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CONVEXITY RESTRICTIONS ON NON-QUADRATIC ANISOTROPIC YIELD CRITERIA

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Abstract—Examination of Hill's 1979 anisotropic yield criterion [*Math. Proc. Camb. Phil. Soc.* **85**, 179 (1979)] shows that for Cases I, II and III, there are combinations of m and R for which the yield loci are outwardly concave or even unbounded. For Case I, all loci are concave unless $m = 2$. For Cases II and III, the combinations of m and R which lead to concavity and unboundedness have been established. Case IV and Hosford's criterion have no problem regions as long as $m \geqslant 1$.

NOTATION

$\sigma_1, \sigma_2, \sigma_3$	principal stresses
σ	stress scaling factor
σ_u	yield strength in uniaxial tension
a, b, c, f, g, h	anisotropic parameters in yield criteria
m	exponent in yield criteria
R	strain ratio, $d\varepsilon_2/d\varepsilon_3$, measured in a one-direction tension test for a material with planar isotropy
α	stress ratio, σ_2/σ_1 (for plane stress, $\sigma_3 = 0$)
Δ	a small positive change in α

INTRODUCTION

Hill [1] proposed, in 1979, a general anisotropic yield criterion of the form:

$$f|\sigma_2-\sigma_3|^m+g|\sigma_3-\sigma_1|^m+h|\sigma_1-\sigma_2|^m+a|2\sigma_1-\sigma_2-\sigma_3|^m + b|2\sigma_2-\sigma_3-\sigma_1|^m+c|2\sigma_3-\sigma_1-\sigma_2|^m=\sigma^m, \tag{1}$$

where σ is a scaling factor for stresses. Four special cases were suggested which can, with certain values of the constants, encompass the so-called 'anomalous' behavior in which some sheet metals with average strain ratios less than unity were found to have biaxial-to-uniaxial yield strength ratios greater than unity [2, 3]. These four special cases, all of which involve the assumption of planar isotropy ($a = b$ and $f = g$) when expressed for plane stress ($\sigma_3 = 0$) reduce to:

Case I ($a = b = 0, f = g, h = 0$)

$$f(|\sigma_1|^m+|\sigma_2|^m)+c|\sigma_1-\sigma_2|^m=\sigma^m; \tag{2}$$

Case II ($a = b, c = 0, f = g = 0$)

$$h|\sigma_1-\sigma_2|^m+a(|2\sigma_1-\sigma_2|^m+|2\sigma_2-\sigma_1|^m)=\sigma^m; \tag{3}$$

Case III ($a = b, c = 0, f = g, h = 0$)

$$f(|\sigma_1|^m+|\sigma_2|^m)+a(|2\sigma_1-\sigma_2|^m+|2\sigma_2-\sigma_1|^m)=\sigma^m; \tag{4}$$

Case IV ($a = b = 0, f = g = 0$)

$$h|\sigma_1-\sigma_2|^m+c|\sigma_1+\sigma_2|^m=\sigma^m. \tag{5}$$

An alternative non-quadratic criterion, which is also a special case of equation (1), was

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proposed by Hosford [4] on the basis of curve fitting yield loci calculated from crystallographic considerations. Although this criterion cannot accommodate 'anomalous' behavior, it does retain some freedom of shape when reduced to isotropy. (This is in contrast to Hill's four special cases which can accommodate isotropy only if $m = 2$, in which event they all reduce to the von Mises criterion.) When this criterion was proposed, the exponent was not regarded as an adjustable parameter. Values of $m = 8$ and 6 were recommended for fcc and bcc metals, respectively, regardless of the R-value. However, for comparison with the other criteria, the exponent is varied here. For planar isotropy and plane stress, this criterion reduces to:

Hosford's criterion $(a = b = c = 0, f = g)$

$$f(|\sigma_1|^m + |\sigma_2|^m) + h|\sigma_1 - \sigma_2|^m = \sigma^m. \tag{6}$$

If we consider a tension test in the one-direction ($\sigma_1 = \sigma_u$, $\sigma_2 = 0$) we can express the scaling factor and the constants in terms of σ_u and the strain ratio, R. The general expression for R is

$$R = [(2^{m-1}+2)a - c + h]/[(2^{m-1}-1)a + 2c + f]. \tag{7}$$

These can be combined to give the following expressions for σ_u/σ_1 in terms of the stress ratio, α:

Case I

$$(\sigma_1/\sigma_u)^m = (R+1)/[(1+2R)(1+|\alpha|^m) - R|1+\alpha|^m]; \tag{8}$$

Case II

$$(\sigma_1/\sigma_u)^m = (R+1)(2^m-2)/\{[R(2^m-2)-2^m-4]|1-\alpha|^m + 2(|2-\alpha|^m + |2\alpha-1|^m)\}; \tag{9}$$

Case III

$$(\sigma_1/\sigma_u)^m = (R+1)(2^m+4)/\{[R(2-2^m)+2^m+4](1+|\alpha|^m) + 2R(|2-\alpha|^m + |2\alpha-1|^m)\}; \tag{10}$$

Case IV

$$(\sigma_1/\sigma_u)^m = 2(R+1)/[(2R+1)|1-\alpha|^m + |1+\alpha|^m]; \tag{11}$$

Hosford

$$(\sigma_1/\sigma_u)^m = (R+1)/(1+|\alpha|^m + R|1-\alpha|^m). \tag{12}$$

CALCULATIONS AND LIMITATIONS

The shapes of the yield loci corresponding to these criteria have been explored by numerical evaluation of equations (8)–(12). The results are plotted in Figs 1–5 for $R = 0.5$ and 2 and various levels of m. Even without considering how well they fit experimental data and analyses based on crystallographic slip, some conclusions can be drawn about their applicability. The first three special cases have regions of outward concavity (hereafter referred to simply as concavity) and are unbounded for some combinations of m and R. These regions are summarized in Fig. 6. The exact limits of unboundedness (at $\alpha = 1$ or -1) can be found simply by substituting the appropriate value of into the yield criterion, setting $\sigma_u/\sigma_1 > 0$ and solving for m (or R).

There are several ways of determining the limiting m and R values for convexity. One involves examining the curvature. For loading paths between pure shear ($\alpha = -1$) and plane strain ($d\varepsilon_2 = 0$), convexity requires

$$d^2\sigma_2/d\sigma_1^2 \geqslant 0 \tag{13}$$

and between plane strain ($d\varepsilon_2 = 0$), and biaxial tension ($\alpha = 1$)

$$d^2\sigma_2/d\sigma_1^2 \leqslant 0. \tag{14}$$

Because of the mathematical complexity of the criteria, it is often simpler to examine the

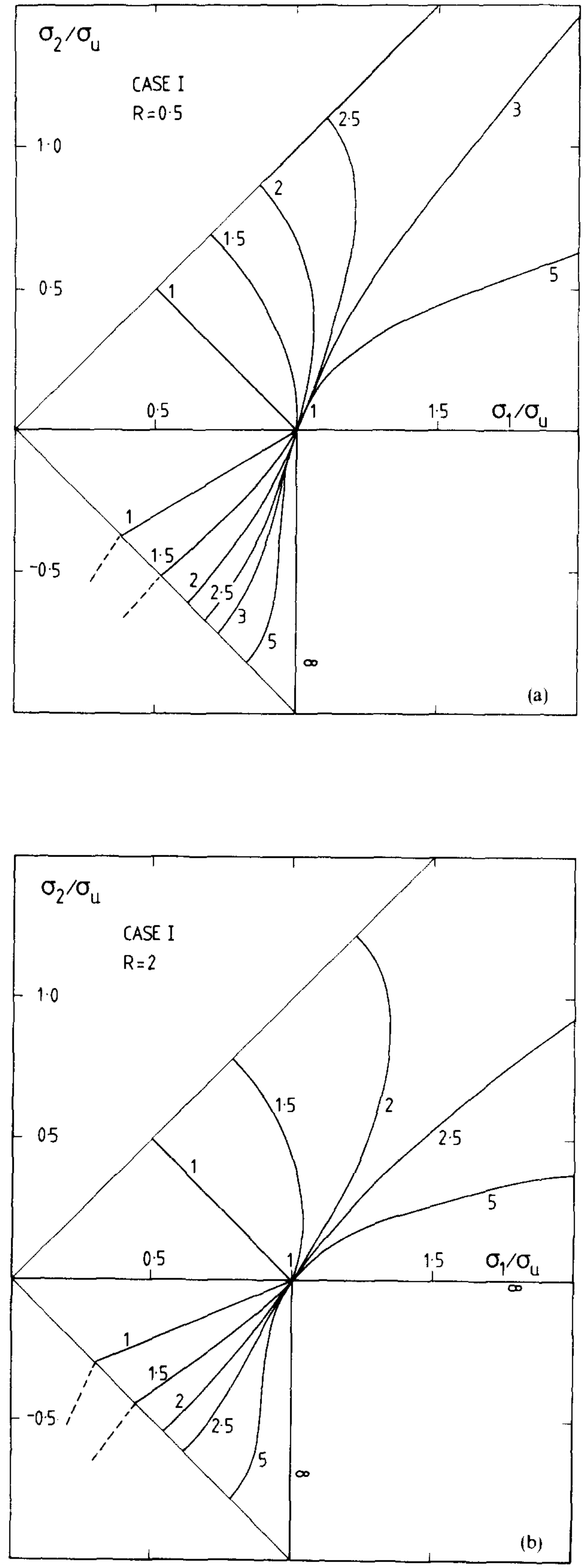

FIG. 1. Yield loci for Case I with (a) $R = 1/2$ and (b) $R = 2$. The numbers refer to the value of the exponent, m. Note the concavities at $\alpha = -1$ and near $\alpha = 0$ for $m > 2$. The loci are unbounded at $\alpha = 1$ for large values of m.

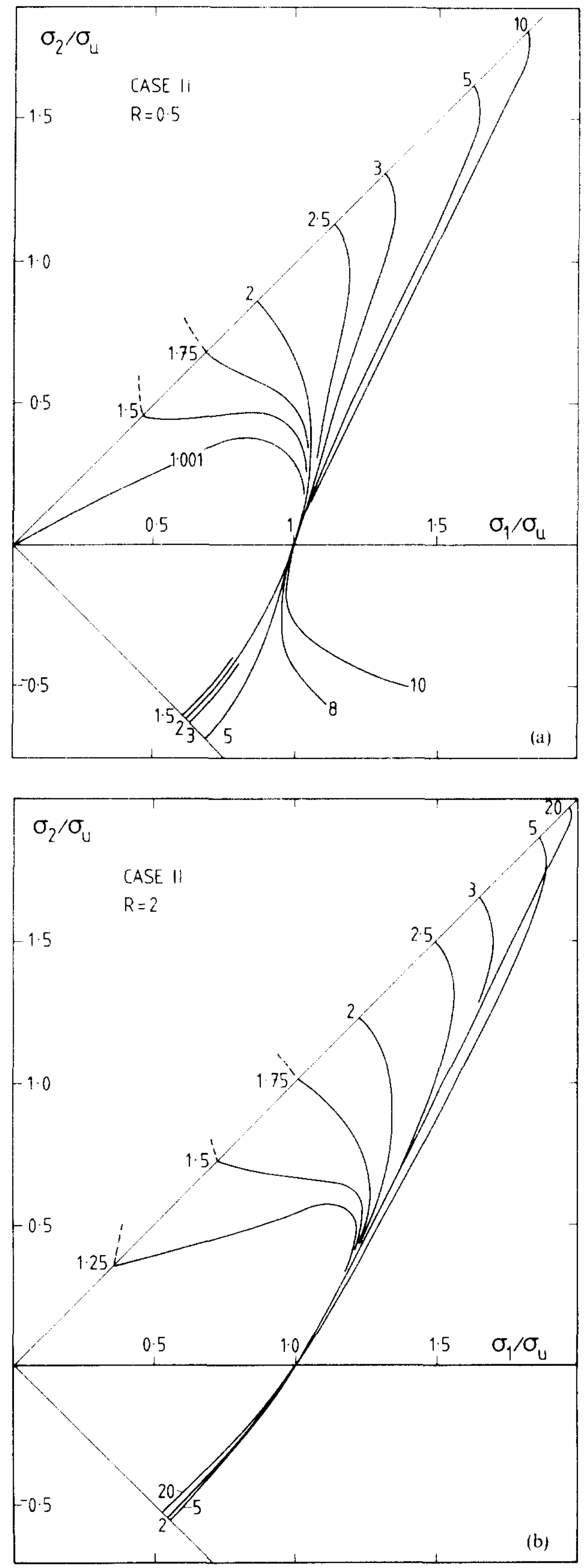

FIG. 2. Yield loci for Case II with (a) $R = 1/2$ and (b) $R = 2$. The numbers refer to the value of the exponent, m. Note the concavities at $\alpha = 1$ for $m > 2$ and at $\alpha = 1/2$ for higher levels of m. The curves for $R = 1/2$ and $m = 8$ and 10 are unbounded at $\alpha = -1$.

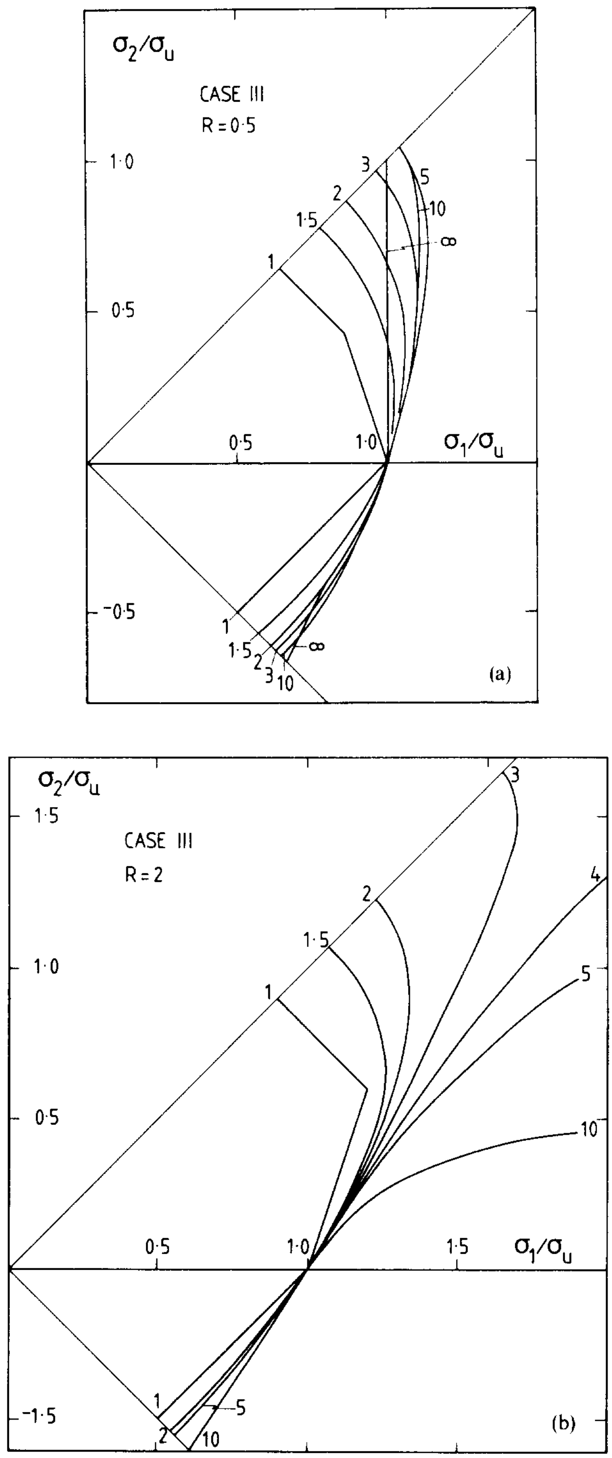

FIG. 3. Yield loci for Case III with (a) $R = 1/2$ and (b) $R = 2$. The numbers refer to the value of the exponent, m. There is no concavity for low values of R, but for high values there is a concavity at $\alpha = 1/2$. The curves for $R = 2$ and $m > 4$ are unbounded.

slopes of the locus in critical regions. Near biaxial tension ($\alpha = 1$),

$$d\sigma_2/d\sigma_1 \leqslant -1 \tag{15}$$

and near pure shear ($\alpha = -1$)

$$d\sigma_2/d\sigma_1 \geqslant 1. \tag{16}$$

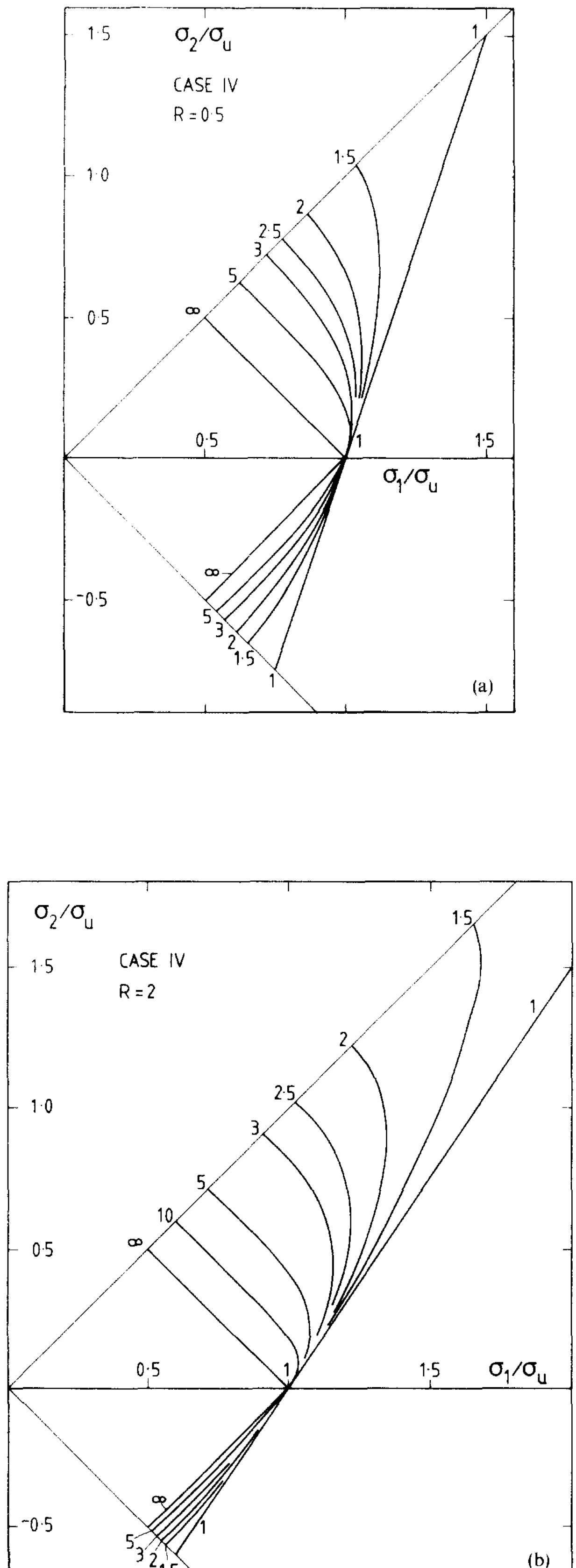

FIG. 4. Yield loci for Case IV with (a) $R = 1/2$ and (b) $R = 2$. The numbers refer to the value of the exponent, m. Note the curves are neither outwardly concave nor unbounded.

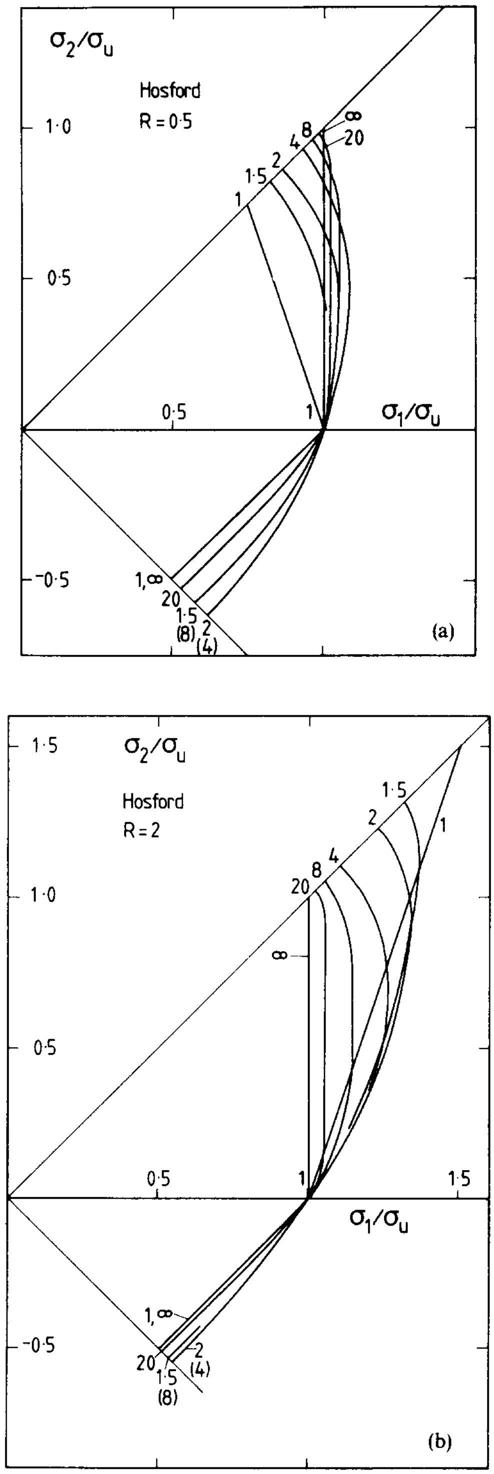

FIG. 5. Yield loci for Hosford's criterion with (a) $R = 1/2$ and (b) $R = 2$. The numbers refer to the value of the exponent, m. Note the curves are neither outwardly concave nor unbounded.

Convexity near $\alpha = 1/2$ requires that if $d\sigma_2/d\sigma_1 > 0$, $d\sigma_2/d\sigma_1$ must increase with α. Mathematically this can be tested by assessing whether

$$(d\sigma_2/d\sigma_1)_{\alpha = 1/2 + \Delta} \geqslant (d\sigma_2/d\sigma_1)_{\alpha = 1/2}, \tag{17}$$

where Δ is small and positive.

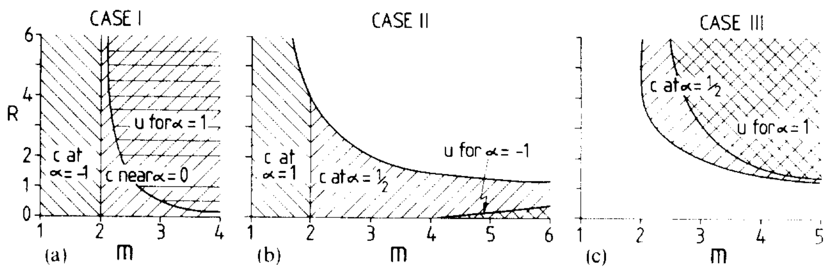

FIG. 6. Summary of the invalid regions for Cases I (a), II (b) and III (c). C means concavity and U means unboundedness.

Case I

(A) If $m > 2$, the locus is concave near $\alpha = 0$. This is because with $m > 2$ and $\alpha = 0$, $d^2\sigma_2/d\sigma_1^2 = -(m-1)(R+1)(2R+1)/R^2$ which cannot be positive as required by inequality (13).

(B) For $m < 2$, the locus is concave near $\alpha = -1$. This can be demonstrated with inequality (16) which reduces to

$$[(1+2R)/2R][1-(1-\Delta)^{m-1}] - \Delta^{m-1} \geqslant 0.$$

This can be satisfied as $\alpha \to -1$ ($\Delta \to 0$) only if $m > 2$.

(C) The locus is unbounded at $\alpha = 1$ for

$$m > \ln[2(1+2R)/R]/\ln 2. \tag{18}$$

Case II

(A) There is a concavity for $\alpha = 1$ if $m < 2$ and

$$m < \ln[2(R+2)/(R-1)]/\ln 2. \tag{19}$$

For $\alpha = 1 - \Delta$, inequality (15) reduces to

$$h/a = 2^{m-1}(R-1) - (R+2) \geqslant (3/2)[(1/\Delta - 2)^{m-1} - (1/\Delta + 2)^{m-1}].$$

With $m < 2$, the RHS $\to 0$ as $\Delta \to 0$, so inequality (19) results.

(B) For $m > 2$, there is a concavity at $\alpha = 1/2$ for low values of R. Inequality (17) may be expressed as

$$h/a \geqslant (12^m/4)/[(3/\Delta - 6)^{m-1} - (3/\Delta - 2)^{m-1} - 4^{m-1}].$$

For $m > 2$, the RHS $\to 0$ as $\Delta \to 0$, so there is a concavity if $h/a < 0$. Substituting $h/a = 2^{m-1}(R-1) - (R+2)$, the condition for concavity may be expressed as

$$2 < m < \ln[2(R+2)/(R-1)]/\ln 2. \tag{20}$$

(C) The locus is unbounded at $\alpha = -1$ if

$$R \leqslant [4 + 2^m - 4(3/2)^m]/(2^m - 2). \tag{21}$$

Case III

(A) There is a concavity at $\alpha = 1/2$ if

$$m > 2 \quad \text{and} \quad m > \ln[2(R+2)/(R-1)]/\ln 2. \tag{22}$$

These limits follow from inequality (17) which is violated if $m > 2$ and $f/a > 0$.

(B) The locus is unbounded at $\alpha = 1$ for

$$m > \ln[4(R+1)/(R-1)]/\ln 2. \tag{23}$$

For Case IV and the alternative criterion, there appear to be no limitations on m and R, except that $m \geqslant 1$ as stated by Hill [1] for the general criterion.

DISCUSSION

That a yield surface may not be concave can be demonstrated by an argument commonly referred to as Drucker's postulate [5]. Bishop and Hill [6] advanced an argument for convexity based on Schmid's law for slip in crystals. Hill [7] has given an improved proof. The convexity requirement puts a very severe limitation on the three special cases of Hill's 1979 non-quadratic criterion.

In a general sense the problems of convexity come about because the special cases violate the important caveat stated by Hill [1], that "in principle, all three stress differences should arguably appear on a broadly similar footing". The trouble appears to arise because of the truncation of the general form in equation (1).

Case I is valid only if $m = 2$, in which case it reduces to Hill's 1948 criterion [8] simplified for planar isotropy. Therefore it should not be regarded as a viable separate criterion or case. 'Anomalous' behavior cannot be accommodated by Case II either because that would require [9] $m > 2$ for $R < 1$ which is a combination for which the criterion is not valid. Anomalous behavior can, however, be accommodated with $m > 2$ because there is no difficulty if $R > 1$. Of course Case IV can also be used.

When the special cases are used to evaluate experimental data, care should be taken to ensure that the data lie within an acceptable m vs R region. This was not done, for example, in the analysis [10] of experimental data for copper, brass and aluminum [11]. The analysis reported levels of m for Cases I and II that lie in the regions for which these cases should not be used and therefore have no validity. The data analyses with Cases III and IV are acceptable.

CONCLUSION

The special cases of Hill's 1979 general anisotropic criterion have been examined. For many combinations of m and R, the predicted loci are concave or even unbounded. Case I is useable only with $m = 2$, and Cases II and III have limited applicability. No concavities were found with Case IV or Hosford's non-quadratic criterion.

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