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# FERMIONS IN A TIME-PERIODIC SU(2) BACKGROUND: The eigenfunctions

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Isospinor fermions are studied in the presence of a covariantly constant, time-periodic SU(2) background field. The eigenfunction spectrum is derived. Part of the modes, termed transverse, decouple from the external field. The non-trivial content is given by the so-called scalar-longitudinal modes, which are quasi-doubly-periodic functions of time.

## 1. Introduction

Since the work of Schwinger [1] there has been great interest in electromagnetic processes in external fields. Many authors [2] have studied in detail the abelian constant uniform field and the plane wave field, where the one-particle Green functions for matter fields can be solved exactly. The most celebrated result is that a constant electric field can create real pairs and therefore the vacuum is unstable. In general one is also interested in the properties of the mass and polarization operators of an electron and a photon in the external and radiation fields. Indeed the problem of calculating synchrotron radiation of photons is essentially equivalent to extracting the imaginary part of the second-order elastic scattering amplitude of the electron [3].

Very few of these results have been generalized to non-abelian gauge theories, despite the role they have played in the theory of elementary particles in recent years. To avoid confusion we try to elucidate this point. An impressive body of results has been obtained in the non-abelian theory but almost all of them involve abelian-like configurations which are solutions of the Maxwell equations embedded within the non-abelian Lie algebra with a fixed orientation. Therefore in this class of problems it is relatively easy to diagonalize the part of the lagrangian quadratic in quantum fluctuations with respect to color. Basically once the direction $V^a$ of the external field in color space is given we only have to find the subalgebra of SU($N$)

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whose elements commute with $V$ and the orthogonal complement What is missing is a complete analysis of the problem for a genuinely non-abelian solution Given such a configuration what are the states which diagonalize the lagrangian? For instance in an external SU(2) field with both neutral and charged components an electron can scatter and transform into a neutrino and what we want are the full propagators (to all orders in the external field) for ee, $\nu\nu$ and e$\nu$ transitions Also we would like to compute electroweak or even strong synchrotron radiation (e $\rightarrow$ W + $\nu$ or q $\rightarrow$ q + g) in non-abelian background fields Even if they turn out to be negligible effects for present day phenomenology their interest is connected with the quest for going beyond the framework of perturbation theory The problem with non-abelian theory is that we have very little available information at the level of classical solutions The most obvious candidate would be a constant field $F^a_{\mu\nu}$ The classification of these fields is due to Brown, Coleman and Weisberger [4] who have proved that they can be produced either by an abelian vector potential linear in $x^\mu$ or by a constant non-abelian vector potential For the second case Brown and Weisberger [4] have been able to calculate the vacuum polarization induced by matter fields However as noticed by these authors the constant vector potential is not covariantly constant, i e is not a solution of the classical equations of motion The question we address is the following Can we find a covariantly constant non-abelian vector potential which retains as many as possible of the characteristics of the constant external field of QED, and what can be said about non-abelian processes around it? In order to solve the classical equations of motion we restrict ourselves to the SU(2) case The main result of our paper can be summarized as follows There is a covariantly constant, time-periodic SU(2) vector potential which gives rise to field strengths invariant under space translations and also under combined space and gauge rotations We discuss its properties in sect 2 In sect 3 isospinor fermions are introduced in the background field and the general structure of the eigenfunction spectrum is studied In particular the set of eigenmodes will be shown to separate into two distinct classes, transverse and scalar-longitudinal Transverse modes and their decoupling from the external field are discussed in sect 4 and finally in sect 5 we analyze the non-trivial content of the scalar-longitudinal sector which gives rise to quasi-doubly-periodic solutions These will be bound on the real $t$-axis only for a particular set of values of $p$, the canonical three-momentum modulus, showing forbidden and permitted zones of propagation

## 2. A periodic SU(2) field

We consider the classical equations of motion for an SU(2) gauge field

$$D^{ab}_\mu F^b_{\mu\nu} = 0\,, \tag{2.1}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc}A^b_\mu A^c_\nu , \tag{2 2a}$$

$$D^{ab}_\mu = \delta^{ab}\partial_\mu + g\varepsilon^{acb}A^c_\mu \tag{2 2b}$$

In order to solve them we make the following ansatz

$$A^a_0 = 0, \qquad A^a_b = \lambda(t)\delta^a_b \qquad (a, b = 1,2,3) \tag{2 3}$$

The corresponding field strengths will be

$$F^a_{0b} = \lambda'\delta^a_b , \qquad F^a_{bc} = g\lambda^2\varepsilon^{abc} \tag{2 4}$$

Defining the current $j^a_\mu = D^{ab}_\nu F^b_{\nu\mu}$ we find

$$j^a_0 = 0, \qquad j^a_b = -(\lambda'' + 2g^2\lambda^3)\delta^a_b \tag{2 5}$$

$A^a_\mu$ as given in eq (2 3) is covariantly constant for

$$\frac{d^2}{dt^2}\lambda = -2g^2\lambda^3 \tag{2 6}$$

Introducing $\tau = \alpha t$ and $\lambda = \beta f$ with the choice $\alpha = \sqrt{2}\, g\beta$ we recognize in eq (2 6) a special case of an elliptic equation, whose solution is a Jacobi elliptic function [5]

$$\lambda(t) = \left(\frac{B}{g}\right)^{1/2} \mathrm{cn}\left((2gB)^{1/2}t, \sqrt{\tfrac{1}{2}}\right), \tag{2 7}$$

where $B = g\beta^2$ In the following we extensively use the theory of elliptic functions All the results quoted are from ref [5]

The function cn is a doubly-periodic function of $t$ with periods $4K$ and $2K + 2iK$ where

$$K = \frac{\Gamma^2(\frac{1}{4})}{4\pi^{1/2}} = 1\ 85407, \tag{2 8}$$

with poles at $\beta_{mn} = 2mK + (2n+1)iK$ and zeros at $\alpha_{mn} = (2m+1)K + 2niK$ Therefore for $t$ real $\lambda(t)$ has a period $T$ and it is bounded

$$T = \frac{2\sqrt{2}\,K}{(gB)^{1/2}}, \qquad |\lambda(t)| \leqslant \left(\frac{B}{g}\right)^{1/2} \tag{2 9}$$

Since $\mathrm{cn}'^2(\tau) = \frac{1}{2}(1 - \mathrm{cn}^4\tau)$ the fields may be cast into the form

$$E^a_b = -\sqrt{2}\, B \,\mathrm{sn}\,\tau\,\mathrm{dn}\,\tau\delta^a_b = \pm B(1 - \mathrm{cn}^4\tau)^{1/2}\delta^a_b ,$$

$$B^a_b = B\,\mathrm{cn}^2\tau\delta^a_b , \tag{2.10}$$

where from now on $\tau = \sqrt{2gB}\,t$ and we drop the $\sqrt{\frac{1}{2}}$ from the arguments of cn The fields are manifestly invariant under space translations and also under simultaneous space and gauge rotations They are the minimum extension to the non-abelian case of the constant field of QED since static, uniform configurations are not solutions of the classical equations of motion Despite the fact that the fields are time dependent we have

$$\sum_a \left[ (\boldsymbol{E}^a)^2 + (\boldsymbol{B}^a)^2 \right] = 3B^2 = \mathrm{const} \tag{2.11}$$

For $gB \ll 1$ the fields are approximately constant in an interval $\Delta t$ such that $\sqrt{2gB}\,\Delta t \ll T$ where they reduce to $E^a_b \approx 0$ and $B^a_b \approx B\delta^a_b$ We notice that this solution is regular everywhere in Minkowski space but cannot be continued to euclidean space where it would serve the role of interpolating field between three-dimensional constant boundary values In fact in euclidean space the continuation of $\mathrm{cn}(t)$ is the Jacobi elliptic function $\mathrm{nc}(x_4)$ which shows poles at the zeros of cn which can be real

Solutions to the Yang-Mills equations of the elliptic type are well known in the literature [6] but they were introduced in an attempt to construct finite-energy classical configurations while here we are primarily concerned with the generalization of the constant field problem It should also be noticed that most of these solutions lead to a complex vector potential, a feature we want to avoid in the present case

According to ref [4] our solution diagonalizes the matrix $A^a \cdot A^b$ This will define the gauge where we are working

Finally besides the solution we have explicitly constructed there are other examples when the matrix $A^a \cdot A^b$ has a lower rank

## 3. Introducing fermions

Having introduced a particular example of an SU(2) external field we move to the problem of studying the behavior of matter fields in this background Consider for definiteness the case of spin-$\frac{1}{2}$, isospin-$\frac{1}{2}$ fermions The conventions we use are $\gamma^+_\mu = \gamma_\mu$, $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$ and $\gamma^5 = \gamma_1\gamma_2\gamma_3\gamma_4$ If $\Psi$ is an isospinor the corresponding Dirac equation in the presence of the external field will be

$$\partial\!\!\!/ \Psi - \tfrac{1}{2} i g A\!\!\!/^a \tau_a \Psi = 0 \tag{3.1}$$

The approach we follow in analyzing the predictions of the model can be termed the "eigenfunction" approach We intend to utilize the Dirac wave functions to calculate processes in the external field Transition amplitudes are computable to first order in $g$ but to all orders in the field, after which we square the amplitude and sum over final states There are other approaches based on Schwinger's source theory which eliminate the need for using wave functions but we have not pursued this issue

Since we are using massless fermions it will be convenient to work with states of definite chirality $\gamma^5\Psi_{L,R} = \pm\Psi_{L,R}$ The Dirac spinors are written in terms of upper and lower components

$$\Psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \tag{3.2}$$

and with our conventions a left-handed fermion corresponds to $\psi^- = -\psi^+ = -\psi$ From now on we concentrate on this case and drop the subscript L Similar results will be valid for right-handed fermions $\psi^- = \psi^+ = \psi$ The Dirac equation becomes

$$i\,\partial_t\psi + i\sigma^a(\partial_a - \tfrac{1}{2}ig\lambda\tau_a)\psi = 0, \tag{3.3}$$

where $\sigma^a$ ($\tau^a$) are the usual Pauli matrices acting on spin (isospin) indices The external field $\lambda(t)$ does not depend on space coordinates and therefore the canonical three-momentum is conserved We write

$$\psi(\boldsymbol{x}, t) = e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\chi(t) \tag{3.4}$$

Eq (3 3) becomes

$$i\frac{\mathrm{d}}{\mathrm{d}t}\chi - \sigma^a(p_a - \tfrac{1}{2}g\lambda\tau_a)\chi = 0 \tag{3.5}$$

Following Jackiw and Rebbi [7] we introduce

$$\chi_{si} = (\eta_s + \eta^a\tau_a)_{sm}(\tau_2)_{mi}, \tag{3.6}$$

where $s$ is a spin index and $i$ an isospin index The degrees of freedom are now described in terms of a scalar mode $\eta_s$ and a vector mode $\boldsymbol{\eta}$ It is irrelevant to further distinguish between different indices and for this reason we denote by $\tau$ all the Pauli matrices Using the relation $\tau_2\tau_a^{\mathrm{T}} = -\tau_a\tau_2$ (T = transpose) we get

$$\left(i\frac{\mathrm{d}}{\mathrm{d}t} - \tau^a p_a\right)(\eta_s + \eta^a\tau_a) - \tfrac{1}{2}g\lambda\tau^a(\eta_s + \eta^b\tau_b)\tau_a = 0 \tag{3.7}$$

In order to diagonalize this equation we introduce the set of matrices $\tau^a(p)$ As usual we write

$$p_{\pm} = \sqrt{\tfrac{1}{2}}\,( p_x \mp i p_y ), \qquad p = |\boldsymbol{p}|$$

Given the unitary transformation $U( p )$

$$U_{11} = U_{22} = N,$$

$$U_{12} = +\sqrt{2}\, N \frac{p_+}{p + p_z}, \qquad U_{21} = -\sqrt{2}\, N \frac{p_-}{p + p_z}, \tag{3.8}$$

with $N^{-2} = 2p/( p + p_z)$, we define

$$\tau^a( p ) = U^+( p )\tau^a U( p ) \tag{3.9}$$

It follows

$$p\tau^3( p ) = \boldsymbol{\tau} \cdot \boldsymbol{p}, \qquad \tau^{1,2}( p ) = \boldsymbol{\tau} \cdot \hat{e}_{1,2}, \tag{3.10a}$$

where

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}, \qquad \hat{e}_i \times \hat{e}_j = \varepsilon_{ijk}\hat{e}_k, \tag{3.10b}$$

with $p\hat{e}_3 = \boldsymbol{p}$. The vector mode $\boldsymbol{\eta}$ is conveniently projected in this frame

$$\boldsymbol{\eta} = \eta_{\mathrm{L}}\hat{e}_3 + \sum_{i=1,2} \eta_{\perp}^i e_i,$$

$$\boldsymbol{\eta} \cdot \boldsymbol{\tau} = \eta_{\mathrm{L}}\tau^3( p ) + \sum_{i=1,2} \eta_{\perp}^i \tau_i( p ) \tag{3.11}$$

By equating the coefficients of 1 and $\tau^a( p )$ in eq. (3.7) we obtain two separate systems of equations relative to the SL (scalar-longitudinal) and T (transverse) modes, terminology introduced by Akhoury and Weisberger [4]

$$i\frac{\mathrm{d}}{\mathrm{d}t}\eta_{\mathrm{S}} - p\eta_{\mathrm{L}} - \tfrac{3}{2}g\lambda\eta_{\mathrm{S}} = 0,$$

$$i\frac{\mathrm{d}}{\mathrm{d}t}\eta_{\mathrm{L}} - p\eta_{\mathrm{S}} + \tfrac{1}{2}g\lambda\eta_{\mathrm{L}} = 0, \tag{3.12a}$$

$$i\frac{\mathrm{d}}{\mathrm{d}t}\eta_{\perp}^1 + ip\eta_{\perp}^2 + \tfrac{1}{2}g\lambda\eta_{\perp}^1 = 0,$$

$$i\frac{\mathrm{d}}{\mathrm{d}t}\eta_{\perp}^2 - ip\eta_{\perp}^1 + \tfrac{1}{2}g\lambda\eta_{\perp}^2 = 0 \tag{3.12b}$$

Solutions of the Dirac equation are classified as follows

$$\chi_{\mathrm{SL}} = (\eta_{\mathrm{S}} + \eta_{\mathrm{L}} \hat{e}_3 \cdot \tau) \tau_2 ,$$

$$\chi_{\perp} = \sum_{i=1,2} \eta^i_{\perp} \hat{e}_i \cdot \tau \tau_2 , \tag{3.13}$$

and from eqs. (3.12) we see that the field equation operator does not mix the two sectors. The properties of this splitting can be described in terms of helicity operators

$$\Sigma^{\mathrm{spin}} \chi \equiv \hat{e}_3 \cdot \tau_{rs} \chi_{si} , \qquad \Sigma^{\mathrm{isospin}} \chi \equiv \chi_{rj} \hat{e}_3 \cdot \tau^{\mathrm{T}}_{ji} \tag{3.14}$$

$\chi_{\mathrm{SL}}$ and $\chi_{\perp}$ satisfy the relations

$$\Sigma^{\mathrm{spin}} \Sigma^{\mathrm{isospin}} \chi_{\mathrm{SL}} = -\chi_{\mathrm{SL}} , \qquad \Sigma^{\mathrm{spin}} \Sigma^{\mathrm{isospin}} \chi_{\perp} = \chi_{\perp} \tag{3.15}$$

This allows us to introduce projection operations $P_{\mathrm{SL}}$ and $P_{\perp}$

$$P_{\mathrm{SL},\perp} = \tfrac{1}{2}(1 \mp \Sigma^{\mathrm{spin}} \Sigma^{\mathrm{isospin}}) \tag{3.16}$$

For an arbitrary spinor $\chi$ we have

$$P_{\mathrm{SL}} \chi = \chi_{\mathrm{SL}} , \qquad P_{\perp} \chi = \chi_{\perp} \tag{3.17}$$

Having classified the solutions of eq. (3.1) according to their properties under the action of the helicity operators (3.14) we proceed to the actual construction of the eigenfunctions.

## 4. The transverse sector

Eqs (3.12b), which describe the time evolution of transverse modes, can be rewritten as

$$i \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \eta^1_{\perp} \\ \eta^2_{\perp} \end{pmatrix} + \tfrac{1}{2} g \lambda \begin{pmatrix} \eta^1_{\perp} \\ \eta^2_{\perp} \end{pmatrix} - p \tau_2 \begin{pmatrix} \eta^1_{\perp} \\ \eta^2_{\perp} \end{pmatrix} = 0 \tag{4.1}$$

To solve the above we use eignstates of $\tau_2$

$$\begin{pmatrix} \eta^1_{\perp} \\ \eta^2_{\perp} \end{pmatrix} = \eta_{\pm} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \tag{4.2}$$

Thus

$$\left( i \frac{\mathrm{d}}{\mathrm{d}t} + \tfrac{1}{2} g \lambda \mp p \right) \eta_{\pm} = 0 . \tag{4.3}$$

which can be integrated to give

$$\ln \eta_{\pm} = \mp ipt + \tfrac{1}{2} ig \int d\tau\, \lambda(\tau) \tag{4.4}$$

The integral in the above equation is easily evaluated in terms of the Jacobi elliptic function d$n$

$$g \int dt\, \lambda(t) = \arccos(dn\, \tau), \qquad \tau = \sqrt{2gB}\, t \tag{4.5}$$

It is convenient to introduce the angle $\theta(t) = \frac{1}{2}\arccos(dn\,\tau)$, which by virtue of the periodic properties of $dn\,\tau$ obeys the inequalities $0 \leqslant \theta(t) \leqslant \frac{1}{8}\pi$. The two solutions in the transverse sector are

$$\psi_{\perp}^{\pm} = \exp\{ i\boldsymbol{p}\cdot\boldsymbol{x} \mp ipt + i\theta(t)\}(\hat{e}_1 \pm i\hat{e}_2)\cdot\tau\tau_2, \tag{4.6}$$

$$\Sigma^{\rm spin}\psi_{\perp}^{\pm} = \pm\psi_{\perp}^{\pm} \tag{4.7}$$

Since the only dependence on the external field is contained in the phase $\theta$ we have shown that transverse modes decouple from $A_{\mu}^{a}$. As a matter of fact there is nothing peculiar to the explicit form of $\lambda(t)$ in this decoupling which works also for constant vector potentials. Part of the eigenfunction spectrum corresponding to parallel components of spin and isospin with respect to the momentum $\boldsymbol{p}$ decouples from the background in the gauge $A_0^a = 0$, $A_b^a = \lambda\delta_b^a$. The same effect can be understood from another point of view. Consider the "longitudinal" subgroup generated by

$$U_{\rm L} = \exp(-\tfrac{1}{2} ig\Lambda_{\rm L}\hat{e}_3\cdot\tau) \tag{4.8}$$

We only need to work with infinitesimal transformations. It is easy to show that

$$U_{\rm L}\psi_{\perp}^{\pm} = (1 \mp \tfrac{1}{2} g\Lambda_{\rm L})\psi_{\perp}^{\pm}, \tag{4.9}$$

and that transverse and scalar-longitudinal modes are not mixed. As a consequence we can always choose a gauge where $\psi_{\perp}^{+}$ or $\psi_{\perp}^{-}$ decouple explicitly.

## 5. The scalar-longitudinal sector

Eqs. (3.12a) are much more complicated to solve than the corresponding transverse equations. First we introduce $a(t) = e^{i\theta(t)}$. Thus

$$i\frac{d}{dt}a = -\tfrac{1}{2} g\lambda a \tag{5.1}$$

Next we use the following substitution

$$\eta_L = a\phi_L, \qquad \eta_s = \frac{1}{a^3}\phi_s \tag{5 2}$$

Eqs (3 12b) become

$$i a^4 \frac{d}{dt}\phi_L - p\phi_s = 0,$$

$$i\frac{d}{dt}\phi_s - pa^4\phi_L = 0 \tag{5 3}$$

We look for a solution $\phi_L = \phi$, $\phi_s = i(a^4/p)(d/dt)\phi$ Thus we are led to the second-order differential equation

$$\frac{d^2}{dt^2}\phi + 2ig\lambda\frac{d}{dt}\phi + p^2\phi = 0 \tag{5.4}$$

Since from eq (2 7), $\lambda(t) = (B/g)^{1/2}\mathrm{cn}\,\tau$ with $\tau = \sqrt{2gB}\,t$ we can rewrite the same equation as

$$\frac{d^2}{d\tau^2}\phi + \sqrt{2}\,i\,\mathrm{cn}\,\tau\frac{d}{d\tau}\phi + q^2\phi = 0, \tag{5 5}$$

with $2gBq^2 = p^2$ Before entering the details of the solution we summarize a few properties of the function cn In a shorthand notation we write $\mathrm{cn}\,\tau = \mathrm{cn}(\tau, \sqrt{\tfrac{1}{2}})$ which is a doubly-periodic function of $\tau$ with periods $4K, 2K + 2iK$ where $K$ is given by eq (2 8) Due to the periodicity we can discuss all properties of elliptic functions in the so-called fundamental period parallelogram which for $\mathrm{cn}\,\tau$ is $\tau = \zeta 4K + \eta(2K + 2iK)$, $0 \leqslant \zeta, \eta < 1$ An irreducible set of poles is given by

$$\beta' = \beta_{10} = 2K + iK, \qquad \text{residue } i\sqrt{2},$$

$$\beta = \beta_{20} = 4K + iK, \qquad \text{residue } -i\sqrt{2}, \tag{5 6}$$

while an irreducible set of zeros is given by $\alpha_{00} = K$ and $\alpha_{10} = 3K$ After those preliminaries we study the singular points of the second-order differential equation (5 5) They are $\tau = \beta'$ and $\tau = \beta$ and their congruent points The corresponding exponents are 0 and 3 for $\beta'$, 0 and $-1$ for $\beta$ Since they are unequal integers we can apply the Hermite, Picard, Mittag-Leffler, Floquet theorem [8] which states that eq (5 5) possesses a fundamental set of solutions which are in general doubly-periodic functions of the second kind Therefore we look for a solution of the form [8]

$$\phi(\tau) = e^{b\tau}\frac{\sigma(\tau - a)}{\sigma(\tau - \beta)}f(\tau), \tag{5 7}$$

where $\beta = 4K + iK$, $a$ and $b$ are constants to be determined and $\sigma(\tau)$ is the weierstrassian $\sigma$-function $f(\tau)$ is an elliptic function

As shown below the $\phi(\tau)$ so constructed is doubly-periodic of second kind Considerations following the value of the exponents relative to the singular points of the equation suggest we try $f = 1$

To prove that $\phi(\tau)$ as given in eq (5 7) is actually a solution of eq (5 5), we need to use the other two weierstrassian functions $P(\tau)$ and $\zeta(\tau)$ Briefly, $P(\tau)$ is an elliptic even function with a double pole at $\tau = 0$

$$P(\tau) = \frac{1}{\tau^2} + \mathrm{O}(\tau^2) \tag{5 8}$$

$\zeta(\tau)$ is an odd function defined by $P(\tau) = -\zeta'(\tau)$ Notice that

$$\zeta(\tau) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\sigma(\tau),$$

$$\zeta(\tau) = \frac{1}{\tau} + \mathrm{O}(\tau^3), \qquad \sigma(\tau) = \tau + \mathrm{O}(\tau^5) \tag{5 9}$$

All the elliptic functions considered in this section have the same periods of $\mathrm{cn}\,\tau$, namely $4K, 2K + 2iK$

$\zeta(\tau)$ and $\sigma(\tau)$ are not elliptic functions but instead

$$\zeta(\tau + 4K) = \zeta(\tau) + 2\zeta(2K) = \zeta(\tau) + 2\eta, \tag{5 10a}$$

$$\sigma(\tau + 4K) = -\exp\{2\eta(\tau + 2K)\}\sigma(\tau), \tag{5 10b}$$

and similarly for the other period It follows that on the real axis

$$\phi(\tau + 4K) = \exp\{4b + 2\eta(\beta - a)\}\phi(\tau) \tag{5 11}$$

and $\phi(\tau)$ is a quasi-doubly-periodic function An important property which follows from the definition of $\phi$ is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\phi = [b + \zeta(\tau - a) - \zeta(\tau - \beta)]\phi \tag{5 12}$$

Eq (5 5) becomes

$$\{[b + \zeta(\tau - a) - \zeta(\tau - \beta)]^2 - P(\tau - a) + P(\tau - \beta)$$
$$+\sqrt{2}\,i\,\mathrm{cn}\,\tau[b + \zeta(\tau - a) - \zeta(\tau - \beta)]\}\phi = -q^2\phi \tag{5 13}$$

Let us consider the function defined by the left-hand side of the previous equation

$f(a, b, \tau)$

$$\{f(a, b, \tau) + q^2\}\phi = 0 \tag{5 14}$$

$\operatorname{cn}\tau$, $P(\tau - a)$, $P(\tau - \beta)$ are elliptic functions Also the difference $\zeta(\tau - a) - \zeta(\tau - \beta)$ is an elliptic function which means that $f$ is elliptic There are in principle three poles for $f(\tau)$ The point $\tau = a$ is a pole for $\zeta(\tau - a)$ and $P(\tau - a)$ while $\tau = \beta$ is a pole for $\zeta(\tau - \beta)$, $P(\tau - \beta)$ and $\operatorname{cn}\tau$ and finally at $\tau = \beta'$, $\operatorname{cn}\tau$ has a pole However given the Laurent expansions for the functions $P$, $\zeta$ and cn we easily verify that $\tau = \beta$ is a regular point for $f(\tau)$ Vice versa, $\tau = a$ is a single pole with residue

$$R_a = 2[b - \zeta(a - \beta)] + \sqrt{2}\, i \operatorname{cn} a \tag{5 15a}$$

and $\tau = \beta'$ is another single pole with residue

$$R_{\beta'} = -2[b + \zeta(\beta' - a) - \zeta(\beta' - \beta)] \tag{5 15b}$$

Thus $f(\tau)$ is an elliptic function of order 2 and as a consequence $R_a + R_{\beta'} = 0$ This can be proved by expressing $\operatorname{cn}\tau$ in terms of $\zeta$-functions We have

$$\operatorname{cn}\tau = \operatorname{cn} a + i\sqrt{2}\,[\zeta(\tau - \beta') - \zeta(\tau - \beta) + \zeta(a - \beta) - \zeta(a - \beta')] \tag{5 16}$$

Taking the limit $\tau \to \beta'$ in the above expression gives

$$\sqrt{\tfrac{1}{2}}\, i \operatorname{cn} a = \zeta(a - \beta) - \zeta(\beta' - \beta) - \zeta(a - \beta'), \tag{5 17}$$

which indeed shows $R_a + R_{\beta'} = 0$ If we require the conditions

$$b = \zeta(a - \beta) - \sqrt{\tfrac{1}{2}}\, i \operatorname{cn} a, \tag{5 18}$$

it follows that $f(\tau)$ is an elliptic function with no poles which by Liouville's theorem is a constant With $b$ given by eq (5 18) we can fix $a$ such that $f = -q^2$ and a solution to eq (5 5) is obtained To verify the correctness of our procedure we have computed

$$f_1(a) = f(\beta, a), \qquad f_2(a) = f(\beta', a), \qquad f_3(a) = f(a, a), \tag{5 19}$$

with the following results

$$f_1(a) = -\tfrac{1}{2}\operatorname{cn}^2 a - P(a - \beta),$$

$$f_2(a) = P(a - \beta') - P(\beta' - \beta),$$

$$f_3(a) = \tfrac{1}{2}\operatorname{cn}^2 a + 3P(a - \beta) - i\sqrt{2}\, sna\ dna \tag{5 20}$$

When we write $-i\sqrt{2}\,\mathrm{sn}a\;\mathrm{dn}a$ this is equivalent to $i\,\mathrm{cn}'a = \pm\sqrt{\tfrac{1}{2}}\,i(1-\mathrm{cn}^4 a)^{1/2}$ which again has periods $4K, 2K+2iK$ By inspection $f_1$ is regular at $a=\beta$ and shows a double pole with zero residue at $a=\beta'$ The same is true for $f_2$ and $f_3$ and moreover $f_1(\beta)=f_2(\beta)=f_3(\beta)=0$ Therefore they are elliptic functions of the argument $a$ with the same periods, a double pole at $a=\beta'$ with the same principal parts Since they have the same value at the point $a=\beta$ they are the same function

A solution to eq (5 5) is specified by

$$\phi(\tau) = e^{b\tau}\frac{\sigma(\tau-a)}{\sigma(\tau-\beta)}, \tag{5 21a}$$

$$\beta = 4K + iK,$$

$$b = \zeta(a-\beta) - \sqrt{\tfrac{1}{2}}\,i\,\mathrm{cn}\,a,$$

$$P(\beta'-a) = e_1 - q^2, \tag{5 21b}$$

where

$$\beta' = 2K + iK, \qquad e_1 = P(2K) \tag{5 21c}$$

$P$ is elliptic of order 2, which means there are two zeros in the fundamental period parallelogram The corresponding two values for $a$ from eq (5 21b) give a fundamental set of solutions to eq (5 5)

For instance for $q^2=0$ we find a solution $a=iK$ and correspondingly $b=-2\eta$ with $\eta$ given by eq (5 10a) In general for $q^2\neq 0$ we have to invert an elliptic function

$$a = \beta' \pm P^{-1}(e_1 - q^2) \tag{5 22}$$

At this point we want to check that for $gB\to 0$ solution (5 21a) goes into a free-particle wave function Since $q^2\to\infty$, $\tau\to 0$ in this limit, with $q\tau = pt$ we have from eq (5 21b)

$$\beta' - a \underset{gB\to 0}{\sim} \frac{i}{q}, \tag{5 23}$$

which gives

$$P(\beta'-a) \underset{gB\to 0}{\sim} -q^2 \tag{5 24}$$

For $q^2\to\infty$ therefore $a=\beta'$ and $\zeta(\beta'-\beta)$ is finite It follows

$$b \underset{gB\to\infty}{\sim} -\sqrt{\tfrac{1}{2}}\,i\,\mathrm{cn}\,a \sim iq \tag{5 25}$$

Also $\sigma(-\beta')$ and $\sigma(\beta'-\beta)$ are non-zero giving from eq (5 21a)

$$\phi(\tau) \underset{gB\to 0}{\sim} \mathrm{const}\, e^{ipt} \tag{5 26}$$

Consider now our solution $\phi(\tau)$ on the real $\tau$-axis From eq (5 11)

$$\phi(\tau + 4K) = \exp\{4b + 2\eta(\beta - a)\}\phi(\tau) \equiv S\phi(\tau), \tag{5 27}$$

for $|S| \neq 1$, $\phi$ increases without bound in one of the directions $\tau \to \pm\infty$, a situation which has a close resemblance to Bloch's spin-waves The condition that $\phi$ be bounded on the real $\tau$-axis defines the permitted zones, very much as for a Schrodinger equation in a periodic potential The modulus of the canonical three-momentum cannot assume arbitrary values but has to satisfy the following relation

$$\mathrm{Re}\left[2\zeta(a - \beta) - i\sqrt{2}\,\mathrm{cn}\,a + (\beta - a)\eta\right] = 0, \tag{5 28}$$

where $\beta = 4K + iK$, $\eta = \zeta(2K)$ and $a(q^2)$ is given in eq (5 22) Collecting our results we find for the SL modes

$$\eta_{\mathrm{L}} = \exp\{b\tau + i\theta(\tau)\}\frac{\sigma(\tau - a)}{\sigma(\tau - \beta)}, \tag{5 29a}$$

$$\eta_{\mathrm{S}} = \frac{i}{p}\eta_{\mathrm{L}}[b + \zeta(\tau - a) - \zeta(\tau - \beta)], \tag{5 29b}$$

which gives an additional constraint. Only solutions with $\mathrm{Im}\,a \neq 0$ are acceptable since $\tau = a$ is a pole for $\eta_{\mathrm{S}}$

In inverting eq (5 21b) we can use one of the several expressions for the $P$-function For instance when $|e_1 - q^2| \gg 1$ we use

$$\beta' - a = u + \mathrm{O}(u^5), \qquad u = \left(\frac{1}{e_1 - q^2}\right)^{1/2}, \tag{5 30}$$

where the coefficients in the expansion are given up to terms $\mathrm{O}(u^{45})$ See Abramowitz and Stegun, ref [5] On the contrary for $q^2$ small, i e $p^2$ small and $gB$ large we use

$$-q^2 = \sum_m c_m u^m, \qquad u = (iK - a)^2, \tag{5 31}$$

where the coefficients $C_M$ are given explicitly up to $M = 7$

## 6. Conclusions

Non-abelian gauge theories have a rich non-perturbative structure and it seems very natural to extend the external field method to them, with the hope of improving upon ordinary perturbation theory Almost all of the non-abelian external field problems considered so far involve abelian-like background configurations where the analysis simplifies considerably For instance such a field induces a vacuum polarization which is immediately related to the vacuum polarization of QED induced by the analogous Maxwell field

Our major purpose in this paper is to start an exploration of simple aspects of non-abelian gauge theories Vacuum polarization in uniform non-abelian fields has been extensively analyzed by considering the effects of a constant (non-abelian) vector potential on quantized matter fields Since this configuration is not covariantly constant we have investigated the possibility of modifying it by allowing for a time dependence in the vector potential As a result we have obtained a time-periodic non-abelian background consisting of a triplet of collinear chromomagnetic and chromoelectric fields which are invariant under space translations and such that any space rotation can be undone by a gauge transformation

After we introduce quantized matter fields the analytical structure of the problem becomes highly complicated and to unravel the physical content of the theory is necessarily a multistep program In this paper we have decided to concentrate on solving the eigenfunction spectrum for an isospinor fermion in the external field This we consider the basic block in approaching a solution to the questions raised in the introduction since in principle when one has the eigenfunctions then perturbation theory in $g$ at all orders in the field follows by application of standard techniques

In solving the eigenfunctions for the isospinor fermions we have found that part of them, namely those corresponding to transverse modes (with respect to the canonical three-momentum), decouple from the background in the sense that a gauge can be chosen where they are free-particle wave functions On the contrary the scalar-longitudinal sector of the theory has a non-trivial content, consisting of quasi-doubly-periodic functions in the $t$-complex plane Wave propagation of these modes is therefore constrained by requiring them to be bounded on the entire $t$-axis, which in turn gives rise to permitted zones very much in the same spirit of solving a Schrodinger equation in a periodic potential The actual form of these zones, i e those values of $p^2/gB$ for which the eigenfunctions do not increase indefinitely for $t \to \pm\infty$, requires the inversion of a weierstrassian elliptic function, and we give few examples of suitable expansions valid in different regions of the parameters

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