A Note on Baer Rings

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INTRODUCTION

In [3] the author gives a very down-to-earth construction of an embedding of an arbitrary reduced commutative ring $R$ into a Baer ring $R^B$ by an $R$-compatible ring homomorphism. However, the mapping property claimed in [3] does not hold in the generality stated there: an extra condition on the ring is necessary.

In this paper our main task is to correct that result.

We achieve this goal in Theorem 2.2 where we prove that $R \preceq R^B$ is a universal embedding if and only if every $R$-compatible homomorphism $h: R \to S$ from $R$ to a Baer ring $S$ satisfies condition (B): for all given elements $r, b_1, \ldots, b_t$ ($t \geq 1$) of $R$, if $r$ belongs to all minimal prime ideals containing $b_i$, $1 \leq i \leq t$, then $h(r)$ belongs to all minimal prime ideals containing $h(b_i)$, $1 \leq i \leq t$, and in Theorem 2.12 where several other conditions are given. We also show that if $R$ is reduced, a polynomial ring over $R$ automatically satisfies these conditions.

In Section 3 we construct a ring which fails to satisfy the conditions of Theorem 2.12 hence proving that the correction is necessary.

We are indebted to K. Prikry for pointing out a gap in the proof in [3] which eventually led to this Note and to M. Hochster for the hospitality and valuable discussions during the preparation of this Note.

SECTION 1

In this section let us briefly recall from [3] some notation, definitions, and the construction of $R^B$.

First of all, we shall deal with commutative rings with unit. If $a$ is an ideal of the ring $R$, $a^\perp = \{ r \in R | ra = 0 \}$ is the annihilator of $a$ and is an ideal. Sometimes we shall write Anna instead of $a^\perp$. For an element $a$ of $R$,
we shall denote by \((a)\) the principal ideal \(Ra\). An element \(e\) of \(R\) such that \(e^2 = e\) is said to be an idempotent. Finally, \(p\) (resp. \(m\)) will denote a prime (resp. maximal) ideal of \(R\).

**Definition 1.1.** A Baer ring is a ring such that the annihilator of every principal ideal is principal and generated by an idempotent element.

**Definition 1.2.** Let \(R, R'\) be rings. A homomorphism \(h: R \to R'\) from \(R\) to \(R'\) is said to be \(R\)-compatible if whenever \((a)^\perp = (b)^\perp\), \(a, b \in R\), then \((h(a))^\perp = (h(b))^\perp\) in \(R'\).

When \((a)^\perp\) is principal and generated by an idempotent, this idempotent is uniquely determined by \(a\), and we denote it \(a^*\). We write \(a^o\) for \(1 - a^*\). Note that \(a\) is idempotent \(\iff a = a^o \iff a^* = 1 - a\). Therefore Definition 1.2 can be rephrased as follows:

\[(C). \ h \ is \ an \ R\text{-}compatible \ ring \ homomorphism \ implies \ that \ if \ \(a^*\) \ exists, \ then \ \(h(a)^*\) \ exists \ and, \ in \ fact, \ \(h(a)^* = h(a^*)\) \ (since \(\(a^o = (a^*)^o = (1 - a^*)^o = h(1 - a^*)^o = h(1) - h(a^*) = (1 - h(a^*))^o\)).\]

If \(R\) is a Baer ring, then \(a^o = b^o \iff a^* = b^* \iff 1 - a^* = 1 - b^* \iff (1 - a^*)^o = (1 - b^*)^o\). Then \(\(C\) \ implies \ \(h(a^*)\) \ generates \ \(h(a)^o\) \ and \ \(h(b^*)\) \ generates \ \(h(b)^o\), \ and since \(a^* = b^*\), \(h(a^*) = h(b^*)\) and \(h(a)^o = h(b)^o\).

**Definition 1.3.** An \(R\)-compatible homomorphism between two Baer rings is termed a Baer homomorphism.

*Construction of \(R^B\) following [3, Theorem 1]*

Let \(R\) be a reduced ring. Set \(X = \text{Min}(R)\) (i.e., the set of all minimal prime ideals of \(R\) endowed with the inherited Zariski-topology). For any \(x \in X\), \(p_x\) will denote the minimal prime ideal of \(R\) corresponding to the point \(x\). Set \(\mathcal{R} = \prod_{x \in X} (R/p_x)\) where \(R/p_x\) is an integral domain. It is not difficult to prove that \(\mathcal{R}\) has the strongest Baer property, that is, the annihilator of every ideal is principal and generated by an idempotent (see [4, Theorem 4.11]). In particular, \(\mathcal{R}\) is a Baer ring.

Of course, the map \(i: R \to \mathcal{R}\), where for each \(x\) \(i(r)_x = r + p_x\), is injective since \(R\) is reduced and is \(R\)-compatible. However, as \(\mathcal{R}\) can be very big if \(\text{Min}(R)\) is not finite, in [3] we aimed to find a smaller Baer ring in between. The construction goes as follows. Let us think of \(R\) as sitting inside \(\mathcal{R}\), i.e., identify \(R\) with \(i(R) \subset \mathcal{R}\). Hence an element \(r \in R\) is a family \((r_x)_{x \in X}\) where \(r_x = r + p_x\). Set

\[r^c = (r^c_x)_{x \in X} \in \mathcal{R} \quad \text{where} \quad r^c_x = \begin{cases} 1 & \text{if } r \notin p_x \\ 0 & \text{if } r \in p_x \end{cases}\]
and then let \( r^* = 1 - r^0 \). The operations \(-^*\) and \(-^0\) are all to be carried in \( \mathcal{R} \). Note that if \( r \in R \) and \( r^0 \) or \( r^* \) exists in \( R \), then \( i(r)^0 = i(r^0) \) and \( i(r)^* = i(r^*) \); hence \( i \) is an \( R \)-compatible monomorphism. Note also that \( (r)^ - = (r^*) \) in \( \mathcal{R} \). Now let us consider \( R^B \), the subring of \( \mathcal{R} \) generated by the elements \( r, r^*, r \in R \). It is shown in [3, Theorem 1] that \( R^B \) is a Baer ring. However, the universal property for the map \( i: R \to R^B \) does not hold under such a general hypothesis on \( R \). Some restriction is needed.

In the next section we shall provide the appropriate correction and, in Section 3, we shall exhibit an example of a ring failing to satisfy the extra condition.

In particular we shall prove (see Theorem 2.2)

**Theorem.** The following conditions on a reduced ring \( R \) are equivalent.

1. For every \( R \)-compatible homomorphism \( h: R \to S \) from \( R \) to a Baer ring \( S \) there is an induced Baer homomorphism \( h^*: R^B \to S \) such that for all \( r \in R \), \( h^*(i(r)) = h(r) \) and \( h^*(i(r)^*) = h(r)^* \).

2. For every integer \( t \geq 1 \) and elements \( r, b_1, ..., b_t \) of \( R \), if \( r \) belongs to all minimal prime ideals of \( R \) containing \( b_i \), \( 1 \leq i \leq t \), then \( h(r) \) belongs to all minimal prime ideals of \( S \) containing \( h(b_i) \), \( 1 \leq i \leq t \).

**Section 2**

In this section our aim is to restate Theorem 1 in [3] correctly. Heading to this goal let us investigate in detail what is needed for the "universal" mapping property to hold.

The question is: Given a reduced ring \( R \) so that one can construct \( R^B \), is it true that, for every Baer ring \( S \) and for every \( R \)-compatible ring homomorphism \( h: R \to S \), \( h \) factors through the \( R \)-compatible monomorphism \( i: R \to R^B \)? In other words, is there a Baer homomorphism \( h^*: R^B \to S \) which extends \( h \)? If that were true as stated in [3], it should also be true that whenever \( \sum_{\text{finite}} r_i a_i^c = 0 \) in \( R^B \), then \( \sum_{\text{finite}} h(r_i) h(a_i)^c = 0 \) in \( S \) and we shall see that this is not so in general (see Sect. 3).

To gain a better insight into the matter let us provide another construction of \( R^B \).

Let \( \{X_a | a \in R \} \) be a family of indeterminates indexed by \( R \). Set \( T = R[X_a | a \in R]/(X_a^2 - X_a, X_a X_b - X_{ab})_{a, b \in R} \). Hence \( T = R + \sum_{a \in R} R x_a \) where \( x_a^2 = x_a \) and \( x_a x_b = x_{ab}, a, b \in R \). Let us observe that \( R^B \) is nothing else than \( R[a^c | a \in R] \) since \( a^c = 1 - a^0 \) and, therefore, there exists a surjective \( R \)-homomorphism \( t: T \to R[a^0 | a \in R] \) given by \( t(x_a) = a^c \). Note that \( (ab)^c = a^0 b^c \). Of course,

\[
\text{Ker}(t) = \{ r + r_1 x_{a_1} + \cdots + r_m x_{a_m} \in T | r + r_1 a_1^c + \cdots + r_m a_m^c = 0 \}.
\]
Set \( j = \text{Ker}(t) \). We obtain Scheme 1. (Note: \( j \) denotes lowercase German "jay."

\[
\text{SCHEME 1}
\]

**Comment.** We first get a map \( \tilde{h} \) from the polynomial ring \( R[X_a | a \in R] \) to \( S \) by \( X_a \mapsto h(a)^\circ \). Since \( h(a)^\circ \) is idempotent, \( \tilde{h} \) kills \( X_a^2 - X_a \); and, since \( (h(ab))^\circ = (h(a)h(b))^\circ = h(a)^\circ g(b)^\circ \), \( \tilde{h} \) kills \( X_{ab} - X_aX_b \). Hence \( \tilde{h} \) induces \( \tilde{h}: T \to S \).

Since \( \zeta: T/j \to R^B \) is an isomorphism, the existence of \( h^*: R^B \to S \) will follow from the existence of \( h^*: T/j \to S \) which makes the above diagram commute. Clearly, \( h^* \) exists if and only if \( \tilde{h} \) kills \( j \).

As it is sufficient to show that \( \tilde{h} \) kills generators of \( j \), let us write the elements of \( j \) as a sum of "simpler" elements.

**Lemma 2.1.** In \( T \) every element can be written as a sum of expressions involving mutually orthogonal idempotents in the sense we make precise below in formula (3).

**Proof.** Pick an element \( x \) of \( T \), hence \( x = r \cdot 1 + r_1x_{a_1} + \ldots + r_mx_{a_m} \). As \( x_a^2 = x_a \), \( x_a^* = x_a \) and \( x_a^* = 1 - x_a \), hence \( x_a^*x_a^* = 0 \) and \( x_a^* + x_a^* = 1 \). Also \( 1 = \prod_{i=1}^m (x_{a_i}^* + x_{a_i}^*) \) or

\[
1 = \sum_{(j_1, \ldots, j_m) \in 2^m} x_{a_1}^{j_1} \cdots x_{a_m}^{j_m}, \tag{1}
\]

where \( 2 = \{0, *\} \) and any two elements in this sum with distinct indices are mutually orthogonal. Therefore, each \( x_{a_i}^v \) (\( v = 0, * \) and \( i = 1, \ldots, m \)) can be written as

\[
x_{a_i}^v = x_{a_i}^* \cdot 1 = \sum_{(j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_m) \in 2^m-1} x_{a_1}^{j_1} \cdots x_{a_{i-1}}^{j_{i-1}} x_{a_i}^v x_{a_{i+1}}^{j_{i+1}} \cdots x_{a_m}^{j_m} \tag{2}
\]
which implies that

\[ x = \sum_{(j_1, \ldots, j_m) \in 2^m} r_{j_1} \cdots j_m x_{a_1}^{j_1} \cdots x_{a_m}^{j_m}. \quad \text{Q.E.D.} \quad (3) \]

In particular, an element \( x \) of \( T \) with a fixed representation as in (3) belongs to \( J_r \) if and only if \( \sum r_{j_1} \cdots j_m a_1^{j_1} \cdots a_m^{j_m} = 0 \) which implies that each term of the sum is 0.

Then, the question of whether \( \bar{h} \) kills \( J_r \) reduces to the question of whether

\[ \text{If } ra_1^* \cdots a_s^* b_1^* \cdots b_t^* = 0 \text{, then } \bar{h}(ra_1^* \cdots a_s^* b_1^* \cdots b_t^*) = 0, \]

i.e.,

\[ h(r) \cdot \left( \prod_{i=1}^s h(a_i)^* \right) \cdot \left( \prod_{j=1}^t 1 - h(b_j)^* \right) = 0. \quad (4) \]

But

\[ r \cdot \left( \prod_{i=1}^s a_i^* \right) \cdot \left( \prod_{j=1}^t b_j^* \right) = 0 \iff \text{For every minimal prime ideal } p \text{ of } R, \text{ either } r \in p \text{ or at least one } a_i \in p \]

or at least \( b_j \notin p \); i.e., for every minimal prime ideal \( p \) of \( R \), either

\[ ra_1 \cdots a_s \in p \text{ or at least one } b_j \notin p. \]

That is,

\[ r \cdot \left( \prod_{i=1}^s a_i^* \right) \cdot \left( \prod_{j=1}^t b_j^* \right) = 0 \iff \text{Every minimal prime ideal of } R \text{ which contains } b_1, \ldots, b_t (t \geq 1) \text{ also contains } ra_1 \cdots a_s (s \geq 1). \]

We have thus shown

**Theorem 2.2.** Let \( R \) be a reduced ring. The following conditions are equivalent:

(i) \( R^B \) is a universal Baer extension of \( R \).

(ii) For every \( R \)-compatible homomorphism \( h: R \to S \) from \( R \) to a Baer ring \( S \), there exists a Baer extension \( h^*: R^B \to S \) of \( h \).

(iii) For all \( r, b_1, \ldots, b_t (t \geq 1) \) elements of \( R \), if \( r \) belongs to all minimal prime ideals of \( R \) containing \( b_i \), \( 1 \leq i \leq t \), then \( h(r) \) belongs to all minimal prime ideals of \( S \) containing \( h(b_i) \), \( 1 \leq i \leq t \).
Throughout we shall refer to (iii) as to condition (B).
Let us head to a characterization of such rings $R$.

**Proposition 2.3.** Let $R$ be a ring, $t$ an element of $R$, and $R_t$ the localization of $R$ at the element $t$. Then

1. The natural map $\phi: R \to R_t$ is $R$-compatible.
2. If $R$ is reduced, then for all $u, v \in R$
   - $(u/v)^{1-} = (v/u)^{1-}$ in $R$,
   - $(v/u)^{1-}$ in $R_t$ implies $v/u^{1-}$ in $R$.
3. If $R$ is a Baer ring, then $\phi$ is a Baer homomorphism.

**Proof.** (1) It is enough to recall from [2, Proposition 3.14] that $S^{-1}(\text{Ann } M) = \text{Ann}(S^{-1}M)$ for all finitely generated $R$-modules $M$.

(2) – $\alpha$. Let us assume that $v \in (tu)^{1-}$ in $R$, i.e., $v(tu) = 0$. Then $\phi(v(tu)) = (v/u)(tu/u) = 0/1$ and $t/1$ invertible imply $v/u \in (u/1)^{1-}$. Conversely, let $v/u \in (u/1)^{1-}$ in $R_t$. Then $t^k(vu) = 0$ in $R$ for some integer $k \geq 0$ implies $v(tu)^k = 0$. Therefore $v(tu) = 0$ as $R$ is a reduced ring, i.e., $v \in (tu)^{1-}$ in $R$. (2) – $\beta$. It follows from (1) and (2) – $\alpha$.

(3) It follows from (1) and Definition 1.3 as it is easy to check that $R_t$ is a Baer ring as well.

**Corollary 2.4.** An $R$-compatible homomorphism $h: R \to R'$ between two reduced rings $R, R'$ induces an $R$-compatible map $h_t: R_t \to R'_t$ for all elements $t$ of $R$.

**Proof.** Let us remark that in $R_t$, $\text{Ann}(b/t) = \text{Ann}(b/1)$ as $1/t$ is invertible. Thus we have to prove that if $\text{Ann}(b/1) = \text{Ann}(c/1)$ in $R_t$, then $\text{Ann}(h(b)/1) = \text{Ann}(h(c)/1)$ in $S_{h(t)}$, for all $b/1, c/1$ in $R_t$. By Proposition 2.3: (2) – $\beta$, $\text{Ann}(b/1) = \text{Ann}(c/1)$ implies $(tb)^{1-} = (tc)^{1-}$ which implies $(h(tb))^{1-} = (h(tc))^{1-}$ by the $R$-compatibility of $h$, and this means $\text{Ann}(h(b)/1) = \text{Ann}(h(c)/1)$ by Proposition 2.3: (2) – $\beta$ already mentioned.

**Definition 2.5.** An ideal $i$ of $R$ is said to be a $B$-ideal if for all elements $u, v$ of $R$, $u^{1-}v^{1-} = v^{1-}u^{1-}$ and $u \in i$, then $v \in i$.

**Examples.** The ring itself, the zero ideal, a minimal prime ideal, and, of course, any intersection of them are $B$-ideals.

**Definition 2.6.** A $B$-ideal of a Baer ring is termed a Baer ideal.

**Definition 2.7.** A dense ideal $b$ of $R$ is an ideal with $b^{1-} = (0)$. 
A few properties of $B$-ideals, Baer ideals, and dense ideals strictly related to our goal are

**Proposition 2.8.** (1) A $B$-ideal of a reduced ring is radical.

(1') An ideal of a Baer ring is a Baer ideal if and only if it is an intersection of minimal prime ideals of the ring.

(2) An ideal of a reduced ring $R$ is a $B$-ideal if and only if it is the kernel of an $R$-compatible ring homomorphism having $R$ as a source.

(2') An ideal of a Baer ring $S$ is a Baer ideal if and only if it is the kernel of a Baer homomorphism from $S$ to a Baer ring $S'$.

(3) If $b = (b_1, \ldots, b_t)$ is a finitely generated ideal of a reduced ring $R$, then $b$ is not dense (i.e., $b^+ \neq (0)$) $\iff \exists a \in R - \{0\}$ such that $ab_i = 0$, $1 \leq i \leq t \iff V^-(b) = \{ p \in \text{Min}(R) | p \supseteq b \} \neq \emptyset$.

**Proof.** (1) For an element $x$ of a reduced ring $R$ we have $(x^n)^+ = x^+$, hence if $x^n \in i$, then $x \in i$ since $i$ is a $B$-ideal, i.e., $i$ is radical.

(2) Let $i$ be a $B$-ideal of $R$. Set $\overline{R} = R/i$. Claim: The natural map $\pi: R \to \overline{R}$ is $R$-compatible.

In fact, let $r, u \in R$ be such that $r^+ = u^-$. Two cases are possible. 1st Case. If $r$ (or $u$) $\in i$, then $u$ (or $r$) $\in i$, hence $\overline{R} = r^+ = u^-$. 2nd Case. Assume $r \notin i$ and $r^+ \neq u^+$. Then there exists an element $\overline{r} \in \overline{R}$ such that $\overline{r} \cdot \overline{r} = 0$ and $\overline{r} \cdot \overline{u} \neq 0$; that is, $tr \in i$ and $tu \notin i$, a contradiction since $r^+ = u^- \Rightarrow (tr)^+ = (tu)^+$ for all $t \in R$, because $R$ is reduced (see Proposition 2.3). Conversely, let $\phi: R \to R'$ be an $R$-compatible homomorphism. Set $i = \text{Ker} \phi$. Let $r, u \in R$ have the property that $r^+ = u^-$. If $r \in i$, then $R' = (\phi r)^+ = (\phi u)^+$ which implies $\phi u = 0$ hence $u \in i$. Note that we do not need $R$ to be reduced in this part.

For the proof of (1') see [8], for the proof of (3) see [1]. (2') follows from (2).

**Definition 2.9.** An $R$-compatible homomorphism $h: R \to S$ from a reduced ring $R$ to a Baer ring $S$ is said to satisfy condition (B) if

(B) For all elements $b_1, \ldots, b_t$ (t $\geq$ 1) of $R$, if no minimal prime ideal of $R$ contains $b_i$, $1 \leq i \leq t$, then no minimal prime ideal of $S$ contains $h(b_i)$, $1 \leq i \leq t$.

Since no minimal prime ideal of $R$ contains $b_i$, $1 \leq i \leq t \iff$ the ideal $b = (b_1, \ldots, b_t)$ is dense, condition (B) says that under $h$ a finitely generated dense ideal of $R$ expands to a dense ideal of $S$.

**Remark 2.10.** Of course, if an $R$-compatible homomorphism $h: R \to S$ from a reduced ring $R$ to a Baer ring $S$ satisfies condition (B), then it
satisfies condition \((B_0)\), since no minimal prime ideal contains \(b_i, 1 \leq i \leq t\) belongs to all minimal prime ideals containing \(b_i, 1 \leq i \leq t\).

**Theorem 2.11.** Let \(R\) be a reduced ring, \(S\) a Baer ring, and \(h: R \rightarrow S\) an \(R\)-compatible homomorphism. TFAE

1. \(h\) satisfies condition \((B)\).
2. \(h_f: R_f \rightarrow S_{h(f)}\) satisfies \((B)\) for all \(f \in R\).
3. \(h_f: R_f \rightarrow S_{h(f)}\) satisfies \((B)\) for all \(f \in R\).

**Proof.** (ii) \(\Rightarrow\) (iii) for all \(f \in R\) by Remark 2.10. (iii) \(\Rightarrow\) (i). If \((B)\) fails, we get elements \(r, b_1, ..., b_t (t \geq 1)\) in \(R\) such that \(r\) belongs to all minimal prime ideals of \(R\) containing \(b_1, ..., b_t\) but \(h(r)\) does not belong to a minimal prime ideal \(q\) of \(S\) containing \(h(b_1), ..., h(b_t)\). In the ring \(R_r, b_1/1, ..., b_t/1\) do not belong to any minimal prime ideal. By \((B)\) for \((R_r, S_{h(r)}, h_r)\) the images \(h_r(b_i)/1, ..., h_r(b_t)/1\) are not in any minimal prime ideal of \(S_{h(r)}\). But \(q \cdot S_{h(r)}\) gives a minimal prime which contains \(h_r(b_i)/1, 1 \leq i \leq t\), a contradiction.

(i) \(\Rightarrow\) (ii). Given \(r/f^m, b_i/f^m, 1 \leq i \leq t\), elements of \(R_f\), to show that if \(r/f^m\) belongs to all minimal primes containing \(b_i/f^m, 1 \leq i \leq t\), then \(h(r)/h(f)^m\) belongs to all minimal primes of \(S_{h(f)}\) containing \(h(b_i)/h(f)^m, 1 \leq i \leq t\), is equivalent to showing that if \(r/1\) belongs to all minimal primes containing \(b_i/1, 1 \leq i \leq t\) in \(R_f\), then \(h(r)/1\) belongs to all minimal primes containing \(h(b_i)/1, 1 \leq i \leq t\), in \(S_{h(f)}\).

If not, choose a minimal prime ideal \(q\) of \(S_{h(f)}\) containing \(h(b_i)/1, 1 \leq i \leq t\) and not containing \(h(r)/1\).

**Claim.** Every minimal prime ideal of \(R\) which contains \(b_i, 1 \leq i \leq t\), contains \(r/1\). Assume not and let \(p\) be a minimal prime ideal containing \(b_i, 1 \leq i \leq t\), and not \(r/1\). Then \(f \notin p\) implies \(pR_f\) is a minimal prime containing \(b_i/1, i \in \{1, 2, ..., t\}\); hence \(pR_f\) contains \(r/1\). This implies \(f^k r \in p \Rightarrow (fr)^k \in p \Rightarrow fr \in p\), a contradiction. Therefore, every minimal prime of \(S\) which contains \(h(b_i), i = 1, ..., t\), contains \(h(rf) = h(r)h(f)\), a contradiction since \(q\) does not contain \(h(f)h(r)\) and is a minimal prime containing \(h(b_i), i = 1, ..., t\). Q.E.D.

For the next result we need some notation. Let \(R\) be a reduced ring. For an element \(r\) of \(R\), set \(Y = \text{Min}(R_r)\), while \(X = \text{Min}(R)\). Let \(X_r = \{ x \in X : r \notin p_x \}\). There is a canonical homomorphism \(\eta\) from \(X_r\) to \(Y\). Let \(\rho: \prod_{x \in X} R/p_x \rightarrow \prod_{x \in X} R/p_x\) be the restriction map.

**Theorem 2.12.** There are natural isomorphisms \((R^B_r) = (R^B)^{\eta(r)} \cong (R_r)^B \cong \rho(R^B_{\rho(\eta(r))})\).
Let \( \rho^B \) be the restriction of \( \rho \) to \( R^B \) so that \( \rho^B: R^B \to \rho(R^B) \). Set \( j = \text{Ker}(\rho^B) \). \( j \) consists precisely of the elements of \( R^B \) vanishing on \( Y \), whence \( j = i(r)^{\perp} = \bigcup \sigma(i(r))^{\perp} \). Therefore the induced map \( \rho^B_{\hat{\iota}(r)}: (R^B)_{\hat{\iota}(r)} \to (\rho(R^B))_{\rho(\hat{\iota}(r))} \) is an isomorphism. Let \( j: R_r \to \prod_{\sigma \in \sigma} R_{x}/p_{\sigma(x)} \) be the map for \( R_r \) which corresponds to the map \( i \) for \( R \) defined earlier. The maps \( \rho_{\hat{\iota}(r)} \) and \( \psi \) in the commutative diagram below

\[
\begin{array}{ccc}
R_r & \xrightarrow{\psi} & \prod_{\sigma \in \sigma} (R_{x}/p_{\sigma(x)}) \\
\downarrow & & \downarrow \\
(R_{r})^{B} & \xrightarrow{\rho_{\hat{\iota}(r)}} & \left( \prod_{\sigma \in \sigma} R_{x}/p_{\sigma(x)} \right)_{\rho(\hat{\iota}(r))}
\end{array}
\]

are easily seen to be isomorphisms. (Here, if \( g \in \prod_{\sigma \in \sigma} R_{x}/p_{\sigma(x)} \), \( \psi(g/1)(\eta(x)) = g(x)/1 \); \( R_{x}/p_{\eta(x)} \) is identical with \( (R_{x}/p_{\sigma(x)})_{\hat{\iota}} \).) By definition, \( (R_{r})^{B} \) is the subring of \( \prod_{\sigma \in \sigma} R_{x}/p_{\sigma(x)} \) generated by the elements \( j(f), f \in R_{r} \), and \( (j(f))^{\hat{\iota}(r) \perp} \), \( f \in R_{r} \), or by the elements \( j(f), f \in R_{r}, 1/j(r) \), and \( (j(f))^{\hat{\iota}(r) \perp} \), \( f \in R \). The image of this subring under \( \psi^{-1} \) in \( (\prod_{\sigma \in \sigma} R_{x}/p_{\sigma(x)})_{\rho(\hat{\iota}(r))} \) is the subring generated by \( \rho(j(f)), f \in R_r, \rho(j(fr))^{\hat{\iota} \perp} = \rho[i(f)]^{\hat{\iota} \perp} \), i.e., \( (R_{r})^{B} \) viewed in \( (\prod_{\sigma \in \sigma} R_{x}/p_{\sigma(x)})_{\rho(\hat{\iota}(r))} \) is the subring generated by \( \rho(i(f)), f \in R_r, 1/\rho(i(r)) \), and \( \rho[i(f)]^{\hat{\iota} \perp} \), \( f \in R \), which is exactly \( \rho(R^B)_{\rho(\hat{\iota}(r))} \). Therefore we have got the isomorphisms \( (R_{r})^{B} \cong \rho(R^B)_{\rho(\hat{\iota}(r))} \cong (R^B)_{\hat{\iota}(r)} \).

PROPOSITION 2.13. If all \( R \)-compatible homomorphisms \( h: R \to S \) from a reduced ring \( R \) to a Baer ring \( S \) satisfy condition (B), then all \( R \)-compatible homomorphisms \( k: R_{r} \to T \) from \( R_{r} \) to a Baer ring \( T \) satisfy (B).

**Proof.** Choose an element \( r \) of \( R \) and let \( k: R_{r} \to T \) be such a homomorphism. Note that \( k(r/1) \) is invertible in \( T \). First we get an \( R \)-compatible map \( h: R \to \phi R_{r} \to T \), hence there exists \( h^{\phi}: R^B \to T \) such that \( h^{\phi} \circ i = k \circ \phi = h \). By localizing \( R^B \) at \( i(r) \) we get a map \( \phi^{*}: R_{r} \to (R^B)_{\hat{\iota}(r)} \) by the universality of \( R_{r} \) and also a map \( (R^B)_{\hat{\iota}(r)} \to T \) since \( h(r) \) is invertible in \( T \). Hence by the isomorphism \( (R^B)_{\hat{\iota}(r)} \cong (R^B)_{\hat{\iota}(r)} \) established earlier we obtain a map \( (R_{r})^{B} \to T \) which says that \( k \) satisfies (B).

Our task is at end since we can prove

**Theorem 2.14.** TFAE on a reduced ring \( R \).

(1) Every \( R \)-compatible homomorphism \( h: R \to S \) from \( R \) to a Baer ring \( S \) satisfies condition (B).

(2) \( R \subseteq R^B \) is a universal \( R \)-compatible embedding.
(3) A proper $B$-ideal of $R$, has no dense finitely generated subideal, for all $r$ in $R$.

(4) A prime $B$-ideal of $R$, has no dense finitely generated subideal, for all $r$ in $R$.

(5) Every $R$-compatible map $R, \rightarrow K$ satisfies condition $(B_2)$ for all fields $K$ and $r$ in $R$.

Proof. (1) $\iff$ (2) by Theorem 2.2. (1)$\implies$ (3), (1)$\implies$ (5) are easy to prove. (3)$\implies$ (4) is trivial.

(4)$\implies$ (1). Let us assume that (4) holds. We want to prove that every $R$-compatible map $h: R, \rightarrow S$, $S$ Baer ring, satisfies condition (B). Claim: It suffices to show that an $R$-compatible map $R, \rightarrow S'$ satisfies condition $(B_2)$ for all Baer rings $S'$ and $r$ in $R$.

Assume not and let $h: R, \rightarrow S'$ fail to satisfy condition $(B_2)$. Then there exists a finitely generated dense ideal of $R$, which does not expand to a dense ideal in $S'$. Say $h = (b_1, ..., b_t)$. Choose a minimal prime ideal $q$ of $S'$ containing $h(b_i), 1 \leq i \leq t$. Claim: $h^{-1}(q)$ is a $B$-ideal of $R$.

If not, let $x^\perp = y^\perp$ in $R$, and $x \in h^{-1}(q), y \notin h^{-1}(q)$. Since $h$ is $R$-compatible, we have $h(x)^- = h(y)^\perp$, a contradiction because $h(x) \in q \implies h(x)^- \notin q$, but $h(y)^\perp \notin q \implies h(y)^\perp = h(x)^- \subseteq q$.

(5)$\implies$ (1). If not, let $h: R, \rightarrow S$ fail to satisfy condition (B), i.e., there exist elements $r, b_1, ..., b_t$ in $R$ such that $r$ belongs to all minimal primes of $R$ containing $b_i, 1 \leq i \leq t$, but $h(r) \notin q$ a minimal prime ideal of $S$ which contains $h(b_i), 1 \leq i \leq t$. By localizing at $r$ and $h(r)$ and then taking the fraction field $K$ of $S_{h(r)/q}S_{h(r)}$, we obtain an $R$-compatible map

$$R, \rightarrow S_{h(r)}, R, \rightarrow S_{h(r)/q}S_{h(r)} \rightarrow K$$

which maps the finitely generated dense ideal $(b_1, ..., b_t)$ to $(0)$, a contradiction.

Next is a result, interesting in itself, which implies that for a reduced ring $R$, the embedding $R[X] \subset R[X]^B$ is automatically universal.

**Theorem 2.15.** Let $R$ be a reduced ring. Then in $R[X], f \in R[X]$, every finitely generated dense ideal contains a nonzerodivisor.

Proof. Suppose that $d_0/f^n, ..., d_r/f^r \in R[X], f$ have no common annihilator. Then the elements $d_i/1, i = 0, 1, ..., r$, have no common annihilator. Claim: If $N > \sup\{\deg d_i, 0 \leq i \leq r\}$, then $\sum_{i=0}^r d_i X^N/1$ is a nonzerodivisor. Proof. Say $g/fj$ kills it. Then $G = f^k g$ kills $D = \sum_{i=0}^r d_i X^N$ in $R[X]$, for some sufficiently large $k$.

It suffices to show that if $G$ kills $\sum_{i=0}^r d_i X^N$ in $R[X]$ then $G$ kills each $d_i (0 \leq i \leq r)$, for then $G/1 = 0$ in $R[X], f$. Let $C_G, C_d$ be the ideals
of $R$ generated by the coefficients of $G$ and $D$ respectively. $GD = 0 \Rightarrow C_G \cdot C_D = (0)$. (If not, choose a minimal prime ideal $p$ of $R$ such that $p \not\subseteq C_G \cdot C_D$, that is, $p \not\subset c_1 \cdot c_2$ for some coefficient $c_1$ of $G$ and some coefficient $c_2$ of $D$. Then $GD \not\equiv 0 \mod p$, a contradiction.) $C_G \cdot C_D = (0)$, however, implies that $C_G \cdot C_d = (0)$ since $C_d \subset C_D$, $0 \leq i \leq r$. Hence $Gd_i = 0$ for $i = 0, 1, \ldots, r$, i.e., $G/1$ kills $d_i/1$ in $R[X]_f$ for all $i$. Thus $G/1 = f^k g^l/1 = 0/1$ in $R[X]_f$, i.e., $g/l = 0/l$ in $R[X]$. Q.E.D.

**Corollary 2.16.** For a reduced ring $R$, then embedding $R[X] \subset R[X]^B$ is universal.

**Proof.** Assume not and let $h: R[X]_f \to S$ fail to satisfy Theorem 2.14: (4). Let $d = (d_0, \ldots, d_r)$ be a finitely generated dense ideal of $R[X]_f$ which expands to a nondense ideal. There exists a minimal prime ideal $q$ of $S$ containing $h(d_0), \ldots, h(d_r)$, whence $h^{-1}(q)$ contains $b = (d_0, \ldots, d_r)$ which contains a nonzerodivisor $\delta$ by Theorem 2.15. But $h$ is $R$-compatible and, therefore, $h(\delta)^{-1} = h(1)^{-1} = (0) \subseteq q$, a contradiction since $h(\delta) \in q$.

**Section 3**

In this section we shall exhibit a ring which fails to satisfy condition $(B_z)$ and hence the conclusion of Theorem 2.12 does not hold for it. Therefore, Theorem 1 as stated in [3] is not correct.

We shall construct a reduced quasilocal ring $(R_\omega, m_\omega)$ and elements $x, y \in m_\omega$ such that $x^* \cap y^{-1} = (x, y)^- = (0)$, but every element of $m_\omega$ is a zerodivisor. It is then immediate that $R_\omega \to K_\omega/m_\omega$ is an $R$-compatible map from $R_\omega$ to a field $K_\omega$ which does not satisfy $(B_z)$ or $(B)$. Hence $R_\omega \subset R'_\omega$ does not have the universal mapping property and this is not the universal Baer embedding of $R_\omega$.

**Lemma 3.1.** Let $(R, m)$ be a quasilocal reduced ring with $x, y \in m$ such that

1. $\text{Ann } x \cap \text{Ann } y = (0)$.
2. If $s \mid x^n$ and $s \mid y^n$, then $s$ is a unit.

Let $u \in m$. Set $R' = R[Z]/j$ where $j = \{w \in R[Z]/\exists N \text{ such that } (xw)^N, (yw)^N \in (uZ)\}$. Then

(a) $R'$ is quasilocal and reduced.
(b) $j \cap R = (0)$ and hence $R \subset R'$ and $m_R \subset m_{R'}$.
(c) The image of $Z$ in $R'$ is not zero, $uZ = 0$ in $R'$, and hence $u$ is a zerodivisor in $R'$.
(d) In $R'$ (1) and (2) hold for the images of $x$ and $y$. 

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Proof. (a) Let \( \bar{x} \in R' \) be such that \( \bar{x} = 0 \), hence \( x \in j \) in \( R[\bar{Z}] \) for some \( i \), that is, \((x',x) = (x'y')^m \in (uZ)\). Set \( N = \max(m, im) \). Then \((zx)^N, (zy)^N \in (uZ)\), whence \( x \in j \), i.e., \( \bar{x} = 0 \). That proves (a) since \( R' \) is clearly quasilocal.

(b) We need to check that \( j \cap R = (0) \). Pick an element \( r \in j \cap R \). Then \((rx)^N, (ry)^N \in (uZ)\). Elements of \((uZ)\) have constant term 0, whence \((rx)^N = (ry)^N = 0\), i.e., \( rx = ry = 0 \), since \( R \) is reduced. Therefore \( r \in \text{Ann } x \cap \text{Ann } y = (0) \) hence \( r = 0 \).

(c) If \( Z \in j \), then \((Zx)^N = uZ \cdot h_1(Z)\) and \((Zy)^N = uZ \cdot h_2(Z)\) and, therefore, \( u \mid x^N \) and \( u \mid y^N \), i.e., \( u \) is a unit. This is a contradiction since \( u \in m \). Thus, \( Z \notin j \) and the image of \( Z \) in \( R' \) is not zero.

(d) (1) Suppose \( f(Z) \in \text{Ann } x \cap \text{Ann } y \) in \( R' \). Then \( f(Z) \cdot x \in j \) and \( f(Z) \cdot y \in j \) in \( R[\bar{Z}] \), hence \((f(Z) \cdot x)^N \in (uZ)\) and \((f(Z) \cdot y)^N \in (uZ)\). Set \( N'' = \max\{2N, 2N'\} \). Then \((f(Z) \cdot x)^N \) and \((f(Z) \cdot y)^N \) belong to \((uZ)\), i.e., \( f(Z) \in j \) hence \( f(Z) = 0 \) in \( R' \).

(2) If \( f(Z) \mid x^N \) and \( f(Z) \mid y^N \) in \( R' \), then \( x^N - f(Z) h(Z) \in j \) and \( y^N - f(Z) h(Z) \in j \), that is, for sufficiently large \( N\) ((\( x^N - f(Z) g(Z) \)) \cdot x)^N = uZ \cdot k_1(Z); ((\( y^N - f(Z) g(Z) \)) \cdot y)^N = uZ \cdot k_2(Z); ((\( y^N - f(Z) h(Z) \)) \cdot y)^N = uZ \cdot t_1(Z)\) and \((y^N - f(Z) h(Z)) \cdot y)^N = uZ \cdot t_2(Z)\). Substituting 0 for \( Z \) we obtain, in \( R, ((x^N - f(0) g(0)) \cdot x)^N = 0 \), i.e., \( (x^N - f(0) g(0)) \cdot x = 0 \), and \((y^N - f(0) g(0)) \cdot y)^N = 0 \), i.e., \( (y^N - f(0) g(0)) \cdot y = 0 \), that is, \( (x^N - f(0) g(0)) \in \text{Ann } x \cap \text{Ann } y = (0) \), hence \( x^N - f(0) g(0) = 0 \) in \( R \). Therefore \( f(0) \) divides \( x^N \).

Similarly \( y^N - f(0) h(0) \Rightarrow f(0) \mid y^N \). Hence \( f(0) \) is a unit in \( R \) and, therefore, \( f(Z) \) is a unit in \( R[\bar{Z}] \) and, of course, \( f(Z) \) is a unit in \( R' \).

**Lemma 3.2.** Let \((R, m)\) be a quasilocal, reduced ring. Let \( x, y \in m \) be such that

1. \( \text{Ann } x \cap \text{Ann } y = (0) \).
2. \( s \mid x^N \) and \( s \mid y^N \Rightarrow s \) is a unit of \( R \).

Then \( R \subseteq R_1 \), where \( R_1 \) is quasilocal, reduced with \( m_{R_1} = m_{R_1} \), (1) and (2) hold in \( R_1 \), and every element of \( m_{R_1} \) is a zerodivisor in \( R_1 \).

Proof. Let \( \lambda \) be an ordinal with first element 0 such that \( A \setminus \{0\} \) is in 1-1 correspondence with \( m_{R_1} \). Construct a chain of rings \( R_\lambda \) indexed by the ordinal \( A \) by transfinite induction. Let \( R_0 = R \). If \( \lambda > 0 \), there are two cases. If \( \lambda \) is a limit ordinal, let \( S_\lambda = \bigcup_{\mu < \lambda} R_\mu \) and then use Lemma 3.1 to enlarge \( S_\lambda \) to a ring \( R_\lambda \) in which \( u_\lambda \) is a zerodivisor and the conditions specified in the conclusion of the Lemma hold. If \( \lambda \) has an immediate predecessor \( \mu \), use Lemma 3.1 likewise to enlarge \( R_\mu \) to an \( R_\lambda \) such that \( u_\lambda \) is a zerodivisor in \( R_\lambda \). Let \( R_1 = \bigcup_{\lambda \in A} R_\lambda \). Q.E.D.
Finally, consider a chain $R \subset R_1 \subset \cdots$ where $R_{n+1} = (R_n)_1$ in the sense of Lemma 3.2, and set $R_\omega = \bigcup_{i \geq 0} R_i$ where $R_0 = R$. Then $R_\omega$ has the following properties:

1. It is quasilocal and reduced.
2. There exist $x, y \in m_\omega$ such that $\text{Ann } x \cap \text{Ann } y = (0)$.
3. Every element of $m_\omega$ is a zerodivisor.

As an example of a ring to start with take $R = K[[X, Y]]$, $K$ a field.

For the ring $(R_\omega, m_\omega)$, the canonical projection $\pi: R_\omega \to R_\omega/m_\omega = K_\omega$ is $R$-compatible in that $a^- = b^-$ in $R_\omega$ implies $\overline{a^-} = \overline{b^-}$, $m_\omega$ is a prime $B$-ideal containing the finitely generated dense ideal $(X, Y)$, hence by Theorem 2.12 the map $R_\omega \to R_\omega^B$ is not universal.

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