THERE MAY BE SIMPLE $P_{\kappa^+}$- and $P_{\kappa^+}$-POINTS 
AND THE RUDIN-KEISLER ORDERING MAY 
BE DOWNWARD DIRECTED

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We prove the consistency, relative to ZFC, of each of the following two (mutually contradictory) statements. (A) Every two non-principal ultrafilters on $\omega$ have a common image via a finite-to-one function. (B) Simple $P_{\kappa^+}$-points and simple $P_{\kappa^+}$-points both exist. These results, proved by the second author, answer questions of the first author and P. Nyikos, who had obtained numerous consequences of (A) and (B), respectively. In the models we construct, the bounding number is $\kappa_1$, while the dominating number, the splitting number, and the cardinality of the continuum are $\kappa_2$.

1. Introduction

The purpose of this paper is to prove the consistency, relative to ZFC, of each of the following two statements.

(A) If $\mathcal{U}_1$ and $\mathcal{U}_2$ are non-principal ultrafilters on $\omega$, then $f(\mathcal{U}_1) = f(\mathcal{U}_2)$ for some finite-to-one $f : \omega \rightarrow \omega$.

(B) There exist both a simple $P_{\kappa^+}$-point and a simple $P_{\kappa^+}$-point.

For any regular uncountable $\kappa$, a simple $P_{\kappa}$-point is an ultrafilter (by which we always mean a non-principal ultrafilter on $\omega$) generated by an almost decreasing (i.e. decreasing modulo finite sets) $\kappa$-sequence of subsets of $\omega$. Clearly, every simple $P_{\kappa}$-point $\mathcal{U}$ is a $P$-point, i.e., for any countably many sets $A_n \in \mathcal{U}$, there is a set in $\mathcal{U}$ almost included in every $A_n$.

It is easy to check that, if $\mathcal{U}$ is a simple $P_{\kappa}$-point, then so is $f(\mathcal{U})$ for any $f : \omega \rightarrow \omega$. It is also easy to check that no ultrafilter can be a simple $P_{\kappa}$-point for two different values of $\kappa$. Thus, the models for (A) and for (B) cannot be the same. Nevertheless, the ideas involved in the two consistency proofs are very similar.

To discuss the origin of the statements (A) and (B) as well as the nature of our models for them, we must recall the definitions of three of the cardinal invariants
of the continuum discussed in [11]: the dominating number, the bounding number, and the splitting number. A family $\mathcal{F}$ of functions $\omega \to \omega$ is dominating (resp. unbounded) if, for each $g : \omega \to \omega$ there exists an $f \in \mathcal{F}$ such that $g(n) < f(n)$ for all but finitely many (resp. for infinitely many) $n$. A family $\mathcal{S}$ of subsets of $\omega$ is splitting if, for each infinite $A \subseteq \omega$, there exists an $S \in \mathcal{S}$ such that both $A \cap S$ and $A - S$ are infinite. The dominating, bounding, and splitting numbers are the smallest cardinalities, $d$, $b$, and $s$, of dominating, unbounded, and splitting families. Both $b$ and $s$ are uncountable and $d \leq \text{cf}(\mathcal{P}(\omega))$ (see [11]), and it was shown in [8] that the relative order of $b$ and $s$ is independent of ZFC. The models constructed in this paper satisfy $b < s$ and thus provide an alternate proof for one of the main results of [8]. Nyikos has shown that $b < s$ is implied by (B); that the same inequality also holds in our model of (A) seems to be due merely to the similarity of the constructions of our two models.

The statement (A), or rather its negation, arose in the work of Blass and Weiss [2] on decompositions of ideals of Hilbert space operators. It was shown there that, if one could refute (A) in ZFC, then one could also eliminate the continuum hypothesis from the main theorem of that paper. Interest in (A) increased when van Douwen [10] pointed out that the negation of (A) was sufficient to prove, as Rudin [6] had done using the continuum hypothesis, that the indecomposable continuum $\beta([0, 1]) - [0, 1)$ has more than one composant. Since then, Blass [1] has proved the converses of these results (the latter was also obtained by Mioduszewski [3]) and several other equivalences involving (A). One of the equivalent formulations of (A) is that every non-principal ultrafilter on $\omega$ has an image, via a finite-to-one function, generated by fewer than $d$ sets. The consistency of $u < d$, where $u$ is the minimal number of sets that can generate an ultrafilter, was proved, using a different model, by the second author shortly before finding the proof presented here. That model is, for the study of $u$ and $d$, preferable to the present one in that it allows $u$ and $d$ to be prescribed cardinals. On the other hand, it will not do for our present purpose, as M. Canjar has pointed out that it does not satisfy (A). The consistency of $u < d$ contrasts with Solomon's theorem [9] that no non-principal ultrafilter on $\omega$ is generated by fewer than $b$ sets. In particular, (A) implies $b < d$.

Statement (B), or rather the statement that simple $P_\kappa$-points exist for two distinct cardinals $\kappa$, is the strongest of several statements whose consistency was asked about by Nyikos [4]. The question arises naturally from his theorem that, if a simple $P_\kappa$-point exists, then $\kappa = b$ or $\kappa = d$. Nyikos [5] has deduced from (B) or from weaker statements numerous consequences concerning the eventual-domination ordering on the set of functions $\omega \to \omega$.

The plan of this paper is as follows. In Section 2, we introduce a notion of forcing $Q$ and establish some of its combinatorial properties, including properness and a partition theorem. In Section 3, we study the generic extension produced by forcing with $Q$. In particular, we show that, if there is a $P$-point in the ground model, then it generates a $P$-point in the extension and every two ultrafilters in
the ground model have a common image via a finite-to-one function in the
extension. In Section 4, we establish that countable-support iteration of proper
forcing preserves the property, "P-points in the ground model generate P-points
in the extension", that was previously established for Q. Using this, we show in
Section 5 that an $\aleph_2$-length countable-support iteration of Q forcing, over a
ground model satisfying CH, yields a model of (A). Finally, in Section 6, we show
how to modify the construction so as to produce a model of (B) instead.

Our notation is standard, except for the following points. Names in forcing
languages are often denoted by boldface symbols, like $x$. The corresponding
lightface symbol then denotes the value of this name with respect to some generic
subset of the forcing notion. When necessary, we shall be more explicit, writing
$x^{(G)}$ for the value of $x$ with respect to the generic set $G$. This notation will also be
used for partial evaluation of names in the context of iterated forcing. Thus, if $x$
is a $P \ast Q$-name and $G$ is a $V$-generic subset of $P$, then $x^{(G)}$ is the $Q$-name such
that, for all $V[G]$-generic $H \subseteq Q$, $x^{(G)[H]} = x^{(G \ast H)}$.

We often let $\chi$ be an unspecified regular cardinal so large that $H(\chi)$, the
collection of sets of hereditary cardinality $<\chi$, contains all sets of interest to us.
In this situation, we often consider (countable) elementary submodels $N$ of $H(\chi)$,
by which we mean that $(N, \in)$ is an elementary submodel of $(H(\chi), \in)$. For all
practical purposes, one can think of $H(\chi)$ as being the whole universe $V$; the
reason for introducing $H(\chi)$ is that the concept of elementary submodel of $V$
cannot be formalized in the usual language of set theory.

The theorems proved in this paper are due to the second author. The first
author's contribution was to fill in some details, to ask the second author to fill in
other details, and to write the paper.

2. The basic forcing

We shall construct a model for (A) by iterating, $\aleph_2$ times, with countable
support, a certain forcing $Q$, which we introduce in this section. The essential
properties of $Q$ are (a) that it is proper, (b) that P-points in the ground model
generate P-points in the generic extension, and (c) that it adjoins a finite-to-one
$f: \omega \rightarrow \omega$ such that all ultrafilters in the ground model have the same $f$-image,
provided a P-point exists in the ground model. In addition, $Q$ has the property,
inessential for (A) but essential for (B), (d) that it adjoins an infinite subset $W$ of
$\omega$ that is not split into two infinite pieces by any set in the ground model; this
property will ensure that $s = \aleph_2$ in the iterated forcing model. The sets on which
the function $f$ of (c) is constant will be the intervals determined by successive
elements of the set $W$ of (d), so we think of forcing with $Q$ as simply adjoining $W$.
In this section, we define $Q$, verify (a), and establish a combinatorial lemma
about $Q$. The proofs of (b), (c), and (d) will occupy Section 3.

For natural numbers $n < m$, let $K_{n,m}$ be the set of all binary relations
$t \subseteq \mathcal{P}(n) \times \mathcal{P}(m)$ such that, for each $a \subseteq n$,

$$(a, a) \in t, \quad \text{and} \quad \text{if } (a, b) \in t, \text{then } b \cap n = a.$$

We think of each $t \in K_{n,m}$ as specifying, for each $a \subseteq n$, some permissible extensions of $a$ to subsets of $m$, each extension being obtained by adjoining to $a$ some members of $[n, m]$; the first requirement on $t$ above says that adjoining nothing is always permissible. If $t \in K_{n,m}$ and $s \in K_{m,l}$, then we write $ts$ for the ordinary composition of these binary relations, so $ts \in K_{n,l}$.

For $t \in K_{n,m}$ and $Y \subseteq [n, m)$, we define $t_Y \in K_{n,m}$ by

$$t_Y = \{(a, b) \in t \mid b - a \subseteq Y\}.$$  

We define the depth $Dp(t)$ of each $t \in K_{n,m}$ by the following induction

$Dp(t) \geq 0$ always.

$Dp(t) \geq 1$ if, for every $a \subseteq n$, there is $b \subseteq m$ such that

$$(a, b) \in t \text{ and } b \neq a.$$  

$Dp(t) \geq d + 1$, for $d \geq 1$, if, for every partition $\{Y, Z\}$ of $[n, m)$, at least one of $Dp(t_Y)$ and $Dp(t_Z)$ is $\geq d$.

Thus, $t$ has depth $\geq d + 1$ if and only if we have a winning strategy in the game played as follows. There are $d$ moves. At each move our opponent partitions $[n, m)$ into two pieces and we choose one of the pieces. After $d$ moves, let $Y$ be the intersection of all the sets we chose. We win if and only if, for every $a \subseteq n$, $t_Y$ contains $(a, b)$ for some $b \neq a$. (Note that the game would be unchanged if, from the second move on, our opponent partitioned the set we had just chosen rather than $[n, m)$.) This game interpretation of depth makes the following lemmas quite easy.

**Lemma 2.1.** For any $t \in K_{n,m}$ and $s \in K_{m,l}$,

$$\max\{Dp(t), Dp(s)\} \leq Dp(ts) \leq 1 + \max\{Dp(t), Dp(s)\}.$$  

**Proof.** For the first inequality, we adopt a strategy of playing the game for $ts$ by using our winning strategy in the game for the deeper of $t$ and $s$, while ignoring the other ‘half’ of $[n, l)$. To prove the second inequality, consider what happens if our opponent’s first move is to partition $[n, l)$ into $[n, m)$ and $[m, l)$. □

**Lemma 2.2.** Let $t \in K_{n,m}$ have depth $\geq 2d + 1$, let $w \subseteq n$, and let $\mathcal{P}(m)$ be the union of two pieces. Then for at least one of the pieces, say $X$, the relation

$$t' = \{(a, b) \in t \mid \text{if } a = w \text{ and } b \neq a, \text{then } b \in X\}$$  

has depth $\geq d + 1$.

**Proof.** Let the two pieces be $X_0$ and $X_1$, and suppose the conclusion fails for both
of the corresponding relations $t_0'$ and $t_1'$. So we do not have a winning strategy in either of the associated $d$-move games described above. Since the games are finite, our opponent has winning strategies, say $\sigma_0$ and $\sigma_1$. Suppose our opponent plays the $2d$-move game for $t$ by using $\sigma_0$ for the first $d$ moves and then using $\sigma_1$ for the last $d$ moves (as if each half were a separate game). No matter how we respond, the intersection $Y$ of our moves will be $Y_0 \cap Y_1$ where $Y_0$, the intersection of our first $d$ moves, is a possible outcome of the $d$-move game with our opponent using $\sigma_0$, and $Y_1$, the intersection of our last $d$ moves is a possible outcome of the $d$-move game with our opponent using $\sigma_1$. As $\sigma_0$ and $\sigma_1$ are winning strategies for our opponent, there exist sets $a_0, a_1 \subseteq \omega$ such that, for $i = 0$ and $1$,

$$t_i' \text{ contains no pair } (a_i, b) \text{ with } b \neq a_i \text{ and } b - a \subseteq Y_i.$$ 

If, for at least one $i$, $a_i \neq w$, then $t$ contains no pair $(a_i, b)$ with $b \neq a_i$ and $b - a \subseteq Y$ (as $Y \subseteq Y_i$ and $t_i'$ agrees with $t$ on pairs whose first component is not $w$), so we have lost the play of the $2d$-move game for $t$. If, on the other hand, $a_i = w$ for both values of $i$, then $t$ contains no pair $(w, b)$ with $b \neq w$ and $w - b \subseteq Y$, for such a pair would be in $t_0'$ or $t_1'$, according to whether $b \in X_0$ or $X_1$. So again we have lost the $2d$-move game for $t$. This shows that our opponent's strategy “first $\sigma_0$ then $\sigma_1$” is a winning one for him. This contradicts the hypothesis $D_p(t) \geq 2d + 1$. □

We are now ready to define the notion of forcing $Q$. A condition in $Q$ is a pair $(w, T)$ consisting of a finite subset $w$ of $\omega$ and a sequence $T = \langle t_l : l \in \omega \rangle$ such that, for some increasing function $n : \omega \rightarrow \omega$,

(a) $w \subseteq n(0)$,

(b) $t_l \in K_{n(l), n(l+1)}$ for each $l$, and

(c) $D_p(t_l) \rightarrow \infty$ as $l \rightarrow \infty$.

(Notice that $(w, T)$ determines the function $n$ uniquely, since $\mathcal{P}(n(l))$ is the domain of $t_l$.) Another such condition $(w', T')$ is an extension of $(w, T)$ if and only if there is an increasing function $k : \omega \rightarrow \omega$ such that, writing $t^*_l$ for $t_{k(l)}t_{k(l)+1} \cdots t_{k(l+1)-1}$, we have

(a) $(w, w') \in t_0, t_1 \cdots t_{k(0)-1}$, by which we mean $w = w'$ if $k(0) = 0$.

(b) $t_l^* \in K_{n(k(l)), n(k(l)+1)}$ for all $l \in \omega$, and

(c) $t_l^* \subseteq t_l^*$ for all $l \in \omega$.

Thus, any extension of $(w, T)$ is obtained by a succession of operations of the following three sorts.

**Compose relations.** Partition the sequence $T$ into finite blocks of consecutive $t_l$'s, and compose the $t_l$'s within each block. Leave $w$ unchanged. (In the description of extensions above, this is the special case where $k(0) = 0$ and $t_l^* = t_l^*$.)

**Shrink relations.** Replace each $t_l$ by a subset $t_l'$ in $K_{n(l), n(l+1)}$, and leave $w$ unchanged. Of course the $t_l'$ must be big enough so that their depths tend to $\infty$ with $l$. (This is the special case where $k(l) = l$ for all $l$.)
Fix values. Replace $w$ with some $w'$ such that $(w, w') \in t_0 \cdots t_{m-1}$ for some $m,$ and delete the initial segment $t_0, \ldots, t_{m-1}$ from $T,$ so $t'_l = t_{k+l}.$ (This is the special case where $k(l) = m+1$ and $t'_l = t^*_l$; it could be replaced by the even more special case where $m = 1$.)

By Lemma 2.1, the relations $t^*_l$ obtained by composing have depth at least equal to the maximum depth of the relations $t_{k(l)}, \ldots, t_{k(l+1)-1}$ being composed. Because of this and the fact that $Dp(t_l) \to \infty$ in any condition, we can always compose relations so as to make $Dp(t^*_l)$ grow as rapidly as we want. Thus, for example, if $f : \omega \to \omega,$ then the conditions with $Dp(t_l) \geq f(l)$ for all $l$ are dense in $Q,$ so are the conditions with $Dp(t_{l+1}) \geq f(Dp(t_l))$ for all $l.$ Furthermore, the extensions witnessing this density can all be taken to be of the ‘compose relations’ type.

We think of a condition $(w, T)$ as providing the following information about the generic $W \subseteq \omega$ being produced:

$W \cap n(0) = w.$

For each $l$, $(W \cap n(l), W \cap n(l+1)) \in t_l.$

(The first of these explains the terminology ‘fixing values’.) Clearly, extensions in $Q$ give more information about $W.$ We call a natural number $x$ possible for $(w, T)$ if there exist $l, v \subseteq \omega$ such that $x \in v$ and $(w, v) \in t_0 \cdots t_l.$ Thus, $x$ is possible for $(w, T)$ if and only if the information that $(w, T)$ gives about $W$ does not preclude the possibility that $x \in W.$ An equivalent formulation is that $(w, T)$ has an extension $(w', T')$ (which can be taken to be a ‘fixing values’ extension) with $x \in w'.$ We write $ps(w, T)$ for $\{x \in \omega \mid x$ is possible for $(w, T)\}.$

It will be helpful to view a condition $(w, T)$ as a labeled tree in which the root (at level 0) is labeled $w$ and, if a node at level $l$ is labeled with a set $a \subseteq n(l),$ then its immediate successors are labeled with the sets $b \subseteq n(l+1)$ such that $(a, b) \in t_l.$ Thus, the set of labels at level $m$ is

$$\text{Lev}_{(w, T)}(m) = \{a \mid (w, a) \in t_0 \cdots t_{m-1}\}.$$ 

We omit the subscript when $(w, T)$ is clear from the context. Note that a set that labels a node at some level also labels successor nodes at all higher levels. (This is why we say that sets label nodes, not that sets are nodes.) We write $\text{Lev}_{(w, T)}(m)$ for the set $\text{Lev}_{(w, T)}(m) - \text{Lev}_{(w, T)}(m-1)$ of new labels at level $m.$ We also write $\text{Tree}(w, T)$ for the set $\bigcup_m \text{Lev}(m)$ of all the labels occurring in the tree. The information in $(w, T)$ about $W$ is that $\{W \cap n(l) \mid l \in \omega\}$ is the set of labels of a path through this tree. Although $(w, T)$ contains information not captured in the tree, e.g., whether $(a, b) \in t_l$ when $a \notin \text{Lev}(l),$ this additional information will be irrelevant for us, so it would do no harm to identify conditions with trees.

Our next goal is to show that fusion arguments can be carried out in $Q,$ from which it will follow that $Q$ is proper.

By an $m$-extension of a condition $(w, T)$ we mean an extension $(w', T')$ such that $w' = w$ and $t'_l = t_l$ for all $l < m.$ (So the trees agree up to level $m.$) In
particular, a \(0\)-extension is an extension with the same first component. Call a subset \(D\) of \(Q\) \(m\)-dense if every condition in \(Q\) has an \(m\)-extension in \(D\).

**Proposition 2.3.** For each \(m \in \omega\), let \(D_m\) be \(m\)-dense and closed under \(m\)-extensions. Then \(\bigcap_{m \in \omega} D_m\) is dense (in fact \(0\)-dense).

**Proof.** Let any condition \((w, T^0)\) be given. We inductively define extensions \((w, T^m)\) such that, for each \(m\),

1. \((w, T^{m+1}) \in D_m\),
2. \((w, T^{m+1})\) is an \(m\)-extension of \((w, T^m)\), and
3. \(D_p(t_{m+1}) \geq m\).

The induction is quite easy. Given \((w, T^m)\), we use the \(m\)-density of \(D_m\) to get an \(m\)-extension \((w, T') \in D_m\). Then we compose relations to obtain \((w, T^{m+1})\) satisfying (iii) as well. Specifically, we compose \(t_m' \cdot t_{m+1}' \cdots t_{r-1}'\) for some \(r\) such that \(D_p(t_r') \geq m\). Then (iii) holds by Lemma 2.1, while (i) and (ii) hold because \((w, T^{m+1})\) is an \(m\)-extension of \((w, T')\).

Having defined the sequence \(\langle (w, T^m) : m \in \omega \rangle\), we obtain a condition \((w, S)\) by setting

\[ s_i = t_i^{r+1}. \]

This is easily seen to be a condition, since, by (iii), \(D_p(s_i) = i \to \infty\). This \((w, S)\) is an \(m\)-extension of \((w, T^m)\) for every \(m\), so it is in \(\bigcap_{m \in \omega} D_m\) and it extends (in fact \(0\)-extends) \((w, T^0)\). \(\square\)

In Proposition 2.3, we can weaken the hypothesis from 'closed under \(m\)-extensions' to 'closed under \((m + 1)\)-extensions' provided we strengthen the density hypothesis to assert that every condition has \(m\)-extensions \((w, T) \in D_m\) with \(D_p(t_m) \gg m\). This is because the extra hypothesis lets us dispense with composing relations; if we choose \(T'\) in the proof in accordance with the stronger hypothesis, then we can set \(T^{m+1} = T'\). Then, as \((w, S)\) is an \((m + 1)\)-extension of \((w, T^{m+1})\), we will have \((w, S) \in D_m\).

In the next proposition, it will be convenient to have a short notation for conditions obtained by fixing values. Suppose \((w', T')\) is obtained from \((w, T)\) by fixing values, so for some (unique) \(r\) we have \((w, w') \in t_0 \cdots t_{r-1}\) and \(t_i' = t_i + r\). In this situation, we write \(T - r\) for \(T'\).

**Proposition 2.4.** Let \(\tau_i (i < \omega)\) be \(Q\)-names of ordinals. Then every condition has a \(0\)-extension \((w, S)\) with the following property. If \(l \in \omega\), if \(n = n(l)\) is the number such that \(s_i\) has domain \(\mathcal{P}(n)\), if \((w, w^*) \in s_0 \cdots s_{l-1}\), if \(i < n\), and if \((w^*, S - l)\) has a \(0\)-extension forcing a particular value for \(\tau_i\), then \((w^*, S - l)\) forces a particular value for \(\tau_i\).

**Proof.** By Proposition 2.3, it suffices to show that, for each \(l\), every condition has
an $l$-extension with the stated property for that particular $l$. So let $l$ be fixed and let $(w, T)$ be given; we must construct an $l$-extension $(w, S)$ of $(w, T)$ satisfying the conclusion of the proposition for the fixed $l$. By definition of $l$-extension, $n(l)$ is the same for $(w, S)$ as for $(w, T)$, so $n$ is also fixed. There are only finitely many $i < n$ and finitely many $w^* \leq n$ to consider. List the $w^*$'s in some arbitrary order. For the first $w^*$ in the list, inductively define conditions $(w, T')$ $(i \leq n)$ by

$$T^0 = T - l.$$ If $(w^*, T')$ has a 0-extension forcing a particular value for $\tau_i$, then let $(w^*, T'^{i+1})$ be such a 0-extension; otherwise let $T'^{i+1} = T^i$.

Let $T' = T^n$. Now repeat the process for the next $w^*$ on the list, starting with $T'$ in place of $T - l$. Continue in the same way with all the other $w^*$'s. If we let $T''_{w^*}$ be the $T'$ obtained at the end of the stage where $w^*$ was used and we let $T''$ be the final $T' (T''_{w^*}$ for the last $w^*$ on the list), then, for each $w^*$:

- if $i < n$ and some 0-extension of $(w^*, T''_{w^*})$ forces a particular value for $\tau_i$, then so does $(w^*, T''_{w^*})$, and $(w^*, T'')$ is a 0-extension of $(w^*, T''_{w^*})$, so if $i < n$ and some 0-extension of $(w^*, T'')$ forces a particular value for $\tau_i$, then so does $(w^*, T'')$.

Finally, we set $s_m = t_m$ for $m < l$ and $s_m = t''_{m-l}$ for $m \geq l$. Then $(w, s)$ is an $l$-extension of $(w, T)$ and $S - l = T''$ has the desired properties.  

**Proposition 2.5.** $Q$ is proper.

**Proof.** Let $\chi$ be a regular cardinal so large that $H(\chi)$, the set of sets of hereditary cardinality $< \chi$, contains all the sets we are interested in. Let $N$ be a countable elementary submodel of $H(\chi)$ containing $Q$. We must show that every condition $(w, T) \in N \cap Q$ has an $(N, Q)$-generic extension.

We shall use the proof of Proposition 2.4 with a little extra caution. That proof involved some arbitrary choices, namely the $T'i$s. Let us fix, once and for all, a choice function (on the nonempty subsets of $Q$, say) in $N$, to be used whenever such choices need to be made. The existence of a choice function in $N$ follows, of course, from the elementarity of the submodel $N$ of $H(\chi)$.

Let $(\tau_i : i < \omega)$ be an enumeration of all the elements of $N$ that are $Q$-names of ordinals. Proposition 2.4 gives us a 0-extension $(w, S)$ of $(w, T)$ having the 'value-deciding' property (as in the proposition) for the sequence $(\tau_i : i < \omega)$. This sequence is, of course, not in $N$, so the construction of $S$ cannot be carried out in $N$, but any finite initial segment of it can be. Indeed, for each fixed $l$, the proof of Proposition 2.4 for that $l$ involves $\tau_i$ only for $i < n$, so it can be done within $N$. It is only in applying Proposition 2.3, i.e., in doing this proof repeatedly for ever larger values of $l$ (hence of $n$), that we need the whole sequence of $\tau_i$'s and must therefore step outside $N$.

For each $i$, the construction (outside $N$) must eventually lead to a condition
forcing a particular value for \( \tau_i \). Indeed, since \( \tau_i \) is a name of an ordinal, some condition extending \((w, S)\) forces a particular value of \( \tau_i \). Extending it further, we may assume that this condition is a 0-extension of \((w^*, S - l)\) for some \( l > i \) and some \( w^* \) with \((w, w^*) \in s_0 \cdots s_{l-1} \). Then \((w^*, S - l)\) forces a particular value for \( \tau_i \). But this occurs at a finite stage of the construction and the construction up to that point could have been carried out in \( N \). Thus, the value forced for \( \tau_i \) is in \( N \).

This shows that every particular value for \( \tau_i \) that is forced by an extension of \((w, S)\) is in \( N \). Since \( \tau_i \) ranges over all names in \( N \) for ordinals, we have shown that \((w, S)\) is \((N, Q)\)-generic. \( \square \)

The following partition theorem is the combinatorial property of \( Q \) on which the rest of the proof hinges.

**Theorem 2.6.** Let \((w, T) \in Q\) and let \( C \) map the finite subsets of \( \omega \) into \([0, 1]\). Then either there is an extension \((w', T')\) of \((w, T)\) such that \( C \) maps \( \text{Tree}(w', T') \) to 0 or there is a 0-extension \((w, T')\) of \( T \) such that \( C \) maps \( \text{Tree}(w, T') - \{w\} \) to 1.

**Proof.** Let \( C \) be given. The theorem asserts that the set of conditions \((w, T)\) such that either \( C \) is constantly 1 on \( \text{Tree}(w, T) - \{w\} \) or there is an extension on whose tree \( C \) is constantly 0 is 0-dense. By the remark following Proposition 2.3, it suffices to show that, for each \( m \), the set

\[
D_m = \{(w, T) \mid \text{either } (w, T) \text{ has an extension on whose tree } \text{C is identically 0 or C is identically 1 on } \text{Lev}_{(w,T)}(m + 1)\}
\]

is \( m \)-dense in the strong sense that every condition \((w, T)\) has an \( m \)-extension \((w, T')\) in \( D_m \) with \( \text{Dp}(t_m) \geq m \). (Note that \( D_m \) is trivially closed under \((m + 1)\)-extensions.) Let \( m \) and \((w, T)\) be given. We construct a suitable \((w, T')\). We may assume, by composing relations if necessary, that \( \text{Dp}(t_l) \geq l \) for all \( l \geq m \).

The core of the construction is the following definition.

Let \( q \) be a positive integer, \( \zeta = \langle \zeta(n) : n \in \omega \rangle \) an increasing sequence of natural numbers, and \( \eta = \langle \eta(n) : n \in \omega \rangle \) a subsequence of \( \zeta \) with \( \eta(0) = \zeta(0) \). We say that \( \eta \) is \( q \)-thin in \( \zeta \) if the hypotheses

\( i) \ \zeta(k) < \zeta(l) \) are consecutive terms in the subsequence \( \eta \),

(\( ii) \ s_m \subseteq t_m \) for every \( m \in [\zeta(k), \zeta(l)] \),

(\( iii) \) for each \( p \in [k, l] \) at least one \( m \in [\zeta(p), \zeta(p + 1)] \) has \( \text{Dp}(s_m) \geq q + m - \zeta(k) \), and

(\( iv) \) \( t^* := \{(a, b) \in s_{\zeta(k)} s_{\zeta(k)+1} \cdots s_{\zeta(l)-1} \mid b = a \) or if \( a \in \text{Tree}(w, T) \) then \( C(b) = 1) \}

imply that \( \text{Dp}(t^*) \geq q \).

We consider two cases according to whether or not every increasing sequence \( \zeta \) of non-negative integers has a 1-thin subsequence.
Case 1: Some $\zeta$ has no 1-thin subsequence.

Fix such a $\zeta$. We attempt to define inductively a 1-thin subsequence by setting $\eta(0) = \zeta(0)$ and, after $\eta(n)$ has been chosen as $\zeta(k)$, say, setting $\eta(n+1) = \zeta(l)$ for some $l$ such that hypotheses (ii) to (iv) in the definition of 1-thin imply $Dp(t^*) \geq q$. By the case hypothesis, this attempt fails; at some stage, no suitable $l$ exists. Consider such a stage and fix $k$ as above, so $\zeta(k)$ is the last $\eta(n)$ defined. For every $l > m$, there exist

$$s^{(l)}_m \subseteq t_m \quad \text{for every } m \in [\zeta(k), \zeta(l))$$

such that, for each $p \in [k, l)$, at least one $m \in [\zeta(p), \zeta(p+1))$ has $Dp(s^{(l)}_m) \geq 1 + m - \zeta(k)$, but $t^*$, defined by (iv), has depth 0. Recall that depth 0 means that, for some $a^{(l)} \subseteq n(\zeta(k)) = \{n \text{ such that } t_{\zeta(k)} \text{ has domain } P(n)\}$, the only $b$ such that $(a^{(l)}, b) \in t^*$ is $b = a_l$. Clearly $a^{(l)}$ must be in $\text{Tree}(w, T)$ for otherwise $t^*$ would contain all pairs $(a^{(l)}, b) \in s^{(l)}_{\zeta(k)} \cdots s^{(l)}_{\zeta(l)-1}$ which has depth $\geq 1$ by hypothesis (iii) and Lemma 2.1. We thus have that, if $(a^{(l)}, b) \in s^{(l)}_{\zeta(k)} \cdots s^{(l)}_{\zeta(l)-1}$ and $b \neq a_l$, then $C(b) = 0$.

As $l$ varies while $m$ is fixed, there are only finitely many possibilities for $s^{(l)}_m$, namely the subsets of $t_m$; similarly, there are only finitely many possibilities for $a^{(l)}$, namely the members of $\text{Lev}(\zeta(k))$. By König's infinity lemma, there is a single infinite sequence $\langle s^*_m : m \geq \zeta(k) \rangle$ and there is a single $a \in \text{Lev}(\zeta(k))$ such that

1. $s^*_m \subseteq t_m$ for all $m$,
2. for each $p \geq k$ there is $m \in [\zeta(p), \zeta(p+1))$ with $Dp(s^*_m) \geq 1 + m - \zeta(k)$,
3. for each $l \geq k$, if $(a, b) \in s^{(l)}_{\zeta(k)} \cdots s^{(l)}_{\zeta(l)-1}$ and $b \neq a_l$, then $C(b) = 0$.

For $p \geq k$, let $s^*_p = s^{(p)}_{\zeta(p)} \cdots s^{(p+1)}_{\zeta(p+1)-1}$, and let $s^*$ be the sequence $\langle s^*_{k+i} : i \in \omega \rangle$. Then $(a, s^*)$ is an extension of $(w, T)$ (obtained by fixing values to $a$, shrinking relations to $s$, and composing relations to $s^*$). Note that (iii') and Lemma 2.1 ensure that $Dp(s^*_{k+i}) \geq 1 + \zeta(k + l) - \zeta(k) \to \infty$, as $l \to \infty$, so $(a, s^*)$ is a condition. Furthermore, (iv') tells us that all the sets $b \in \text{Tree}(a, s^*)$ except $a$ itself have $C(b) = 0$. The exception can be eliminated by extending $(a, s^*)$ by fixing values. So we have an extension of $(w, T)$ on whose tree $C$ is identically 0. So $(w, T) \in D_m$.

Case 2: Every $\zeta$ has a 1-thin subsequence.

Lemma 2.7. If $\theta$ is a $q$-thin subsequence of $\eta$ and $\eta$ is a $q$-thin subsequence of $\zeta$, then $\theta$ is a $q + 1$-thin subsequence of $\zeta$.

Proof. Let $\zeta(k)$, $\zeta(l)$, $s_m$, and $t^*$ be as in hypotheses (i) to (iv) with $\theta$ in place of $\eta$ and $q + 1$ in place of $q$. We must show that $Dp(t^*) \geq q + 1$. Let $\{Y, Z\}$ be a partition of $[n(\zeta(k)), n(\zeta(l))]$ into two pieces; we must show that either $t^*_Y$ or $t^*_Z$
has depth $\geq q$. Observe that
\[ t^*_y = \{(a, b) \in (s_{\zeta(k)})_Y, \ldots, (s_{\zeta(l)-1})_Y \mid b = a \text{ or if } a \in \text{Tree}(w, T), \text{ then } C(b) = 1\}, \]
and similarly for $t^*_z$.

By hypothesis (iii) and definition of depth, we have, for each $p \in [k, l]$, at least one $m = m^p \in [\zeta(p), \zeta(p + 1))$ and at least one $X = X^p \in \{Y, Z\}$ such that $Dp((s_m)_Y) \geq q + m - \zeta(k)$.

If there are $k' < l' \in [k, l]$ such that $\zeta(k')$ and $\zeta(l')$ are consecutive terms of $\eta$ and $X^{p^p} = Y$ for all $p \in [k', l')$, then $k', l', (s_m)_Y$, and $t^*_y$ satisfy hypotheses (i) to (iv) in the definition of "$\eta$ is $q$-thin in $\zeta\)", so we have $Dp(t^*_y) \geq q$, as desired.

Otherwise, let $k''$ and $l''$ be such that $\eta(k'') = \zeta(k)$ and $\eta(l'') = \zeta(l)$. Clearly, $k''$, $l''$, $(s_m)_Z$, and $(t^*_z)_Z$ satisfy hypotheses (i), (ii), and (iv) in the definition of "$\theta$ is $q$-thin in $\eta\)". If we verify that (iii) also holds, then that definition allows us to conclude $Dp(t^*_z) \geq q$, as desired. To verify (iii), let $p \in [k'', l'')$ be given, and let $k'$ and $l'$ be such that $\zeta(k') = \eta(p)$ and $\zeta(l') = \eta(p + 1)$. Since the hypothesis of the preceding paragraph fails, there must be $p' \in [k', l')$ with $X^{p'} = Z$. So there is, by definition of $X^{p'}$, at least one $m \in [\zeta(p'), \zeta(p' + 1)) \in [\zeta(k'), \zeta(l')) = [\eta(p), \eta(p + 1))$ with $Dp((s_m)_Z) \geq q + m - \zeta(k)$, so (iii) is verified and the lemma is proved. \(\square\)

**Corollary 2.8.** For every $q$, every increasing sequence $\zeta$ has a $q$-thin subsequence.

**Proof.** Immediate by the lemma and the case hypothesis. \(\square\)

Let $\xi$ be the sequence $(m, m + 1, m + 2, \ldots)$ so $\xi(k) = m + k$, and let $\eta$ be an $m$-thin subsequence. Apply the definition of $m$-thin with $k = 0$ (so $\xi(k) = m = \eta(0)$) and $l$ such that $\xi(l) = m + l = \eta(1)$, and with $s_m = t_m$. Hypotheses (i) and (ii) are clear, and (iii) asserts that for each $p \in [0, l)$
\[ Dp(t_{p+m}) \geq m + (p + m) - m = p + m, \]
which is true by our initial normalization of $T$. So, if we define $t^*$ by (iv), we have $Dp(t^*) \geq m$. Define $T'$ by
\[
t' = \begin{cases} t_r & \text{if } r < m, \\
t^* & \text{if } r = m, \\
t_{r-l} & \text{if } r > m. \end{cases} \]

Then $(w, T')$ is an $m$-extension of $(w, T)$, has $Dp(t'_m) \geq m$, and lies in $D_m$ because $C$ is identically 1 on $\text{Lev}(w, T')(m + 1)$ by definition of $t^*$. This completes the proof of the theorem. \(\square\)

We apply Theorem 2.6 to obtain an improvement of Proposition 2.4 in the case that the names $\tau_i$ have only finitely many values.
Proposition 2.9. Let $A$ be a $Q$-name for a subset of $\omega$. Then every condition has a 0-extension $(w, S)$ with the following property. If $l \in \omega$, if $n = n(l)$ is the number such that $s_l$ has domain $\mathcal{P}(n)$, if $(w, w^*) \in s_0 \cdots s_{l-1}$, and if $i < n$, then $(w^*, S - l)$ decides whether $i \in A$.

Proof. Let $\tau_i$ be the $Q$-name forced (by all conditions) to be 1 if $i \in A$ and 0 if $i \notin A$. So to decide whether $i \in A$ is the same as to force a particular value for $\tau_i$. Let $(w, S)$ be as in Proposition 2.4 for this sequence $(\tau_i : i \in \omega)$. Let $l, n, w^*$, and $i$ be as in the present proposition. The desired conclusion would follow, by 2.4, if we knew that some 0-extension of $(w^*, S - l)$ decides whether $i \in A$.

We apply Theorem 2.6 with $(w^*, S - l)$ in the role of $(w, T)$ and with the function $C$ defined by

$$C(a) = 1 \text{ iff for some } r, (a, S - r) \text{ decides whether } i \in A.$$ 

The first alternative in the theorem is that $(w^*, S - l)$ has an extension $(w', T')$ on whose tree $C$ is identically zero. Let $(a, T'')$ be an extension of $(w', T')$ deciding whether $i \in A$. Then $a$ is in the tree of $(w', T')$, so $C(a) = 0$. But $(a, T'')$ is an extension of $(w^*, S - l)$, hence is a 0-extension of $(a, S - r)$ for some $r$ (the lowest level of $a$ in the tree of $(w^*, S)$). So $C(a) = 1$. This contradiction shows that the first alternative does not occur.

So we have the second alternative in the theorem. That is, we have a 0-extension $(w^*, T')$ of $(w^*, S - l)$ such that $C$ is identically 1 on the set $L = \text{Tree}(w^*, T') - \{w^*\}$. Thus, for each $a \in L$, $(a, S - r)$ decides a value $v(a) \in \{0, 1\}$ for $\tau_i$, provided $r$ is large enough. Our choice of $S$ ensures that we can take the smallest possible $r$, namely the one with $a \in \text{Lev}_{(w, S)}(r)$, and still have $(a, S - r)$ forcing $\tau_i = v(a)$.

Apply Lemma 2.2 to each of the terms in the sequence $S - l$, using $w^*$ in the role of $w$ and using the partition given by the function $v$. the result is a sequence of sub-relations $s_k \subseteq s_k$ for $k \geq l$, with $\text{Dp}(s_k') \geq \frac{1}{2} \text{Dp}(s_k)$, such that if $(w^*, b) \in s_k'$ and $w^* \neq b$, then $v(b) = \bar{v}$ where $\bar{v}$ depends only on $k$. Since $\text{Dp}(s_k') \to \infty$, $(w^*, (s_k', r \in \omega)) = (w, S')$ is a condition, a 0-extension of $(w^*, S)$. Composing relations, we can arrange that $\bar{v}$ is independent of $k$.

We complete the proof by checking that $(w^*, S')$ forces "$\tau_i = \bar{v}$". If not, then some extension $(b, S'')$ forces "$\tau_i = 1 - \bar{v}$", and, by extending further (fixing values) we may assume $b \neq w^*$. But then $v(b) = \bar{v}$, so $(b, S - r)$ forces "$\tau_i = \bar{v}$". This is absurd, as $(b, S')$ is an extension of $(b, S - r)$. This contradiction completes the proof of the proposition. $\square$

3. The extension by $Q$

This section is devoted to the study of the forcing extension $V[G]$ produced by adjoining to the universe $V$ a $V$-generic subset $G$ of $Q$. In this extension, we
define
\[ W = \bigcup \{ w \mid (w, T) \in G \}. \]
A trivial genericity argument shows that \( W \) is an infinite subset of \( \omega \).

**Proposition 3.1.** For every set \( X \subseteq \omega \) in the ground model \( V \), either \( W \cap X \) or \( W - X \) is finite.

**Proof.** Let \( X \) be given; we show that the conditions forcing the desired conclusion form a dense set. Let \( (w, T) \) be any condition. For each \( k \), let \( t'_k = (t_k)_X \) or \( (t_k)_{\omega - X} \), whichever has the greater depth (either one in case of equal depth). By definition of depth, \( Dp(t'_k) \geq Dp(t_k) - 1 \), so \( (w, T') \) is a condition extending \( (w, T) \). Suppose \( t'_k = (t_k)_X \) for infinitely many \( X \). (The \( \omega - X \) case is analogous.) Let \( T'' \) be the subsequence of \( T' \) consisting of only those \( t'_k \) that equal \( (t_k)_X \). (The codomain of such a \( t'_k \) may need to be defined differently in \( T'' \) than in \( T' \), to match the domain of the next term in \( T'' \).) Then \( (w, T'') \) is an extension of \( (w, T') \) (obtainable by composing and then shrinking relations), and it clearly forces \( W - X \subseteq w \). \( \Box \)

The main theorem of this section will assert that any \( P \)-point ultrafilter in \( V \) generates a \( P \)-point in \( V[G] \). We first check that it will suffice to prove that it generates an ultrafilter.

**Lemma 3.2.** Let \( \mathcal{U} \) be a \( P \)-point, and let \( H \) be a \( V \)-generic subset of some proper notion of forcing. If \( \mathcal{U} \) generates an ultrafilter \( \mathcal{U} \) in \( V[H] \), then this \( \mathcal{U} \) is a \( P \)-point in \( V[H] \).

**Proof.** Since every set in \( \mathcal{U} \) has a subset in \( \mathcal{U} \), it suffices to show that, if \( \langle X_n : n \in \omega \rangle \) is a sequence in \( V[H] \) of sets in \( \mathcal{U} \), then some set \( Y \in \mathcal{U} \) is almost included (i.e., included modulo a finite set) in every \( X_n \). By [7, III.1.16], there is a countable set \( S \in V \) such that each \( X_n \in S \). As \( \mathcal{U} \) is a \( P \)-point in \( V \), there is \( Y \in \mathcal{U} \) almost included in every set in \( S \cap \mathcal{U} \), hence almost included in every \( X_n \). \( \Box \)

As \( Q \) is proper, Lemma 3.2 can be applied to it. We have stated the lemma for arbitrary proper forcing notions in order to apply it to iterations of \( Q \) in the next section.

**Theorem 3.3.** Every \( P \)-point in \( V \) generates a \( P \)-point in \( V[G] \).

**Proof.** Let \( \mathcal{U} \) be a \( P \)-point in \( V \). By the preceding lemma, we need only show that the filter \( \mathcal{U} \) generated by \( \mathcal{U} \) in \( V[G] \) is an ultrafilter. By genericity, it suffices to show that, if \( (w, T) \) forces "\( A \subseteq \omega \)", then some extension forces either
"B \subseteq A" or "B \cap A = \emptyset" for some \( B \in \mathcal{U} \). According to Proposition 2.9, we may assume that, if \( l \in \omega \), if \( n = n(l) \) is the number such that \( t_l \) has domain \( \mathcal{P}(n) \), if \( (w, w^*) \in t_0 \cdot \cdots \cdot t_{l-1} \), and if \( i < n \), then \( (w^*, T - l) \) decides whether \( i \in A \).

Consider any \( w^* \in \text{Tree}(w, T) \). Then \( w^* \in \text{Lev}_{(w, T)}(l) \) for all sufficiently large \( l \). Thus, for any fixed \( i \in \omega \), \( (w^*, T - l) \) will decide whether \( i \in A \) once \( l \) is large enough; of course the decisions agree as \( l \) varies, since \( (w^*, T - l') \) extends \( (w^*, T - l) \) (by fixing values) if \( l' \geq l \). Let \( A(w^*) \) be the set of those \( i \in \omega \) for which the decision is positive, i.e., \( (w^*, T - l) \models "i \in A" \) for all sufficiently large \( l \). Partition \( \text{Tree}(w, T) \) by putting into one class all those \( w^* \) for which \( A(w^*) \in \mathcal{U} \). By Theorem 2.6, we can extend \( (w, T) \) to arrange that all of \( \text{Tree}(w, T) \) is in a single class. Note that, when we form this extension, we do not destroy the fact that, for \( i \in A(w^*) \) (resp. \( i \notin A(w^*) \)), \( (w^*, T - l) \models "i \in A" \) ("i \notin A") for all sufficiently large \( l \). We assume henceforth that \( A(w^*) \in \mathcal{U} \) for all \( w^* \in \text{Tree}(w, T) \); the case that \( \omega - A(w^*) \in \mathcal{U} \) for all \( w^* \in \text{Tree}(w, T) \) is handled analogously, with \( A \) replaced by its complement. As \( \mathcal{U} \) is a P-point, let \( B \in \mathcal{U} \) be almost included in each \( A(w^*) \).

Inductively define a sequence \( \langle \zeta(n) : n \in \omega \rangle \) of natural numbers, starting with \( \zeta(0) = 0 \), and increasing so rapidly that, if \( w^* \in \text{Lev}_{(w, T)}(\zeta(n)) \), then

(i) \( B - A(w^*) \subseteq \zeta(n + 1) \), and

(ii) if \( i \in A(w^*) \) and \( i < \zeta(n) \), then \( (w^*, T - \zeta(n + 1)) \models "i \in A" \).

Think of \( \zeta \) as partitioning \( \omega \) into blocks \( [\zeta(n), \zeta(n + 1)] \) and consider the four sets obtainable by taking the union of every fourth block:

\[ X_j = \bigcup \{ [\zeta(n), \zeta(n + 1)] : n = j \mod 4 \} \]

As \( \mathcal{U} \) is an ultrafilter, it contains exactly one of these sets. By omitting a few terms (at most 3) from the \( \zeta \) sequence, we may assume \( X_2 \in \mathcal{U} \). Replacing \( B \) with \( X_2 \cap B \), which is also in \( \mathcal{U} \), we may assume \( B \subseteq X_2 \).

We define an extension \( (w, T') \) of \( (w, T) \) as follows. Let \( n \) be the function such that \( t_l \in K_{n(l), n(l) + 1} \) for all \( l \). Then \( t'_k \) is to be the element of \( K_{n(\zeta(4k)), n(\zeta(4k + 4))} \) given by \( t'_k = t_{\zeta(4k)} \) (as relations). This defines a condition because \( Dp(t'_k) = Dp(t_{\zeta(4k)}) \to \infty \). Notice that \( \text{Lev}_{(w, T')}((k + 1)) \subseteq \text{Lev}_{(w, T)}(\zeta(4k + 1)) \subseteq \text{Lev}_{(w, T)}(\zeta(4k + 1)) \).

To complete the proof of the theorem, we show that \( (w, T') \) forces "\( B \subseteq A \)". Suppose it did not, and fix an element \( i \in B \) and an extension \( (v, S) \) of \( (w, T') \) forcing "\( i \notin A \)". Since \( B \subseteq X_2 \), let \( k \) be such that \( i \in [\zeta(4k + 2), \zeta(4k + 3)) \). Let \( w^* = v \cap n(\zeta(4k + 4)) \). So \( w^* \in \text{Lev}_{(w, T')}((k + 1)) \subseteq \text{Lev}_{(w, T)}(\zeta(4k + 1)) \) and \( (w^*, T' - (k + 1)) \), being compatible with \( (v, S) \), cannot force "\( i \in A \)". We will obtain a contradiction by showing that \( (w^*, T - \zeta(4k + 4)) \), of which \( (w^*, T' - (k + 1)) \) is an extension, does force "\( i \in A \)". Since \( w^* \in \text{Lev}_{(w, T)}(\zeta(4k + 1)) \) and \( i \in B \) and \( i \geq \zeta(4k + 2) \), clause (i) in the definition of \( \zeta \) implies that \( i \in A(w^*) \). Since also \( w^* \in \text{Lev}_{(w, T)}(\zeta(4k + 3)) \) and \( i \leq \zeta(4k + 3) \), clause (ii) in the same definition implies that \( (w^*, T - \zeta(4k + 4)) \models "i \in A" \), the desired contradiction. \( \square \)
Our next goal is to show, under suitable hypotheses, that every two ultrafilters in $V$ have a common finite-to-one image in $V[G]$. In fact, for this purpose, a single finite-to-one function works for all ultrafilters simultaneously. Specifically, let $f \in V[G]$ be a function from $\omega$ to $\omega$ that is constant precisely on intervals $[a, b)$ where $a$ and $b$ are consecutive elements of $W$ (or $a = 0$ and $b$ is the first element of $W$); for example

$$f(x) = \text{the number of elements of } W \text{ that are } \leq x.$$ 

**Proposition 3.4.** Let $X$ be an infinite subset of $\omega$ in $V$, and let $\mathcal{U}$ be an ultrafilter in $V$. There exists $Y \in \mathcal{U}$ such that $f(X) \equiv f(Y)$.

**Proof.** Let $X$, $\mathcal{U}$, and a condition $(w, T) \in Q$ be given. We shall find $Y \in \mathcal{U}$ and an extension of $(w, T)$ forcing "$f(X) \equiv f(Y)$"; by genericity of $G$, this will suffice to prove the proposition. By a preliminary extension of $(w, T)$ (fixing values), we may assume that the first element of $W$ is forced to be a particular number $p$, and we decide to put no smaller numbers into $Y$. Then the statement to be forced, "$f(X) \equiv f(Y)$", is equivalent to "If $a$ and $b$ are consecutive elements of $W$ and if $Y$ has elements in $[a, b)$, then so does $X$".

Extending $(w, T)$ by composing relations and fixing values, we may assume that the function $n : \omega \to \omega$, such that $t_1$ has domain $\mathcal{P}(n(l))$, grows so rapidly that each interval $[n(l), n(l + 1))$ meets $X$ and that $n(0) > p$. Let $Y_0$ and $Y_1$ be defined by

$$Y_i = \bigcup \{[n(l), n(l + 1)) \mid l \equiv i \pmod{2}\}.$$ 

These two sets constitute a partition of $[p, \omega)$, so one of them in $\mathcal{U}$; assume for notational simplicity that it is $Y_1$. Let $t_1' = t_2$. Then $(w, T')$ is an extension of $(w, T)$. Notice that the set $ps(w, T')$ of possible elements of $W$ (given $(w, T')$) is included in $n(0) \cup Y_0$. Thus, $(w, T')$ forces "$W \subseteq n(0) \cup Y_0$", which implies "If $y \in [n(k), n(k + 1))$ with $k$ odd and if $a \leq y \leq b$ with $a, b \in W$, then $a < n(k)$ and $b \geq n(k + 1)$", which in turn implies "If $y \in Y_1 \cap [a, b)$ with $a, b \in W$, then $[a, b)$ includes an interval $[n(k), n(k + 1))$ for some $k$". Finally, since every interval $[n(k), n(k + 1))$ meets $X$, we see that $(w, T')$ forces "If $Y_1$ meets $[a, b)$ with $a, b \in W$, then $X$ also meets $[a, b)$", as desired. □

If $\mathcal{U}$ is an ultrafilter in $V$, let $\hat{\mathcal{U}}$ be the filter it generates in $V[G]$. Observe that, for any function $g : \omega \to \omega$, $g(\hat{\mathcal{U}})$ is generated by the sets $g(X)$ with $X \in \mathcal{U}$.

**Corollary 3.5.** Assume that there is a $P$-point in $V$. Then, for any two ultrafilters $\mathcal{U}_1$ and $\mathcal{U}_2$ in $V$, $f(\mathcal{U}_1) = f(\mathcal{U}_2)$.

**Proof.** Since "$f(\mathcal{U}_1) = f(\mathcal{U}_2)$" is an equivalence relation on ultrafilters, it suffices to prove the corollary under the additional hypothesis that $\mathcal{U}_2$ is itself a $P$-point. By Theorem 3.3, the filter $\hat{\mathcal{U}}_2$ is an ultrafilter (in fact a $P$-point); hence so is
f(\mathcal{U}_2). So it suffices to prove that \( f(\mathcal{U}_2) \subseteq f(\mathcal{U}_1) \), and by the observation immediately preceding the corollary, it suffices to show that \( f(X) \in f(\mathcal{U}_1) \) for every \( X \in \mathcal{U}_2 \). But this is immediate by Proposition 3.4. □

**Proposition 3.6.** There is no function \( g : \omega \rightarrow \omega \) in \( V \) such that, for all \( n \in \omega \), the \( n \)th element of \( W \) is \( \leq g(n) \).

**Proof.** Suppose \((w, T)\) forced "for all \( n \in \omega \), the \( n \)th element of \( W \) is \( \leq g(n) \)" for a certain \( g \). Let \( n_0 \) be larger than the number of elements of \( w \). For sufficiently large \( k \) (e.g., any \( k \geq g(n_0) \)), the extension \((w, T - k)\) of \((w, T)\) has \( ps(w, T - k) \) consisting of elements of \( w \) and numbers larger than \( g(n_0) \). Thus, \((w, T - k)\) forces "the \( n_0 \)th element of \( W \) is \( > g(n_0) \)" a contradiction. □

### 4. Preservation of P-points in iterations

In this section, which is nearly independent of the preceding ones, we show that, for proper notions of forcing, countable-support iteration preserves the property that a P-point in the ground model generates a P-point in the extension.

Throughout this section, \( \langle P_\alpha, Q_\alpha : \alpha < \lambda \rangle \) is a countable-support proper forcing iteration of limit length \( \lambda \). That is,

- \( P_0 \) is the trivial notion of forcing (with just one element),
- \( P_{\alpha + 1} = P_\alpha \ast Q_\alpha \) (the two-step iteration),
- \( P_\beta = \) direct limit of \( (P_\alpha)_{\alpha < \beta} \) if \( \beta \) has uncountable cofinality,
- \( P_\beta = \) inverse limit of \( (P_\alpha)_{\alpha < \beta} \) if \( \beta \) has cofinality \( \omega \), and
- \( P_\alpha \forces \"Q_\alpha is a proper notation of forcing\".\)

We write \( P_\lambda \) for the inverse or direct limit of \( (P_\alpha)_{\alpha < \lambda} \) according as the cofinality of \( \lambda \) is \( \omega \) or larger. By [7, III.3.2], each \( P_\alpha (\alpha \leq \lambda) \) is proper.

**Theorem 4.1.** For any countable-support proper forcing iteration \( \langle P_\alpha, Q_\alpha : < \lambda \rangle \) and any P-point \( \mathcal{U} \), if, for each \( \alpha < \lambda \),

\[
P_\alpha \forces \"\mathcal{U} generates a P-point\"
\]

then also

\[
P_\lambda \forces \"\mathcal{U} generates a P-point\"
\]

**Proof.** By Lemma 3.2, it suffices to prove that

\[
P_\lambda \forces \"\mathcal{U} generates an ultrafilter\"
\]

The case \( cf(\lambda) > \omega \) is an easy consequence of the fact that, in this case, every real added by \( P_\lambda \) is already added by \( P_\alpha \) for some \( \alpha < \lambda \) [7, V.4.4], hence either
includes or is disjoint from a member of \( \mathcal{U} \). We may therefore assume that \( \text{cf}(\lambda) = \omega \). Indeed we may assume that \( \lambda = \omega \) by passing to a cofinal \( \omega \)-subsequence of the iteration. More precisely, let \( \langle \alpha_n : n \in \omega \rangle \) be an increasing \( \omega \)-sequence cofinal in \( \lambda \), let \( P'_n = P_{\alpha_n} \) and let \( Q'_n = P_{\alpha_{n+1}} \). Then \( \langle P'_n, Q'_n : n < \omega \rangle \) is a proper forcing iteration of length \( \omega \) with (inverse) limit \( P'_\omega = \mathcal{P} \), and each \( P'_n \) forces "\( \mathcal{U} \) generates a \( \mathcal{P} \)-point". So if the theorem is true for \( \omega \)-length iterations, we obtain that \( P'_\omega (= P) \) forces "\( \mathcal{U} \) generates a \( \mathcal{P} \)-point", as desired.

Henceforth, we assume \( \lambda = \omega \). We are given that
\[ P, \mathcal{U} \text{ generates a } \mathcal{P} \text{-point} \]
for each \( n \in \omega \), and we wish to prove, for the limit forcing,
\[ P \text{ forces } "\mathcal{U} \text{ generates an ultrafilter}". \]

For this we consider an arbitrary \( P_\omega \)-name \( A \) and an arbitrary condition \( p \in P_\omega \) forcing "\( A \subseteq \omega \)". Then we find a set \( B \in \mathcal{U} \) and an extension \( q \) of \( p \) forcing that "\( B \subseteq A \text{ or } B \cap A = \emptyset \)".

We claim that it will suffice to carry out the proof under the following additional assumption.

**Hypothesis 4.3.** For every \( k < \omega \), every generic (over \( V \)) \( G_k \subseteq P_k \) containing \( p \upharpoonright k \), and every \( r \in P_\omega / G_k \) extending \( p \upharpoonright [k, \omega) \), there exists \( B \in \mathcal{U} \) such that, for each \( n \in \omega \), some extension \( r' \) of \( r \) in \( P_\omega / G_k \) forces "\( B \cap n \subseteq A(G_k) \)".

To see that this hypothesis entails no loss of generality, suppose that the desired result had been established under the hypothesis but that, for the particular \( p \) and \( A \) under consideration, the hypothesis fails. Let \( k, G_k, \) and \( r \) constitute a counterexample to the hypothesis. We work temporarily in \( V' = V[G_k] \). In this universe, we have a \( P \)-point \( \mathcal{U}' \) generated by \( \mathcal{U} \), an \( \omega \)-length iteration of proper forcing
\[ \langle P'_n, Q'_n : n \in \omega \rangle = \langle P_{k+n}/G_k, Q'_{k+n}^{(G_k)} : n \in \omega \rangle \]
with inverse limit \( P'_\omega = P_\omega / G_k \), a condition \( p' = r \in P'_\omega \) (the \( r \) from the failure of Hypothesis 4.3), and a \( P'_\omega \)-name \( A' \) for the complement of \( A(G_k) \) (so all conditions force "\( A' = \omega - A(G_k) \)"). Consider Hypothesis 4.3', obtained by putting these primed objects in place of the corresponding unprimed objects in Hypothesis 4.3.

If 4.3' holds, then, by our supposition, so does the primed version of the desired conclusion. That is, we have \( B' \subseteq A' \) and an extension \( q' \) of \( p' \) in \( P'_\omega \) forcing "\( B' \subseteq A' \text{ or } B' \cap A' = \emptyset \)". As \( \mathcal{U} \) generates \( \mathcal{U}' \), let \( B \) be a subset of \( B' \). Then we have (still in \( V' = V[G_k] \))
\[ q' \text{ extends } p' \text{ in } P'_\omega, \text{ and } \]
\[ q' \text{ forces } "B \subseteq A' \text{ or } B \cap A' = \emptyset". \]

Returning to \( V \), find \( s \in G_k \) forcing the facts just displayed. As \( p \upharpoonright k \in G_k \), we
may assume that \( s \) extends \( p \). Then \( q = (s, q') \) is a condition in \( P_\omega \) extending \( p \) (as \( q \upharpoonright k = s \) extends \( p \upharpoonright k \) and \( s \) forces \( q \upharpoonright [k, \omega) = q' \) to extend \( p' = r \) which extends \( p \upharpoonright [k, \omega) \)) and forcing \( "B \cap A = \emptyset \) or \( B \subseteq A" \). Thus, \( B \) and \( q \) are as desired.

There remains the case that Hypothesis 4.3' fails. Let \( k', G_{k'}, r' \) be a counterexample in \( V' = V[G_k] \). Thus, \( G_{k'} \) is a \( V' \)-generic subset of \( P'_{k'} = P_{k+k'}/G_k \) containing \( p' \upharpoonright k' = r \upharpoonright [k, k+k') \), and \( r' \) is an extension of \( p' \upharpoonright [k', \omega) = r \upharpoonright [k', \omega) \) in \( P'_{\omega}/G_{k'} = (P_{\omega}/G_k)/G_{k'} \). Let \( H \) be the \( V \)-generic subset \( G_k \ast G_{k'} \) of \( P_{k+k'} \). Then \( r' \in P_{\omega}/H \). Working in \( V[H] = V[G_k, G_{k'}] \), construct a sequence \( \langle q_n : n \in \omega \rangle \) of conditions in \( P_{\omega}/H \) such that:

\[
q_0 = r',
\]

\[
q_{n+1} \text{ extends } q_n.
\]

For each \( m < n \), \( q_n \Vdash m \in A^{(H)} \) or \( q_n \Vdash m \notin A^{(H)} \).

Let \( \hat{A} = \{ m \in \omega \mid \text{for some, hence for all, } n > m, q_n \Vdash m \in A^{(H)} \} \). Thus, \( q_n \Vdash A^{(H)} \cap n = \hat{A} \cap n \). Since \( \hat{A} \) is in \( V[H] \) and \( \mathcal{U} \) generates an ultrafilter there, we have a \( B \in \mathcal{U} \) included in or disjoint from \( \hat{A} \).

Suppose first that \( B \subseteq \hat{A} \). For every \( n \in \omega \), \( q_n \) forces \( A^{(H)} \cap n = \hat{A} \cap n \supseteq B \cap n \). That \( q_n \) extends \( r' \) and forces \( B \cap n \subseteq A^{(H)} \), being true in \( V[H] = V[G_k, G_{k'}] \), must be forced over \( V[G_k] \) by some \( s \in G_{k'} \). This implies that the condition \( (s, q_n) \in P_{\omega}/G_k \) forces (over \( V[G_k] \)) \( B \cap n \subseteq A^{(G_k')} \). We can arrange for \( s \) to be an extension of \( p' \upharpoonright k' \) as \( p' \upharpoonright k' \in G_{k'} \). But then \( (s, q_n) \) is an extension of \( r \) (as \( s \) extends \( p' \upharpoonright k' = r \upharpoonright [k, k+k') \) and \( s \) forces \( q_n \) to extend \( r' \) which extends \( r \upharpoonright [k+k', \omega) \)) forcing \( B \cap n \subseteq A^{(G_k')} \). That this can be done for every \( n \in \omega \) contradicts our supposition that \( k', G_{k'}, r \) constitute a counterexample to 4.3.

There remains the case that \( B \) is disjoint from \( \hat{A} \). Now, for every \( n \in \omega \), \( q_n \) extends \( r' \) and forces \( B \cap n \subseteq \omega - A^{(H)} = A^{(G_k)} \), that this can be done for every \( n \in \omega \) contradicts our supposition that \( k', G_{k'}, r \) constitute a counterexample to 4.3'.

These contradictions show that, if 4.3 fails then 4.3' must hold, and so we always get the desired extension of \( p \) forcing \( "B \subseteq A \) or \( B \cap A = \emptyset" \).

Thus, we may, and henceforth do, assume Hypothesis 4.3. Before constructing, under this hypothesis, the desired \( B \in \mathcal{U} \) and extension \( q \) of \( p \) forcing \( "B \subseteq A" \) (the alternative \( B \cap A = \emptyset \) was needed only to make 4.3 and 4.3' symmetric), we need some more preliminary information.

**Lemma 4.4.** If \( \mathcal{U} \) is a \( P \)-point and \( X_n \in \mathcal{U} \) for each \( n \in \omega \), then there exists \( Y \in \mathcal{U} \) such that, for infinitely many \( n \in \omega \), \( Y - n \subseteq X_n \).

**Proof.** As \( \mathcal{U} \) is a \( P \)-point, there exists \( Z \in \mathcal{U} \) almost included in each \( X_n \). Inductively define an increasing sequence of natural numbers \( n(l) \) by setting \( n(0) = 0 \) and choosing \( n(l+1) \) so large that \( Z - n(l+1) \subseteq X_{n(l)} \). Being an
ultrafilter, \( \mathcal{U} \) must contain either \( \bigcup_I [n(2l), n(2l + 1)) \) or its complement \( \bigcup_I [n(2l + 1), n(2l + 2)) \). Assume the latter; the other case is analogous. Then \( \mathcal{U} \) contains the set \( Y - Z \cap \bigcup_I [n(2l + 1), n(2l + 2)) \). Since \( Y \) is disjoint from \([n(2l), n(2l + 1))\) for every \( l \), we have
\[
Y - n(2l) = Y - n(2l + 1) \subseteq Z - n(2l + 1) \subseteq X_{n(2l)}.
\]
So \( Y \) is as required. \( \square \)

Fix a regular cardinal \( \chi \) big enough so that all the sets we shall need to consider belong to \( H(\chi) \). Fix a countable elementary submodel \( N \) of \( H(\chi) \) that contains the ultrafilter \( \mathcal{U} \), the forcing sequence \( \langle P_n, Q_n : n \in \omega \rangle \), the name \( A \), and the condition \( p \in P_\omega \) fixed earlier. Since \( N \) is countable and \( \mathcal{U} \) is a \( P \)-point, choose \( B^* \in \mathcal{U} \) such that, for all \( X \in \mathcal{U} \cap N \), \( B^* \) is almost included in \( X \). Let \( B \) be as in Hypothesis 4.3 with \( k = 0 \) (hence \( G_k \) trivial) and \( r = p \); replacing \( B^* \) with \( B^* \Delta B \), we can arrange that

4.5. For each \( n \in \omega \), some extension \( r \) of \( p \) in \( P_\omega \) forces \( "B^* \cap n \subseteq A" \).

Lemma 4.6. Let \( k \in \omega \), let \( q \in P_k \) be \( (N, P_k) \)-generic, let \( G_k \) be a \( V \)-generic subset of \( P_k \) containing \( q \), and let \( \langle X_n : n \in \omega \rangle \) be a sequence in \( N[G_k] \) of sets \( X_n \in \mathcal{U} \). Then, for infinitely many \( n \in \omega \), \( B^* - n \subseteq X_n \).

Proof. Since the forcing \( P_k \) is proper, [7, III.2.11] tells us that \( N[G_k] \) is an elementary submodel of \( H(\chi)^{V[G_k]} \). Therefore, it is true in \( N[G_k] \) that \( \mathcal{U} \) generates a \( P \)-point. Applying Lemma 4.4 in \( N[G_k] \), we obtain \( Y \) in \( \mathcal{U} \cap N[G_k] \) such that \( Y - n \subseteq X_n \) for infinitely many \( n \). Note that we can take \( Y \) to be in \( \mathcal{U} \), not merely in the ultrafilter in \( N[G_k] \) generated by \( \mathcal{U} \), by shrinking \( Y \) if necessary. Since \( Y \in \mathcal{U} \cap N[G_k] \), we have, by [7, III.2.12(c)], \( Y \in N \). Therefore \( B^* \) is almost included in \( Y \). For the infinitely many \( n \) that satisfy \( Y - n \subseteq X_n \) and are larger than all the (finitely many) elements of \( B^* - Y \), we have \( B^* - n \subseteq X_n \). \( \square \)

We shall complete the proof of Theorem 4.1 by constructing an extension \( q \) of \( p \) in \( P_\omega \) forcing \( "B^* \subseteq A" \). The construction is an inductive one, producing one component of \( q \) at a time. After \( k \) steps, we shall have an approximation \( p^k \) to \( q \), correct in the first \( k \) components. In detail, we shall define a sequence of conditions \( p^k \in P_\omega \), starting with \( p^0 = p \), and satisfying, for all \( k \in \omega \),

1. \( p^{k+1} \) is an extension of \( p^k \), and \( p^{k+1} \upharpoonright k = p^k \upharpoonright k \),
2. \( p^k \upharpoonright "B^* \cap k \subseteq A" \),
3. \( p^k \upharpoonright k \) is \( (N, P_k) \)-generic,
4. \( p^k \upharpoonright k \Vdash "p^k \upharpoonright [k, \omega) \in N[G_k]\" \), and
5. \( p^k \upharpoonright k \Vdash "for every \( n \in \omega \), there is an extension \( t \) of \( p^k \upharpoonright [k, \omega) \) in \( P_\omega / G_k \) forcing \( "B^* \cap n \subseteq A^{(G_k)} \" \)."

Before proceeding with the construction, we make a few explanatory remarks.
First, as indicated by the context, \( \Vdash \) refers to forcing with \( P_\omega \) in (2) and to forcing with \( P_k \) in (4) and (5); the word 'forcing' in (5) refers to forcing with \( P_\omega / G_k \).

Second, the choice of \( p^0 \) as \( p \) is consistent with requirements (1) to (5). Indeed, (1) to (3) are trivial for \( k = 0 \) (as \( P_0 \) is trivial), (4) says \( p \in N \) which is true by our choice of \( N \), and (5) is exactly fact 4.5 above. Third, (1) implies that, for each \( k \), \( p'' \upharpoonright k \) is independent of \( n \) once \( n \equiv k \). Since \( P_\omega \) is an inverse limit, we can define \( q \in P_\omega \) by

\[
q \upharpoonright k = p^k \upharpoonright k;
\]

then \( q \) extends every \( p^k \), in particular \( p \), and it forces "\( B^* \subseteq A \)" by (2). Thus, \( q \) will be as desired, so the proof will be complete once we construct the \( p^k \)'s.

Suppose that, for a certain \( k \), \( p^k \) has been constructed and satisfies (2) through (5). We wish to construct \( p^{k+1} \) so that the induction hypotheses continue to hold. By (1) we have no choice about the first \( k \) components; \( p^{k+1} \upharpoonright k \) must equal \( p^k \upharpoonright k \). The rest of \( p^{k+1} \), which we must construct, is best viewed as consisting of two parts, the component \( r = p^{k+1}(k) \) and the rest \( s = p^{k+1} \upharpoonright [k+1, \omega) \). Here

\[
p^k \upharpoonright k \Vdash "r \in Q_k \text{ and } r \Vdash 's \in (P_\omega / G_k)/H'"\]

where \( G_k \) and \( H \) are the names of the canonical generic subsets of \( P_k \) and \( Q_k \) respectively. In terms of \( r \) and \( s \), the five requirements on \( p^{k+1} \) are as follows.

1. \( p^k \upharpoonright k \Vdash "(r, s) \text{ extends } p^k \upharpoonright [k, \omega)""\),
2. \( p^k \upharpoonright k \Vdash "(r, s) \Vdash 'B^* \cap (k + 1) \subseteq A^{(G_k)}'""\),
3. \( (p^k \upharpoonright k, r) \) is \( (N, P_{k+1}) \)-generic,
4. \( p^k \upharpoonright k \Vdash "r \Vdash 's \in N[G_{k+1}]'""\), and
5. \( p^k \upharpoonright k \Vdash "r \Vdash 'for every } n \in \omega, \text{ there is an extension } t \text{ of } s \text{ in } P_\omega / G_{k+1} \text{ forcing } \langle B^* \cap n \subseteq A^{(G_{k+1})} \rangle'""\).

We can make several simplifications here. The only requirement not of the form "\( p^k \upharpoonright k \Vdash '\cdots' \)" is (3), which follows from

\[
p^k \upharpoonright k \Vdash "r \text{ is } (N[G_k], Q_k)-\text{generic}, \quad (3')\]

by [7, p. 91], since \( p^k \upharpoonright k \) is \((N, P_k)\)-generic. To satisfy all these requirements, it suffices to work in a generic extension \( V[G_k] \) where \( G_k \subseteq P_k \) is a \( V \)-generic set containing \( p^k \upharpoonright k \); if we can produce \( r \) and \( s \) in \( V[G_k] \), having all the five properties that we want \( p^k \upharpoonright k \) to force, then the "forcing = truth" and maximum principles produce \( P_k \)-names forced by \( p^k \upharpoonright k \) to denote such \( r \) and \( s \). Henceforth, we work in \( V[G_k] \), with \( G_k \) as above, and we adopt the notational convention that the value, with respect to \( G_k \), of a \( P_k \)-name such as \( Q_k \) will be denoted by the same symbol in lightface, e.g., \( Q_k \). Names that were not boldface to begin with, like \( p^k(k) \), will have their \( G_k \)-values denoted by the same symbol. (This ambiguity seems to cause less difficulty than any attempt to resolve it would.)

In \( V[G_k] \), we seek \( r \) and \( s \) such that

1. \( (r, s) \text{ extends } p^k \upharpoonright [k, \omega) \text{ in } P_\omega / G_k \),
2. \( (r, s) \Vdash "B^* \cap (k + 1) \subseteq A^{(G_k)}", \)
(3') \( r \) is \((N[G_k], Q_k)\)-generic,
(4) \( r \Vdash "s \in N[G_k][H]" \), and
(5) \( r \Vdash "for every \( n \in \omega \), there is an extension \( t \) of \( s \) in \((P_\omega/G_k)/H \) forcing \( B^* \cap n \subseteq A^{(G_k)(H)} \)".
(Here \( H \) is, in accordance with our convention, the \( G_k \)-value of \( H \). Though it is lightface, it is still a name, the \( Q_k \)-name of the canonical generic subset of \( Q_k \).)

We simplify the problem further by replacing (3') and (4) by
(6) \( (r, s) \in N[G_k] \).

(6) implies (4) by definition of \( N[G_k][H] \). It does not imply (3'), but if we have (6), or just \( r \in N[G_k] \), then we can obtain (3') by extending \( r \), since \( Q_k \) is proper and \( N[G_k] \) is an elementary submodel of \( H^{V[G_k]}(\chi) \) by [7, III.2.11]. Of course extending \( r \) preserves the truth of (1), (2), (4), and (5).

Thus, our goal is to produce some \( r \) and \( s \) satisfying requirements (1), (2), (5), and (6) in \( V[G_k] \). Because \( p^k \upharpoonright k \in G_k \), induction hypotheses (2), (4), and (5) imply the following facts, in which we have abbreviated \( p^k \upharpoonright [k, \omega) \) as \( \bar{p} \) and \( A^{(G_k)} \) as \( \bar{A} \).

4.7. \( \bar{p} \Vdash "B^* \cap k \subseteq \bar{A}" \).

4.8. \( \bar{p} \in N[G_k] \).

4.9. For every \( n \in \omega \), there is an extension \( t \) of \( \bar{p} \) in \( P_\omega/G_k \) forcing "\( B^* \cap n \subseteq \bar{A}" \).

We obtain \( r \) and \( s \) by first constructing a multitude of candidates and then selecting appropriate ones. For each finite sequence \( \eta \) of zeros and ones, choose, if possible, an extension \( p_\eta \) of \( \bar{p} \) in \( P_\omega/G_k \) that forces \( \eta \) to be an initial segment of (the characteristic function of) \( \bar{A} \), i.e., forcing

"\( \bar{A} \cap \text{length}(\eta) = \{i < \text{length}(\eta) \mid \eta(i) = 1\} \)."

Let \( T \) be the set of those \( \eta \) for which \( p_\eta \) exists. Clearly \( T \) is a tree (closed under initial segments). Being defined from \( P_\omega/G_k, \bar{A} \), and \( \bar{p} \), all of which are in \( N[G_k] \) (as \( P_\omega, A \in N \) and 4.8 holds), \( T \) belongs to \( N[G_k] \). By 4.9, every initial segment of (the characteristic function of) \( B^* \) is majorized componentwise by at least one \( \eta \in T \); in particular, \( T \) has infinite height.

For each \( \eta \in T \), consider an arbitrary \( V[G_k] \)-generic \( H \subseteq Q_k \) with \( p_\eta(k) \in H \). We wish to apply hypothesis 4.3 with \( k + 1 \) in place of \( k \), \( G_k \star H \) in place of \( G_k \), and \( p_\eta \upharpoonright [k + 1, \omega) \) in place of \( r \). To see that the hypothesis is applicable, we must check that \( p \upharpoonright (k + 1) \in G_k \star H \) and that \( p_\eta \upharpoonright [k + 1, \omega) \) extends \( p \upharpoonright [k + 1, \omega) \). Both verifications are easy because \( G_k \) contains both \( p^k \upharpoonright k \), an extension of \( p \upharpoonright k \), and conditions forcing \( p_\eta \) to extend \( \bar{p} \) which in turn extends \( p \upharpoonright [k, \omega) \).

Applying 4.3, we obtain a set \( C_\eta \in \mathcal{G} \) such that

4.10. For each \( n \in \omega \), some extension \( t \) of \( p_\eta \upharpoonright [k + 1, \omega) \) in \((P_\omega/G_k)/H \) forces "\( C_\eta \cap n \subseteq \bar{A}(H) \)".
Since this statement is true in $V[G_k][H]$, it is forced over $V[G_k]$ by some $r_\eta \in H$; since $p_\eta(k) \in H$, we can arrange that $r_\eta$ extends $p_\eta(k)$.

Since $N[G_k]$ is an elementary submodel of $H^{V[G_k]}(\chi)$, we can and do take $\langle p_\eta, C_\eta, r_\eta : \eta \in T \rangle$ to be in $N[G_k]$.

For each $m \in \omega$, let $C_m$ be the intersection of the $C_\eta$'s for $\eta \in T$ of length $m$. As an intersection of finitely many sets in $\mathcal{U}$, $C_m$ is also in $\mathcal{U}$. Furthermore, the sequence $\langle C_m : m \in \omega \rangle$ is in $N[G_k]$. By Lemma 4.6, with $p^k \upharpoonright k$ in place of $q$, there are infinitely many $m \in \omega$ with $B^* - m \subseteq C_m$. Fix such an $m > k$, and fix an $\eta \in T$ of length $m$ majorizing the characteristic function of $B^*$ (restricted to $m$) componentwise, i.e., $B^* \cap m \subseteq \{i < m \mid \eta(i) = 1\}$. Recall that such an $\eta$ exists by (4.9). We set $r = r_\eta$ and $s = p_\eta \upharpoonright [k + 1, \omega)$. We arranged in the preceding paragraph that these be in $N[G_k]$, so (6) holds. Since $r$ extends $p_\eta(k)$, $(r, s)$ extends $p_\eta$ which extends $\bar{p} = p^k \upharpoonright [k, \omega)$, so (1) holds. Since $m > k$,

$$B^* \cap (k + 1) \subseteq B^* \cap m \subseteq \{i < m \mid \eta(i) = 1\}$$

and this last set is forced to be a subset of $\bar{A}$ by $p_\eta$, hence also by the extension $(r, s)$. Since $\bar{A} = A^{(G_k)}$, we have verified requirement (2). It remains to check (5):

$$r \Vdash \text{"For every } n \in \omega, \text{ there is an extension } t \text{ of } s \text{ in } (P_\alpha/G_k)[H] \text{ forcing } \langle B^* \cap n \subseteq \bar{A}^{(H)} \rangle."$$

If $B^*$ were changed to $C_\eta$, the resulting statement would be true simply by definition of $r = r_\eta$. Now $m$ was chosen so that

$$B^* - m \subseteq C_m \subseteq C_\eta.$$

Thus, to obtain (5), i.e., to change $C_\eta$ back to $B^*$, it suffices to prove that

$$r \Vdash \langle B^* \cap m \subseteq \bar{A}^{(H)} \rangle."$$

But $B^* \cap m \subseteq \{i < m \mid \eta(i) = 1\}$, and $r$ extends $p_\eta(k)$, so it more than suffices to prove

$$p_\eta(k) \Vdash \langle s \Vdash \{i < m \mid \eta(i) = 1\} \subseteq \bar{A}^{(H)} \rangle."$$

Since $s = p_\eta \upharpoonright [k + 1, \omega)$, this reduces to

$$p_\eta \Vdash \langle \{i < m \mid \eta(i) = 1\} \subseteq \bar{A} \rangle,"$$

which is true by definition of $p_\eta$.

This completes the proof that $r$ and $s$ have the required properties; it thus completes the proof of Thoerem 4.1. \qed

5. The consistency of (A)

Assume the continuum hypothesis in the ground model $V$. Let $\langle P_\alpha, Q_\alpha : \alpha < \kappa_2 \rangle$ be a countable support forcing iteration in which each $Q_\alpha$ denotes the basic forcing $Q$ of Section 2 in the forcing extension $V^{\mathcal{F}_\kappa}$. Let $P = P_{\kappa_2}$ be the (direct)
limit of this iteration, and let $G$ be a $V$-generic subset of $P$. Our goal in this section is to prove

**Theorem 5.1.** If $\mathcal{U}_1$ and $\mathcal{U}_2$ are non-principal ultrafilters in $V[G]$, then there exists a finite-to-one $f : \omega \to \omega$ in $V[G]$ such that $f(\mathcal{U}_1) = f(\mathcal{U}_2)$.

Before embarking on the proof, we fix some notation and obtain some preliminary information about $P$ and the forcing extension $V[G]$. For each $\alpha < \aleph_2$, let $G_\alpha$ be the restriction of $G$ to $P_\alpha$, a $V$-generic subset of $P_\alpha$. Let $Q_\alpha \in V[G_\alpha]$ be the $G_\alpha$-value of the name $Q_\alpha$, and let $H_\alpha$ be the $V[G_\alpha]$-generic subset of $Q_\alpha$ given by $G_\alpha$; thus $G_{\alpha+1} = G_\alpha \ast H_\alpha$. Since $Q_\alpha$ is the $Q$ of Section 2, as calculated in $V[G_\alpha]$, all the results about forcing with $Q$ in Sections 2 and 3 are applicable to the forcing by $Q_\alpha$ over $V[G_\alpha]$ that produces $V[G_{\alpha+1}]$.

By Proposition 2.5, $P_\alpha$ is "proper" for every $\alpha < \aleph_2$. By [7, III.3.2], each $P_\alpha$, including $P_{\aleph_2} = P$, is proper.

By Theorem 3.3, every $P$-point in $V[G_\alpha]$ generates a $P$-point in $V[G_{\alpha+1}]$. Using this fact at successor stages and Theorem 4.1 at limit stages, one sees by induction on $\alpha$ that

5.2. Every $P$-point in $V$ generates $P$-points in all $V[G_\alpha]$ ($\alpha < \aleph_2$) and in $V[G]$. 

The definition of $Q$ in Section 2 makes it obvious that $Q$ has the cardinality of the continuum; thus $P_\alpha$ is "$Q_\alpha$ has the cardinality of the continuum". Since we have assumed the continuum hypothesis in the ground model $V$, we find by [7, III.4.1.], that:

5.3. For every $\alpha < \aleph_2$, $P_\alpha$ has a dense subset of cardinality $\leq \aleph_1$. So $P_\alpha \models 2^{\aleph_0} = \aleph_1$.

5.4. $P$ satisfies the $\aleph_2$ chain condition.

(In [7, III.4.1], a cardinality bound on $Q_j$ in $V^{P_i}$ is assumed; in the present context, this seems to require that we know the continuum hypothesis in $V^{P_i}$ a priori, so the proof seems circular. But in fact, in order to get the continuum hypothesis in $V^{P_i}$ one needs the cardinality bounds only for $Q_j$ in $V^{P_i}$ for $j < i$, so the apparent circularity reduces to a legitimate induction.) The chain condition (5.4) implies that forcing with $P$ preserves all cardinals $\geq \aleph_2$; properness implies that it also preserves $\aleph_1$. Thus,

5.5. Cardinals are absolute between $V$ and $V[G]$.

We shall also need the following immediate consequence of [7, V.4.4].

5.6. For any real $x \in V[G]$, there exists $\alpha < \aleph_2$ with $x \in V[G_\alpha]$, and the smallest such $\alpha$ has cofinality $\leq \omega$. 
The next lemma is implicit in the proofs of III.3.2 and III.4.1 in [7]. For all \( \alpha \leq \aleph_2 \), we consider \( P_\alpha \)-names \( x \) of reals (viewed as functions \( \omega \to 2 \)) as being specified by giving, for each \( n \in \omega \), a maximal antichain \( A(x, n) \) in \( P_\alpha \) and, for each \( p \in A(x, n) \), a value \( v(x, n, p) \in \{0, 1\} \) such that \( p \Vdash \langle x(n) = v(x, n, p) \rangle \). It is well known that every name of a real is equivalent, in the sense of equality forced by all conditions, to one of this sort. When we are interested only in conditions extending a particular \( q \), then the antichains \( A(x, n) \) need to be maximal only in the weaker sense that no extension of \( q \) can be added to them; we then refer to \( x \) as a name for a real relative to \( q \). We call such a name \( x \) \textit{hereditarily countable} if, for each \( n \), \( A(x, n) \) is countable and all the \( P_\alpha \)-names of reals, occurring in the conditions \( p_\beta \) constituting any \( p = \langle p_\beta : \beta < \alpha \rangle \in A(x, n) \), are hereditarily countable.

As usual, let \( \chi \) be a regular cardinal so large that \( H(\chi) \) contains all sets of interest to us.

**Lemma 5.7.** Let \( N \) be a countable elementary submodel of \( H(\chi) \) that contains \( P \), and let \( p \) be an \((N, P)\)-generic condition. Then for every ordinal \( \alpha \leq \aleph_2 \) in \( N \) and every \( P_\alpha \)-name \( x \in N \) for a real, there is a hereditarily countable \( P_\alpha \)-name \( y \) relative to \( p \) such that \( p \Vdash "x = y" \).

**Proof.** The \((N, P)\)-genericity of \( p \) implies that \( p \upharpoonright \alpha \) is \((N, P_\alpha)\)-generic for all \( \alpha \in N \). The lemma is proved by induction on \( \alpha \). To obtain \( y \), first replace each of the antichains \( A(x, n) \) by its intersection with \( N \). Genericity ensures that these intersections are maximal relative to \( p \upharpoonright \alpha \), in the sense described above, and that the name \( x' \) obtained in this way is forced by \( p \upharpoonright \alpha \) to equal \( x \). Since \( N \) is countable, the antichains \( A(x', n) = A(x, n) \cap N \) are countable. If \( q = \langle q_\beta : \beta < \alpha \rangle \) is in one of these antichains, hence in \( N \), then \( N \) also contains an enumeration, in an \( \omega \)-sequence, of all the (countably many, as the iteration has countable support) non-trivial components \( q_\beta \). Thus, each of these components \( q_\beta \) is in \( N \) and can therefore, by induction hypothesis, be replaced by a hereditarily countable \( P_\beta \)-name. Doing this simultaneously for all such \( q \) and \( \beta \), we obtain the desired name \( y \). \( \square \)

**Corollary 5.8.** If \( x \) is a \( P \)-name for a real, then the set of conditions that force \( x = y \) for some hereditarily countable \( y \) is dense.

**Proof.** Given \( x \) and an arbitrary condition \( q \), let \( N \) be a countable elementary submodel of \( H(\chi) \) containing \( P, q, \) and \( x \). As \( P \) is proper, \( q \) has an \((N, P)\)-generic extension \( p \). By the lemma, \( p \Vdash "x = y" \) for some hereditarily countable \( y \). \( \square \)

**Lemma 5.9.** For each \( \alpha < \aleph_2 \), there are only \( \aleph_1 \) hereditarily countable \( P_\alpha \)-names, relative to any particular condition.
Proof. We proceed by induction on $\alpha$. A hereditarily countable name $x$ is determined by countably many countable antichains $A(x, n)$ and the function $v(x, n, p)$. Since we are assuming the continuum hypothesis, and therefore $\kappa^\omega = \kappa_1$, it suffices to check that there are at most $\kappa_1$ conditions $p = \langle p_\beta : \beta < \alpha \rangle$ that can occur in these antichains. Since our iteration uses countable supports, we need only check that there are, for each $\beta$, at most $\kappa_1$ possibilities for $p_\beta$. But $p_\beta$ is required to be a hereditarily countable name, so the induction hypothesis gives us what we need. □

Lemma 5.10. Let $\mathcal{F} \in V[\mathcal{G}]$ be a set of reals. There is an $\aleph_1$-closed unbounded set of ordinals $\alpha < \kappa_2$ for which $\mathcal{F} \cap V[G_\alpha] \in V[G_\alpha]$.

Proof. Fix a $P$-name $\mathcal{F}$ for $\mathcal{F}$. For each $\gamma < \kappa_2$ and each hereditarily countable $P_\gamma$-name $y$ for a real, consider a maximal antichain $B(\gamma)$ of conditions that decide whether $y \in \mathcal{F}$. By 5.4 and 5.9, there are only $\kappa_1$ conditions altogether in these antichains. Since $P$ is the direct limit of the $P_\gamma$'s, there is a single $P_\beta$ ($\gamma < \beta < \kappa_2$) containing all these conditions. We claim that $\mathcal{F} \cap V[G_\gamma] \in V[G_\beta]$. Indeed, we claim that $\mathcal{F} \cap V[G_\gamma] = \{y^{(G_\gamma)} : y$ is a hereditarily countable $P_\gamma$-name and some $p \in G_\beta$ forces $y \in \mathcal{F}\}$. To see this, note first that the '⊆' direction is obvious. For the converse, consider any element of $\mathcal{F} \cap V[G_\gamma]$. By Corollary 5.8 and genericity of $G$, it has a hereditarily countable name $y$. By the choice of $\beta$ and genericity of $G_\beta$, some condition $p$ in $G_\beta$ decides whether $y \in \mathcal{F}$. Since $y^{(G)} = y^{(G_\gamma)} \in \mathcal{F} = \mathcal{F}^{(G)}$, no condition in $G_\beta \subseteq G$ can force $y \notin \mathcal{F}$, so $p$ must force $y \in \mathcal{F}$. Therefore $y^{(G_\gamma)} \in \mathcal{F}$, as desired.

For each $\gamma < \kappa_2$, let $h(\gamma)$ be a $\beta < \kappa_2$ as in the preceding paragraph. Let $C$ be the set of ordinals $< \kappa_2$ that are closed under $h$. Then $C$ is closed and unbounded, so the set of ordinals in $C$ with cofinality $\kappa_1$ is $\kappa_1$-closed and unbounded. Consider any such ordinal $\alpha$. By 5.6, each real in $V[G_\alpha]$ is already in $V[G_\gamma]$ for some $\gamma < \alpha$. Therefore, by choice of $h$, $\mathcal{F} \cap V[G_\alpha] = \bigcup_{\gamma < \alpha} \{y^{(G_\gamma)} : y$ is a hereditarily countable $P_\gamma$-name and some $p \in G_{h(\gamma)}$ forces $y \in \mathcal{F}\}$, which is in $V[G_\alpha]$ since $h(\gamma) < \alpha$ for all $\gamma < \alpha$. □

Proof of Theorem 5.1. For each $\alpha < \kappa_2$, let $W_\alpha = \bigcup \{w : (w, T) \in H_\alpha\}$, and, for $x \in \omega$, $f_\alpha(x) = \text{the number of elements of } W_\alpha \text{ that are } \preceq x$. 

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Thus, these are the $W$ and $f$ of Section 3, in the context of the $Q_\alpha$-forcing that adjoins $H_\alpha$ to $V[G_\alpha]$. Each $f_\alpha$ is a finite-to-one function \( \omega \to \omega \), and we shall find, for each pair of ultrafilters, $\mathcal{U}_1$ and $\mathcal{U}_2$, in $V[G]$, an $\alpha$ such that $f_\alpha(\mathcal{U}_1) = f_\alpha(\mathcal{U}_2)$.

Since we are assuming the continuum hypothesis in the ground model, there is a $P$-point in $V$, and it generates a $P$-point $\mathcal{U}_0$ in $V[G]$ by 5.2. To complete the proof, it suffices to find, for any ultrafilter $\mathcal{U}_1$ in $V[G]$, an $\aleph_1$-closed unbounded set of $\alpha$'s $< \aleph_2$ for which $f_\alpha(\mathcal{U}_1) = f_\alpha(\mathcal{U}_0)$. Indeed, if we do this for both $\mathcal{U}_1$ and $\mathcal{U}_2$, then, since the two $\aleph_1$-closed unbounded sets intersect (in an $\aleph_1$-closed unbounded set) there exists (an $\aleph_1$-closed unbounded set of) $\alpha < \aleph_2$ with $f_\alpha(\mathcal{U}_1) = f_\alpha(\mathcal{U}_0) = f_\alpha(\mathcal{U}_2)$.

So let an ultrafilter $\mathcal{U}_1 \in V[G]$ be given. By Lemma 5.10, there is an $\aleph_1$-closed unbounded set of $\alpha < \aleph_2$ such that $\mathcal{U}_1 \cap V[G] \subseteq V[G_\alpha]$. For such an $\alpha$, $\mathcal{U}_1 \cap V[G_\alpha]$ is clearly an ultrafilter in the sense of $V[G_\alpha]$. So is $\mathcal{U}_0 \cap V[G_\alpha]$, since the $P$-point in $V$ that generates $\mathcal{U}_0$ also generates $P$-points in every $V[G_\alpha]$, by 5.2. By Corollary 3.5, applied to $Q_\alpha$-forcing over $V[G_\alpha]$,

$$f_\alpha(\mathcal{U}_0 \cap V[G_\alpha]) = f_\alpha(\mathcal{U}_1 \cap V[G_\alpha]) \in f_\alpha(\mathcal{U}_1).$$

Since $\mathcal{U}_0 \cap V[G_\alpha]$ generates $\mathcal{U}_0$, $f_\alpha(\mathcal{U}_0 \cap V[G])$ generates $f_\alpha(\mathcal{U}_0)$. Therefore $f_\alpha(\mathcal{U}_0) \subseteq f_\alpha(\mathcal{U}_1)$. Finally, since $f_\alpha(\mathcal{U}_0)$ is an ultrafilter and $f_\alpha(\mathcal{U}_1)$ a filter, it follows that $f_\alpha(\mathcal{U}_0) = f_\alpha(\mathcal{U}_1)$.

We conclude this section with some additional information about the model $V[G]$.

**Theorem 5.2.** In $V[G]$, the following are true.

(a) $2^{\aleph_0} = \aleph_2$.

(b) There is a $P$-point generated by $\aleph_1$ sets.

(c) $b = \aleph_1$.

(d) $d = \aleph_2$.

(e) $s = \aleph_2$.

**Proof**

(a) By 5.6, all reals of $V[G]$ are in some $V[G_\alpha]$, and by 5.3 each of the $\aleph_2$ $V[G_\alpha]$'s contributes only $\aleph_1$ reals. So in $V[G]$, $2^{\aleph_0} \leq \aleph_2$. On the other hand, each step in the iteration adds a new real (cf. 3.1) and cardinals are preserved (5.5), so $2^{\aleph_0} = \aleph_2$ in $V[G]$.

(b) The $\mathcal{U}_0$ in the proof of Theorem 5.1 is generated by sets in the ground model, and there are only $\aleph_1$ such sets.

(c) $b$ is always uncountable, and, by a theorem of Solomon [9], no ultrafilter is generated by fewer than $b$ sets. So (c) follows from (b).

(d) By 5.6, any family of $\aleph_1$ reals is included in $V[G_\alpha]$ for some $\alpha < \aleph_2$. But then by 3.6 that family fails to dominate the function

$$n \mapsto \text{the } n\text{th element of } W_\alpha.$$
(e) Again, any family of $\mathbb{N}_1$ subsets of $\omega$ lies in some $V[G_\alpha]$ and therefore, by 3.1, fails to split $W_\alpha$. □

6. The consistency of (B)

In this section, we show how to modify the previous construction so as to obtain a model containing both a simple $P_{\mathbb{N}_1}$-point and a simple $P_{\mathbb{N}_2}$-point.

As before, we assume the continuum hypothesis in the ground model $V$, and we let $G$ be a $V$-generic subset of the direct limit $P = P_{\mathbb{N}_0}$ of a certain countable support iteration $\langle P_\alpha, Q_\alpha : \alpha < \mathbb{N}_2 \rangle$. This iteration differs from the previous one in that, in $V^{P_\alpha}$, $Q_\alpha$ is not the $Q$ described in Section 2 but rather the subset

$$\{(w, T) \mid \text{for each } \beta < \alpha, \text{ there exists an extension } (w', T') \text{ of } (w, T) \text{ such that } ps(w', T') \subseteq^* W_\beta\}.$$ 

Here, $\subseteq^*$ denotes inclusion modulo a finite set, and $W_\beta$ is the $P_{\beta+1}$-name (hence also $P_\alpha$-name) for the subset $\bigcup \{w \mid (w, T) \in G_\beta\}$ adjoined to $V^{P_\beta}$ by forcing with $Q_\beta$. The ordering of $Q_\alpha$ is the same as for $Q$.

Let $G$ be a $V$-generic subset of $P$, and, as in the previous section, let $G_\alpha$ and $H_\alpha$ be the corresponding $V$-generic subset of $P_\alpha$ and $V[G_\alpha]$-generic subset of $Q_\alpha = Q^{(G_\alpha)}_\alpha$, respectively.

**Theorem 6.1.** In $V[G]$, there exist simple $P_{\mathbb{N}_1}$-points and simple $P_{\mathbb{N}_2}$-points.

Before starting the proof of this theorem, we need some preliminary information about the forcing notions involved.

To say that a condition $(w, T)$ in $Q$ has an extension $(w', T')$ with $ps(w', T') \subseteq^* X$ implies that we can obtain, by composing relations in $(w, T)$ a condition $(w, T^*)$ such that $Dp((t^*_n)_X)$ is an unbounded function of $n$. To see this, note that any extension $(w', T')$ is obtainable from a composing-relations extension by shrinking relations and fixing values. These last two operations cannot convert a $(w, T^*)$ with $Dp((t^*_n)_X)$ bounded to a $(w', T')$ with $Dp((t^*_n)_X)$ unbounded, but this unboundedness is needed if we are to have $ps(w', T') \subseteq^* X$. On the other hand, if $(w, T)$ has a composing-relations extension $(w, T^*)$, with $Dp((t^*_n)_X)$ unbounded, then we can find, for any prescribed $m$, an $n$-extension $(w, T')$ of $(w, T)$ with $ps(w, T') \subseteq^* X$. To see this, first note that $T^*$ can be taken to agree with $T$ in the first $m$ components, since only the behavior for large $n$ of $t^*_n$ matters. Then compose relations again, beyond $t^*_n$, to arrange that $Dp((t^*_n)_X) \to \infty$. Finally replace $t^*_n$ with $t'_n = (t^*_n)_X$ for all $n \geq m$. Observe also that, if $Y \subseteq^* X$, and $Dp((t^*_n)_Y)$ is unbounded, then $Dp((t^*_n)_Y)$ is also unbounded, because this unboundedness is preserved both by composition of relations and by shrinking $t^*_n$ to $(t^*_n)_X$.

**Lemma 6.2.** If $\beta < \gamma < \mathbb{N}_2$, then $W_\gamma \subseteq^* W_\beta$. 


Proof. We show by induction on \( \alpha \) that \( \beta < \gamma < \alpha \) implies \( W_\gamma \subseteq^* W_\beta \). It suffices to deduce, from this induction hypothesis, that \( W_\beta \subseteq^* W_\alpha \) for all \( \beta < \alpha \). We work in \( V[G_\alpha] \) and consider any \( (w, T) \in Q_\alpha \). Given any \( \beta < \alpha \), we have an extension \( (w', T') \in Q \) with \( ps(w', T') \subseteq^* W_\beta \). By induction hypothesis, it follows that \( ps(w', T') \subseteq^* W_\delta \) for all \( \delta < \beta \). The induction hypothesis and the remark preceding the lemma show that we can construct \( (w', T') \) so that it has extensions satisfying \( ps(w'', T'') \subseteq W_\gamma \) for any prescribed \( \gamma \in [\beta, \alpha) \). Thus, we can construct \( (w', T') \) so that it is in \( Q_\alpha \). Thus, we have shown that, for each \( \beta < \alpha \), the set

\[
\{(w', T') \in Q_\alpha \mid ps(w', T') \subseteq^* W_\beta\}
\]

is dense in \( Q_\alpha \). Every condition in this dense set clearly forces "\( W_\alpha \subseteq^* W_\beta \)", so \( W_\alpha \subseteq^* W_\beta \). \( \Box \)

The core of the proof of Theorem 5.1 is the following proposition, which carries over to the present forcing construction some crucial properties of the construction in the preceding section.

**Proposition 6.3.** For each \( \alpha < \aleph_2 \) and each non-limit \( \beta < \alpha \):

(a) \( P_{\alpha+1}/P_\beta \) is a proper notion of forcing in \( V^{P_\beta} \).

(b) Every \( \Pp \)-point in \( V^{P_\beta} \) generates a \( \Pp \)-point in \( V^{(P_{\alpha+1})} \).

(c) For every \( A \subseteq \omega \) in \( V[G_\alpha] \), either \( W_\alpha \subseteq A \) or \( W_\alpha \subseteq^* \omega - A \).

**Proof.** We remark first that (a) is sufficient to ensure that \( P_\beta / P_\beta \) is proper, for \( \beta < \lambda \), when \( \lambda \) is a limit ordinal and \( \beta \) is not. This can be seen by looking carefully at the proof [7, III.3.2] of the preservation of properness under iteration; that proof essentially uses only (a), not properness of \( Q_\alpha \). See also [7, X.2.3 and X.2.6]. Similar remarks apply to the conjunction of (a) and (b); \( \Pp \)-points in \( V[G_\beta] \) for non-limit \( \beta \) generate \( \Pp \)-points in \( V[G_\gamma] \) for limit \( \lambda > \beta \), by essentially the same proof as in Section 4.

To prove 6.3, we proceed by induction on \( \alpha \).

If \( \alpha \) is a successor, say \( \alpha = \gamma + 1 \), then the forcing notion \( Q_\alpha \) in \( V[G_\alpha] \) consists of all \( (w, T) \) having an extension satisfying \( ps(w', T') \subseteq W_\gamma \), because of Lemma 6.2. By the proof of that lemma, this \( (w', T') \) can be taken to be in \( Q_\alpha \) also. Thus,

\[
Q'_\alpha = \{(w, T) \mid ps(w, T) \subseteq W_\gamma\}
\]

is dense in \( Q_\alpha \). The extensions of any \( (w, T) \in Q'_\alpha \) are the same in \( Q'_\alpha \) as in \( Q \). Thus, forcing with \( Q_\alpha \), with \( Q'_\alpha \), and with \( Q \) are all equivalent. Now (a) follows from the properness of \( P_{\gamma+1}/P_\beta = P_\alpha/P_\beta \) and Proposition 2.5. (b) follows from \( \Pp \)-point preservation from \( V[G_\beta] \) to \( V[G_\alpha] \) and Theorem 3.3. And (c) follows from Proposition 3.1.

If \( \alpha \) is a limit ordinal of cofinality \( \omega \), let \( (\gamma(n) : n \in \omega) \) be cofinal in \( \alpha \). \( Q_\alpha \) consists, by Lemma 2.2, of all \( (w, T) \) having, for each \( n \), an extension with
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As we saw before Lemma 2.2, each such $(w, T)$ has, for each $n \in \omega$, an $n$-extension $(w', T') \in Q_\alpha$ with $\psi(w', T') \subseteq^* W_{\gamma(n)}$. This fact allows us to do a fusion argument (cf. Proposition 2.3) to obtain an extension $(w'', T'')$ with $\psi(w'', T'') \subseteq^* W_{\gamma(n)}$ for all $n \in \omega$ simultaneously. Thus,

$$Q_\alpha' = \{(w, T) \mid \forall n \in \omega, \psi(w, T) \subseteq^* W_{\gamma(n)}\}$$

is dense in $Q_\alpha$. The extensions in $Q$ of any $(w, T) \in Q_\alpha'$ are all in $Q'$, so forcing with $Q_\alpha$, with $Q_\alpha'$, and with $Q$ are all equivalent. As in the successor case, this completes the proof.

Finally, we consider the case that $\alpha$ has cofinality $\omega_1$. Let $K_\beta$ be any $V$-generic subset of $P_\beta$. To prove (a), it suffices, since $K_\beta$ is arbitrary, to show that $P_{\alpha+1}/K_{\beta}$ is proper in $V[K_{\beta}]$. In $V[K_{\beta}]$, let $N$ be a countable elementary submodel of $H(\chi)$ containing $P_{\alpha}/K_{\beta}$, where $\chi$ is, as usual, a sufficiently large regular cardinal. Let a condition $(q^0, r^0) \in (P_{\alpha+1}/K_{\beta}) \cap N$ be given, where $q^0 \in P_{\alpha}/K_{\beta}$ and $q^0 \Vdash \text{"}r^0 \in Q_\alpha\text{"}$. Since $q^0$ has countable support and $\alpha$ has uncountable cofinality, $q^0 \in P_{\alpha}/K_{\beta}$ for some $\gamma < \alpha$. Since $N$ is an elementary submodel of $H(\chi)$, we can take $\gamma$ to be an element of $N$. Let $\mu$ be the supremum of $\alpha \cap N$. Since $N$ is countable and $\text{cf}(\alpha)$ is not, and since $\gamma \in N$, we have $\gamma < \mu < \alpha$. By the induction hypothesis and the first paragraph of this proof, $P_{\alpha}/K_{\beta}$ is proper. Let $q$ be an $(N, P_{\alpha}/K_{\beta})$-generic extension of $q^0$ in $P_{\alpha}/K_{\beta}$. We intend to define $p$ and $r$ such that $q \Vdash \text{"}p \in P_{\alpha}/G_{\mu}\text{"}$, $(q, p) \Vdash \text{"}r$ is an extension of $r^0$ in $Q_\alpha\text{"}$, and $(q, p, r)$ is $(N, P_{\alpha+1}/K_{\beta})$-generic. This will suffice to complete the proof of (a), since $(q, p, r)$ will then be an extension of $(q^0, r^0)$. To obtain such $p$ and $r$, we work with an arbitrary $V[K_{\beta}]$-generic $K_\mu \subseteq P_{\alpha}/K_{\beta}$ containing $q$, and we find, in $V[K_{\beta}, K_\mu]$, a $p \in P_{\alpha}/(K_{\beta} * K_\mu)$ and an $r$ such that $p \Vdash \text{"}r$ is an extension of $r^0$ in $Q_\alpha\text{"}$, and $(p, r)$ is $(N[K_\mu], P_{\alpha+1}/K_{\beta} * K_{\mu})$-generic. Carrying out this construction in $V[K_{\beta}]([p, K_\mu])$ with the canonical generic subset of $P_{\alpha}/K_{\beta}$ as $K_\mu$, we would then get the required names $p$ and $r$. (See [7, p. 91] for genericity of $(q, p, r)$.)

Let $(\tau_i : i < \alpha)$ be an enumeration of all the $(P_{\alpha+1}/K_{\mu})$-names in $N[K_\mu]$ of ordinals. Since $q^0 \Vdash \text{"}$\(r^0 \in Q_\alpha\text{"}$ and $K_\mu$ contains the extension $q$ of $q^0$, the $K_\mu$-value $r^0$ of $r^0$ is a member $\langle w^0, T^0 \rangle$ of $Q_\alpha$. Combining the fusion argument in the $\text{cf}(\alpha) = \omega$ case above with the fusion argument in the proof of Proposition 2.4 (by interleaving the steps of the two constructions), we obtain an extension $(w, S)$ of $r$ such that $\psi(w, S) \subseteq^* W_\delta$ for all $\delta \in \alpha \cap N$ (countably many $\delta$'s) and such that the conclusion of 2.4 holds. As in the proof of Proposition 2.5, the fusion argument can be carried out so that every finite part of it takes place in $N[K_\mu]$ and therefore the particular values of the $\tau_i$'s forced by extensions of $(w, S)$ all lie in $N[K_\mu]$. If $(w, S)$ were in $Q_\alpha$, we could use it as $r$ (i.e., use its standard name as $r$) and take $p$ to be trivial. Unfortunately, although we have $\psi(w, S) \subseteq^* W_\xi$ for all $\xi < \mu$ since $\alpha \cap N$ is cofinal in $\mu$, we do not know that $(w, S)$ has extensions with $\psi(w', S') \subseteq^* W_\delta$ for $\mu \leq \xi < \alpha$. In other words, $(w, S)$ is in $Q_\alpha$, but it is not forced by the trivial condition in $P_{\alpha}/K_\mu$ to be in $Q_\alpha$. This difficulty is remedied by using a non-trivial condition as $p$, namely $(w, S)$ itself. We define $p \in P_{\alpha}/K_\mu$ to be
(w, S) ∈ Qμ = Pμ+1/Kμ ⊆ Pα/Kμ (so p is the sequence consisting of (w, S) in Qμ followed by trivial conditions in Qζ for μ < ζ < α) and we define r to be the standard (Pα/Kμ)-name for (w, S). Since p forces all Wζ for μ < ζ < α to be subsets of ps(w, S) modulo finite sets, it forces r ∈ Qα, so p and r are as required to finish the proof of (a).

To prove (b), let ℱ be a P-point in V[Kp] and let A be a (Pα+1/Kp)-name for a subset of ω. Starting with any condition (q0, r0) in (Pα+1/Kp), we first proceed as in the proof of (a), using an N that contains ℱ and the name A, until we obtain (w, S) as there. Then we follow, in N[Kμ], the proof of Theorem 3.3 to obtain an extension (w', S') of (w, S) in Qμ forcing A to be included in or disjoint from some set in the ultrafilter in N[Kμ] generated by ℱ. (It does generate an ultrafilter, by the induction hypothesis, the first paragraph of this proof, and the fact that N[Kμ] is an elementary submodel of H(χ)(V[Kp,Kμ]) because Kμ contains the (N, Pμ/Kp)-generic condition q.) Finally, as in the proof of (a), we extend q by putting (w', S') in the μth and αth components and trivial conditions between them. The resulting condition in Pα+1/Kp forces A to include or be disjoint from a set in ℱ.

The proof of (c) is just like that of (b) except that, instead of using the proof of Theorem 3.3 to extend (w, S) to (w', S'), we use the proof of Proposition 3.1. This completes the proof of Proposition 6.3. □

**Corollary 6.4.** (a) P is proper.

(b) Every P-point in V generates a P-point in V[G].

**Proof.** See the first paragraph of the proof of Proposition 6.3. □

**Proof of Theorem 6.1.** Just as in Section 5, we see that P satisfies the κ2 chain condition by [7, III.4.1]. Therefore each real in V[G] is in V[Gα] for some α < κ2.

Since we are assuming the continuum hypothesis in the ground model V, there is a P-point, in fact (necessarily) a simple Pκ1-point, ℱ in V. By Corollary 6.4(b), ℱ generates a P-point ℱ̂ in V[G], and ℱ̂ is a simple Pκ1-point because it is generated by the same almost decreasing κ1-sequence as ℱ̂. (κ1 is absolute by Corollary 6.4(a) or because an ultrafilter ℱ̂ cannot be generated by countably many sets.)

Since every real in V[G] is in some V[Gα], Proposition 6.3(c) implies that the sets Wα (α < κ2) and the cofinite sets generate an ultrafilter in V[G]. By Lemma 6.2, the Wα constitute a strictly almost decreasing κ2-sequence, so this ultrafilter is a simple Pκ2-point. □

As in Section 5, the model V[G] is easily seen to satisfy 2κ0 = κ2. We can also show that b = κ1 and d = s = κ2 by the same method as in Section 5, but in fact these equations follow from the existence of simple Pκ1- and Pκ2-points, by work of Nyikos [5].
References