - \bar{u}(t, x)| \leqslant \eta, \qquad (t, x) \in K. \tag{4.16}$$

Remark. This theorem recovers in the case of systems, the two phenomena which we saw in the simple explicit example in the Introduction, namely, L^1 convergence for $t > 0$ and uniform convergence away from the flow out of the singular set.

Proof of the Theorem. Let $v^\varepsilon = u^\varepsilon - \bar{u}$ and $L = \partial_t + \Lambda\partial_x$. Then

$$Lv^\varepsilon = f(t, x, \bar{u} + v^\varepsilon) - f(t, x, \bar{u}) \equiv \tilde{f}(t, x, v^\varepsilon)$$

$$v^\varepsilon(0, x) = h^\varepsilon.$$

Since $\bar{u}$ is bounded in R, $\tilde{f}$ satisfies the non-explosive hypothesis (4.7) and the superlinear dissipative condition (4.10). Furthermore $\tilde{f}$ satisfies (4.6) and $\tilde{f}(t, x, 0) = 0$. We have thus reduced to the case where $g \equiv 0$ and $f(t, x, 0) = 0$. Let a small $\eta > 0$ and a small time $\bar{t} > 0$ be given and let $S^{\varepsilon,\eta}$, $\mathscr{S}_j^{\varepsilon,\eta}$, $\mathscr{S}_j^{\varepsilon,\eta}(t)$, $\mathscr{S}^{\varepsilon,\eta}$ be as defined in (2.6), (2.7). Then for ε small:

$$m(\mathscr{S}^{\varepsilon,\eta}(t)) \leqslant c\eta, \qquad m(\mathscr{S}_j^{\varepsilon,\eta}) \leqslant c\eta \tag{4.17}$$

where c is independent of η. Through a series of estimates we will show that (4.16) and

$$\int_{\mathscr{S}^{\varepsilon,\eta}(t)} |v^\varepsilon(x, t)|\, dx \leqslant c\eta \qquad \text{if} \quad t > \bar{t} \tag{4.18}$$

hold for ε small enough. This immediately implies the first part of the theorem. For the second part we simply note that for each fixed ε_1 and η we have $\mathscr{S}^{\varepsilon,\eta} \subset \mathscr{T}^{\varepsilon_1}$ for ε small enough, so (4.16) gives the result.

Throughout the following arguments it is useful to think of the special case where $S^{\varepsilon,\eta}$ is a small interval (see Fig. 3). We already know from the non-explosive hypothesis that $|v_j^\varepsilon| \leqslant cM_\varepsilon$, where M_ε is the sup of the initial data. The proof proceeds by improving this estimate, using the hypotheses on $\{h^\varepsilon\}$, the superlinear dissipation, and the geometry of the characteristics. Since η is fixed, we write simply S^ε instead of $S^{\varepsilon,\eta}$. In the arguments that follow the constants c do not depend on η or ε.

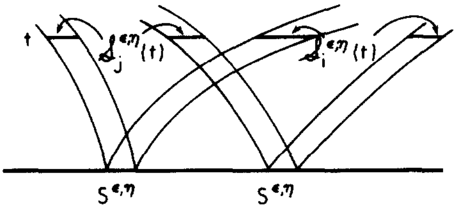

FIGURE 3

First, we note that from the non-explosive hypothesis (4.7) it follows that $v^\varepsilon \in C([0, T]: L^1(R_t))$ and $\|v^\varepsilon\|_{L^1(R_t)} \leqslant ce^{ct}$. Thus,

$$\|v^\varepsilon\|_{L^1(R)} \leqslant c. \tag{4.19}$$

If we multiply the ith equation by $\operatorname{sgn}(v_i)$, we have by (4.7),

$$\begin{aligned}(\partial_t + \lambda_i \partial_x)|v_i^\varepsilon| &= \operatorname{sgn}(v_i^\varepsilon)\, \mathscr{f}_i(v^\varepsilon)\\ &\leqslant c\,|v^\varepsilon|,\end{aligned} \tag{4.20}$$

since $\mathscr{f}_i(0) = 0$. Integrating this relation over the triangular region bounded by the axis $\{t = 0\}$ and the backward i and j characteristics yields (see Fig. 4)

$$\|v_i^\varepsilon\|_{L^1(\mathscr{C}_j)} - \|v_i^\varepsilon\|_{L^1(I_0)} \leqslant \|v^\varepsilon\|_{L^1} \leqslant c. \tag{4.21}$$

Thus,

$$\int_{\mathscr{C}_j} |v_i^\varepsilon| \leqslant c \qquad \text{for all} \quad i \neq j, \tag{4.22}$$

where $\mathscr{C}_j$ is a j characteristic in R.

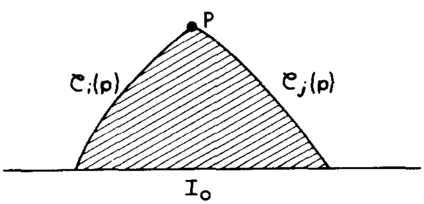

FIGURE 4

Second, using (4.20) and (4.7),

$$|v_j^\varepsilon(p)| = |v_j^\varepsilon(q)| + \int_{\mathscr{C}_j} \operatorname{sgn}(v_j)\, \tilde{f}_j(v^\varepsilon)$$

$$\leqslant |v_j^\varepsilon(q)| + \int_{\mathscr{C}_j} c\,|v^\varepsilon|$$

$$\leqslant |v_j^\varepsilon(q)| + c + c\int_{\mathscr{C}_j} |v_j^\varepsilon| \tag{4.23}$$

because of (4.22). If $p \notin \mathscr{S}_j^\varepsilon$ then $|v_j^\varepsilon(q)| \leqslant \eta$, so by Gronwall's inequality

$$|v_j^\varepsilon(p)| \leqslant c \qquad \text{for all } p \notin \mathscr{S}_j^\varepsilon \text{ for each } j. \tag{4.24}$$

Now we use the superlinear dissipation to control the values of v_j^ε in $\mathscr{S}_j^\varepsilon$. Let $q \in S^\varepsilon$. There are two cases. First, suppose that there is a point r on $\mathscr{C}_j(q)$ so that

$$|v_j^\varepsilon(r)| \leqslant \eta/m(T^\varepsilon).$$

Then, applying Gronwall's inequality to (4.23) (with r replacing q), we see that

$$|v_j^\varepsilon(p)| \leqslant \frac{\eta}{m(T^\varepsilon)}\, c + c \qquad \forall p \in \mathscr{C}_j(q). \tag{4.25}$$

The other case is where

$$|v_j^\varepsilon(r)| \geqslant \eta/m(T^\varepsilon) \qquad \text{for all} \quad r \in \mathscr{C}_j(q). \tag{4.26}$$

We will assume v_j^ε is large and positive. The other case is handled similarly. Let $B = \bigcup_{i \neq j} \mathscr{S}_i^\varepsilon$. Then $m(B \cap \mathscr{C}_j) \to 0$ as $\varepsilon \to 0$ and for $p \in \mathscr{C}_j \backslash B$, the other components $v_i^\varepsilon(p)$ take values in a compact set (by (4.24) for $i \neq j$). Choose L so that $e^{-L/2} \leqslant \eta$. Then choose ε small enough that (4.26) implies

$$\frac{\tilde{f}_j(t, x, v^\varepsilon(p))}{v_j^\varepsilon(p)} \leqslant -L \tag{4.27}$$

by the superlinear dissipation hypothesis (4.10). Thus,

$$v_j^\varepsilon(p) = v_j^\varepsilon(q) + \int_{\mathscr{C}_j(q) \cap B} \tilde{f}_j(v^\varepsilon) + \int_{\mathscr{C}_j(q) \backslash B} \tilde{f}_j(v^\varepsilon)$$

$$\begin{aligned}|v_j^\varepsilon(p)| &\leqslant |v_j^\varepsilon(q)| + \int_{\mathscr{C}_j(q)\cap B} c|v^\varepsilon| - L\int_{\mathscr{C}_j(q)\setminus B} |v_j^\varepsilon| \\ &\leqslant |v^\varepsilon(q)| + c + c\int_{\mathscr{C}_j(q)\cap B} |v_j^\varepsilon| - L\int_{\mathscr{C}_j(q)\setminus B} |v_j^\varepsilon| \\ &\leqslant |v_j^\varepsilon(q)| + c + \int_{\mathscr{C}_j(p,q)} (c\chi_B - L\chi_{\mathscr{C}_j(q)\setminus B})\,|v_j^\varepsilon|.\end{aligned}$$

For ε small, $m(B\cap\mathscr{C}_j)\leqslant \bar{t}/2$. This implies

$$\int_{\mathscr{C}_j(p,q)} c\chi_B - L\chi_{\mathscr{C}_j\setminus B} \leqslant -L/2$$

if $m(\mathscr{C}_j(p,q))\geqslant \bar{t}$. So, by Gronwall's inequality,

$$|v_j^\varepsilon(p)| \leqslant (|v_j^\varepsilon(q)| + c)\,e^{-L/2} \tag{4.28}$$

for such p. Now choose ε_1 small enough so that $m(T^{\varepsilon_1})<\eta$, $\varepsilon_1<\eta$, $\max_{i,q}\{m(B\cap\mathscr{C}_i(q))\}\leqslant \bar{t}/2$, and (4.26) implies (4.27) by the superlinear dissipation hypothesis for all $\varepsilon\leqslant\varepsilon_1$. Then for all ε small enough so that $S^\varepsilon\subset T^{\varepsilon_1}$ and all $p\in S_j^\varepsilon$ such that the time at p is $\geqslant\bar{t}$ we have either

$$|v_j^\varepsilon(p)| \leqslant \frac{\eta}{m(T^\varepsilon)}\,c + c$$

or $\qquad$ (4.29)

$$|v_j^\varepsilon(p)| \leqslant \eta(h_j^\varepsilon(q) + c).$$

Since the L^1 norms of h^ε are uniformly bounded, this proves (4.18). In fact it proves a little more; namely, if $\mathscr{C}_i$ is any piece of a characteristic curve of another family in the region $t>\bar{t}$ then

$$\int_{\mathscr{C}_i\cap\mathscr{S}_j^\varepsilon} |v_j^\varepsilon| \leqslant c\eta. \tag{4.30}$$

Now, suppose $K\subset R\setminus\mathscr{T}$ is compact. Let K_j denote the intersection of the flowbacks of K under $\partial_t+\lambda_i\,\partial_x$ with the line $\{t=0\}$. Since $\bigcup K_j$ and T are compact and $\bigcup K_j\cap T=\varnothing$, hypothesis (2.7) implies that we can choose $\bar{\varepsilon}$ so that $\bigcup K_j\cap T^{\bar{\varepsilon}}=\varnothing$. Choose an open set $O\supset T^{\bar{\varepsilon}}$ so that $\bigcup K_j\cap O=\varnothing$ and $m(O\setminus T^{\bar{\varepsilon}})\leqslant\eta$ and set $K^O=I_0\setminus O$. Let $\mathscr{K}^0$ be the intersection of the flow forwards of $\mathscr{K}^O$ in R. Then $\mathscr{K}^O\supset K$ and $R=\mathscr{K}^O\cup\mathscr{O}$, where $\mathscr{O}$ is the union of the flow forwards of O. Define $\gamma(t)$ by

$$\gamma(t)=\sum_l \sup_{R_l\setminus\mathscr{O}_l} |v_l(t,x)|.$$

If $p \notin \mathcal{O}_j$ and $q \in \{t=0\}$, then $\mathcal{C}_j(p, q)$ can intersect $\mathcal{O}_i$ for $i \neq j$ but not $\mathcal{O}_j$. The non-explosive hypothesis yields

$$|v_j^\varepsilon(p)| \leqslant |v_j^\varepsilon(q)| + c \int_{\mathcal{C}_j(p,q)} \sum |v_l^\varepsilon|$$
$$\leqslant \eta + c \left\{ \int_{\mathcal{C}_j} |v_j^\varepsilon| + \sum_{l \neq j} \int_{\mathcal{C}_j \setminus \mathcal{O}_l} |v_l^\varepsilon| \right\} + \sum_{l \neq j} \int_{\mathcal{C}_j \cap \mathcal{O}_l} |v_l^\varepsilon|$$
$$\leqslant \eta + c \int_0^t \gamma(s)\, ds + \sum_{l \neq j} \int_{\mathcal{C}_j \cap \mathcal{O}_l} |v_l^\varepsilon|.$$

Now, choose $\bar{t} > 0$ so that the forward characteristics from K^O do not meet $\mathcal{T}^{\bar{\varepsilon}}$ until after $\bar{t}$. Since the $\{T_\varepsilon\}$ are nested, this will also be true for all $\varepsilon < \bar{\varepsilon}$. Thus if ε is sufficiently small

$$\int_{\mathcal{C}_j \cap \mathcal{T}_l^\varepsilon} |v_l^\varepsilon| \leqslant c\eta$$

by (4.30) and

$$\int_{\mathcal{C}_j \cap (\mathcal{O}_l \setminus \mathcal{T}_l^\varepsilon)} |v_l^\varepsilon| \leqslant c\eta,$$

since $m(\mathcal{C}_j \cap (\mathcal{O}_l \setminus \mathcal{T}_l^\varepsilon)) \leqslant c\eta$ and $|v_l^\varepsilon|$ is uniformly bounded outside of $\mathcal{T}_l^\varepsilon$ by (4.24). Thus

$$\gamma(t) \leqslant \eta(1+c) + c \int_0^t \gamma(s)\, ds,$$

which implies $\gamma(t) \leqslant c\eta$ for $t \in [0, T]$ by Gronwall's inequality. This proves (4.16). To conclude the proof of L^1 convergence we note

$$\int_{R_t} |v^\varepsilon| \leqslant \int_{R_t \cap \mathcal{K}^0} |v^\varepsilon| + \sum_{l=1} \int_{R_t \cap \mathcal{T}_l^\varepsilon} |v_l^\varepsilon| + \sum_{l=1} \int_{R_t \cap (\mathcal{O}_l \setminus \mathcal{T}_l^\varepsilon)} |v_l^\varepsilon|.$$

For ε small, the first term is less than $c\eta$ by (4.16), the second term is less than $c\eta$ for $t > \bar{t}$ by (4.30), and the third term is less than $c\eta$ by (4.24) and the fact that the measure of $\mathcal{O}_l \setminus \mathcal{T}_l^\varepsilon$ is small. ▌

It may seem at first that in the presence of the condition of superlinear dissipation one should not need the non-explosive hypothesis (4.10) at all. But actually (4.10) is not a strong hypothesis since it is only required to hold for large u_j if the other u_i are in a compact set. Nothing is said in the hypothesis about what happens if all the variables are big. The following example ($\Box u + 8u_t^3 = 0$ written as a system) makes this situation clear:

$$(\partial_t + \partial_x)w = -(w+v)^3$$
$$(\partial_t - \partial_x)v = -(w+v)^3.$$

The non-linearity is easily seen to satisfy the superlinear dissipation hypotheses but it does not satisfy the non-explosive condition. Note that $\partial_t(w-v) + \partial_x(v+w) = 0$ so $\int (w-v)\,dx$ is a conserved quantity. Therefore if $v^\varepsilon(0,x) \equiv 0$ and $w^\varepsilon(0,x) = j_\varepsilon(x)$ we will have $\int w^\varepsilon(t,x) - v^\varepsilon(t,x)\,dx = 1$, so we cannot have $(w^\varepsilon, v^\varepsilon) \to 0$ in L^1 for $t>0$. This shows that superlinear dissipation alone is not sufficient to imply the conclusion $u^\varepsilon \to \bar{u}$ in Theorem 4.1.

Unfortunately, for a strictly hyperbolic system not in the characteristic form, the invariant form of our hypotheses is somewhat awkward. If $\pi_i(t,x)$ are the spectral projections of $A(t,x)$, then the non-explosive hypothesis becomes

$$\operatorname{sgn}(\pi_j u)(\pi_j f) \leqslant c(1 + |u|)$$

and the superlinear dissipation hypothesis is

$$\lim_{|\pi_j u| \to 0} \pi_j f(t,x,u)/\pi_j u = -\infty,$$

when $(I - \pi_j)u$ remains in a compact set. Note that by strict hyperbolicity the π_j have rank one so these conditions make sense.

References

1. N. Dunford and J. T. Schwartz, "Linear Operators," Pt. 1, p. 298, Interscience, New York, 1963.
2. M. Oberguggenberger, Weak limits of solutions of semilinear hyperbolic systems, *Math. Ann.* **274** (1986), 599–607.
3. J. Rauch and M. Reed, Propagation of singularities for semilinear hyperbolic equations in one space variable, *Ann. of Math.* **111** (1980), 531–532.
4. J. Rauch and M. Reed, Jump discontinuities of semilinear, strictly hyperbolic systems in two variables: Creation and propagation, *Comm. Math. Phys.* **81** (1981), 203–207.
5. J. Rauch and M. Reed, Nonlinear microlocal analysis of semi-linear hyperbolic systems in one space dimension, *Duke Math. J.* **49** (1982), 397–476.
6. E. Stein, "Singular Integrals and Differentiability Properties of Functions," Princeton Univ. Press, Princeton, NJ, 1970.
7. M. Taylor, "Pseudodifferential Operators," Princeton Univ. Press, Princeton, NJ 1981.