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A Note on Baer Rings

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INTRODUCTION

In [3] the author gives a very down-to-earth construction of an embedding of an arbitrary reduced commutative ring R into a Baer ring R^B by an R-compatible ring homomorphism. However, the mapping property claimed in [3] does not hold in the generality stated there: an extra condition on the ring is necessary.

In this paper our main task is to correct that result.

We achieve this goal in Theorem 2.2 where we prove that $R \hookrightarrow R^B$ is a universal embedding if and only if every R-compatible homomorphism $h: R \to S$ from R to a Baer ring S satisfies condition (B): for all given elements $r, b_1, ..., b_t$ $(t \geqslant 1)$ of R, if r belongs to all minimal prime ideals containing b_i, $1 \leqslant i \leqslant t$, then $h(r)$ belongs to all minimal prime ideals containing $h(b_i)$, $1 \leqslant i \leqslant t$, and in Theorem 2.12 where several other conditions are given. We also show that if R is reduced, a polynomial ring over R automatically satisfies these conditions.

In Section 3 we construct a ring which fails to satisfy the conditions of Theorem 2.12 hence proving that the correction is necessary.

We are indebted to K. Prikry for pointing out a gap in the proof in [3] which eventually led to this Note and to M. Hochster for the hospitality and valuable discussions during the preparation of this Note.

SECTION 1

In this section let us briefly recall from [3] some notation, definitions, and the construction of R^B.

First of all, we shall deal with commutative rings with unit. If $\mathfrak{a}$ is an ideal of the ring R, $\mathfrak{a}^{\perp} = \{r \in R \mid r\mathfrak{a} = 0\}$ is the annihilator of $\mathfrak{a}$ and is an ideal. Sometimes we shall write Ann$\mathfrak{a}$ instead of $\mathfrak{a}^{\perp}$. For an element a of R,

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we shall denote by (a) the principal ideal Ra. An element e of R such that $e^2 = e$ is said to be an idempotent. Finally, $\mathfrak{p}$ (resp. $\mathfrak{m}$) will denote a prime (resp. maximal) ideal of R.

DEFINITION 1.1. A *Baer ring* is a ring such that the annihilator of every principal ideal is principal and generated by an idempotent element.

DEFINITION 1.2. Let R, R' be rings. A homomorphism $h: R \to R'$ from R to R' is said to be *R-compatible* if whenever $(a)^\perp = (b)^\perp$, $a, b \in R$, then $(h(a))^\perp = (h(b))^\perp$ in R'.

When $(a)^\perp$ is principal and generated by an idempotent, this idempotent is uniquely determined by a, and we denote it a^*. We write a° for $1 - a^*$. Note that a is idempotent $\Leftrightarrow a = a^\circ \Leftrightarrow a^* = 1 - a$. Therefore Definition 1.2 can be rephrased as follows:

(C). h is an R-compatible ring homomorphism implies that if a^* exists, then $h(a)^*$ exists and, in fact, $h(a)^* = h(a^*)$ (since $(a)^\perp = (a^*) = (1 - a^*)^\perp \mapsto h(a)^\perp = h(1 - a^*)^\perp = (1 - h(a^*))^\perp$).

If R is a Baer ring, then $a^\perp = b^\perp \Leftrightarrow a^* = b^* \Leftrightarrow 1 - a^* = 1 - b^* \Leftrightarrow (1 - a^*)^\perp = (1 - b^*)^\perp$ and $a^\perp = (1 - a^*)^\perp$, $b^\perp = (1 - b^*)^\perp$. Then $(\mathrm{C}) \mapsto h(a^*)$ generates $h(a)^\perp$ and $h(b^*)$ generates $h(b)^\perp$, and since $a^* = b^*$, $h(a^*) = h(b^*)$ and $h(a)^\perp = h(b)^\perp$.

DEFINITION 1.3. An R-compatible homomorphism between two Baer rings is termed a *Baer homomorphism.*

Construction of R^B following [3, *Theorem* 1]

Let R be a reduced ring. Set $X = \mathrm{Min}(R)$ (i.e., the set of all minimal prime ideals of R endowed with the inherited Zariski-topology). For any $x \in X$, $\mathfrak{p}_x$ will denote the minimal prime ideal of R corresponding to the point x. Set $\mathscr{R} = \prod_{x \in X} (R/\mathfrak{p}_x)$ where $R/\mathfrak{p}_x$ is an integral domain. It is not difficult to prove that $\mathscr{R}$ has the strongest Baer property, that is, the annihilator of every ideal is principal and generated by an idempotent (see [4, Theorem 4.11]). In particular, $\mathscr{R}$ is a Baer ring.

Of course, the map $i: R \to \mathscr{R}$, where for each x $i(r)_x = r + \mathfrak{p}_x$, is injective since R is reduced and is R-compatible. However, as $\mathscr{R}$ can be very big if $\mathrm{Min}(R)$ is not finite, in [3] we aimed to find a smaller Baer ring in between. The construction goes as follows. Let us think of R as sitting inside $\mathscr{R}$, i.e., identify R with $i(R) \subset \mathscr{R}$. Hence an element $r \in R$ is a family $(r_x)_{x \in X}$ where $r_x = r + \mathfrak{p}_x$. Set

$$r^\circ = (r_x^\circ)_{x \in X} \in \mathscr{R} \qquad \text{where} \qquad r_x^\circ = \begin{cases} 1 & \text{if} \quad r \notin \mathfrak{p}_x \\ 0 & \text{if} \quad r \in \mathfrak{p}_x \end{cases}$$

and then let $r^* = 1 - r^\circ$. The operations $-^*$ and $-^\circ$ are all to be carried in $\mathscr{R}$. Note that if $r \in R$ and r° or r^* exists in R, then $i(r)^\circ = i(r^\circ)$ and $i(r)^* = i(r^*)$; hence i is an R-compatible monomorphism. Note also that $(r)^\perp = (r^*)$ in $\mathscr{R}$. Now let us consider R^B, the subring of $\mathscr{R}$ generated by the elements r, r^*, $r \in R$. It is shown in [3, Theorem 1] that R^B is a Baer ring. However, the universal property for the map $i: R \to R^B$ does not hold under such a general hypothesis on R. Some restriction is needed.

In the next section we shall provide the appropriate correction and, in Section 3, we shall exhibit an example of a ring failing to satisfy the extra condition.

In particular we shall prove (see Theorem 2.2)

THEOREM. *The following conditions on a reduced ring R are equivalent.*

(1) For every R-compatible homomorphism $h: R \to S$ from R to a Baer ring S there is an induced Baer homomorphism $h^\#$: $R^B \to S$ such that for all $r \in R$, $h^\#(i(r)) = h(r)$ and $h^\#(i(r)^\circ) = h(r)^\circ$.

(2) For every integer $t \geqslant 1$ and elements $r, b_1, ..., b_t$ of R, if r belongs to all minimal prime ideals of R containing b_i, $1 \leqslant i \leqslant t$, then $h(r)$ belongs to all minimal prime ideals of S containing $h(b_i)$, $1 \leqslant i \leqslant t$.

SECTION 2

In this section our aim is to restate Theorem 1 in [3] correctly. Heading to this goal let us investigate in detail what is needed for the "universal" mapping property to hold.

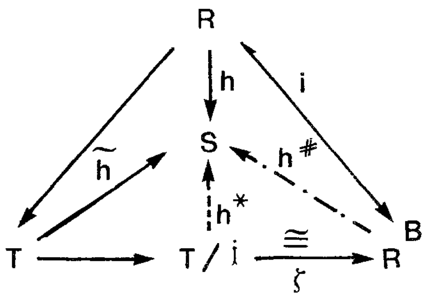
The question is: Given a reduced ring R so that one can construct R^B, is it true that, for every Baer ring S and for every R-compatible ring homomorphism $h: R \to S$, h factors through the R-compatible monomorphism $i: R \to R^B$? In other words, is there a Baer homomorphism $h^\# : R^B \to S$ which extends h? If that were true as stated in [3], it should also be true that whenever $\sum_{\text{finite}} r_i a_i^\circ = 0$ in R^B, then $\sum_{\text{finite}} h(r_i)\, h(a_i)^\circ = 0$ in S and we shall see that this is not so in general (see Sect. 3).

To gain a better insight into the matter let us provide another construction of R^B.

Let $\{X_a \mid a \in R\}$ be a family of indeterminates indexed by R. Set $T = R[X_a \mid a \in R]/(X_a^2 - X_a,\ X_a X_b - X_{ab})_{a,\, b \in R}$. Hence $T = R + \sum_{a \in R} R x_a$ where $x_a^2 = x_a$ and $x_a x_b = x_{ab}$, a, $b \in R$. Let us observe that R^B is nothing else than $R[a^\circ \mid a \in R]$ since $a^* = 1 - a^\circ$ and, therefore, there exists a surjective R-homomorphism $t: T \twoheadrightarrow R[a^\circ \mid a \in R]$ given by $t(x_a) = a^\circ$. Note that $(ab)^\circ = a^\circ b^\circ$. Of course,

$$\operatorname{Ker}(t) = \{r + r_1 x_{a_1} + \cdots + r_m x_{a_m} \in T \mid r + r_1 a_1^\circ + \cdots + r_m a_m^\circ = 0\}.$$

Set $\mathfrak{j} = \mathrm{Ker}(t)$. We obtain Scheme 1. (Note: $\mathfrak{j}$ denotes lowercase German jay.)



SCHEME 1

Comment. We first get a map $\check{h}$ from the polynomial ring $R[X_a \mid a \in R]$ to S by $X_a \mapsto h(a)^c$. Since $h(a)^c$ is idempotent, $\check{h}$ kills $X_a^2 - X_a$: and, since $(h(ab))^c = (h(a)\,h(b))^c = h(a)^c\,g(b)^c$, $\check{h}$ kills $X_{ab} - X_a X_b$. Hence $\check{h}$ induces $\tilde{h}$: $T \to S$.

Since ζ: $T/\mathfrak{j} \to R^B$ is an isomorphism, the existence of $h^{\#}$: $R^B \to S$ will follow from the existence of h^*: $T/\mathfrak{j} \to S$ which makes the above diagram commute. Clearly, h^* exists if and only if $\tilde{h}$ kills $\mathfrak{j}$.

As it is sufficient to show that $\tilde{h}$ kills generators of $\mathfrak{j}$, let us write the elements of $\mathfrak{j}$ as a sum of "simpler" elements.

LEMMA 2.1. *In T every element can be written as a sum of expressions involving mutually orthogonal idempotents in the sense we make precise below in formula* (3).

Proof. Pick an element x of T, hence $x = r \cdot 1 + r_1 x_{a_1} + \cdots + r_m x_{a_m}$. As $x_a^2 = x_a$, $x_a^c = x_a$ and $x_a^* = 1 - x_a$, hence $x_a^c x_a^* = 0$ and $x_a^c + x_a^* = 1$. Also $1 = \prod_{i=1}^m (x_{a_i}^c + x_{a_i}^*)$ or

$$1 = \sum_{(j_1, \dots, j_m) \in \mathbf{2}^m} x_{a_1}^{j_1} \cdots x_{a_m}^{j_m}, \tag{1}$$

where $\mathbf{2} = \{0, *\}$ and any two elements in this sum with distinct indices are mutually orthogonal. Therefore, each $x_{a_i}^\nu$ ($\nu = 0$, $*$ and $i = 1, \dots, m$) can be written as

$$x_{a_i}^\nu = x_{a_i}^\nu \cdot 1 = \sum_{(j_1, \dots, \hat{j}_i, \dots, j_m) \in \mathbf{2}^{m-1}} x_{a_1}^{j_1} \cdots x_{a_{i-1}}^{j_{i-1}} x_{a_i}^\nu x_{a_{i-1}}^{j_{i-1}} \cdots x_{a_m}^{j_m} \tag{2}$$

which implies that

$$x = \sum_{(j_1, \ldots, j_m) \in 2^m} r_{j_1 \cdots j_m} x_{a_1}^{j_1} \cdots x_{a_m}^{j_m}. \quad \text{Q.E.D.} \tag{3}$$

In particular, an element x of T with a fixed representation as in (3) belongs to $\mathfrak{j}$ if and only if $\sum r_{j_1 \cdots j_m} a_1^{j_1} \cdots a_m^{j_m} = 0$ which implies that each term of the sum is 0.

Then, the question of whether $\tilde{h}$ kills $\mathfrak{j}$ reduces to the question of whether

$$\begin{aligned} &\text{If } ra_1^\circ \cdots a_s^0 b_1^* \cdots b_t^* = 0, \text{ then is } \tilde{h}(ra_1^\circ \cdots a_s^\circ b_1^* \cdots b_t^*) = 0, \\ &\text{i.e., is } h(r) \cdot \left(\prod_{i=1}^{s} h(a_i)^\circ \right) \cdot \left(\prod_{j=1}^{t} 1 - h(b_j)^\circ \right) = 0. \end{aligned} \tag{4}$$

But

$r \cdot \left(\prod_{i=1}^{s} a_i^\circ \right) \cdot \left(\prod_{j=1}^{t} b_j^* \right) = 0 \Leftrightarrow$ For every minimal prime ideal $\mathfrak{p}$ of R, either $r \in \mathfrak{p}$ or at least one $a_i \in \mathfrak{p}$ or at least $b_j \notin \mathfrak{p}$; i.e., for every minimal prime ideal $\mathfrak{p}$ of R, either $ra_1 \cdots a_s \in \mathfrak{p}$ or at least one $b_j \notin \mathfrak{p}$.

That is,

$r \cdot \left(\prod_{i=1}^{s} a_i^\circ \right) \cdot \left(\prod_{j=1}^{t} b_j^* \right) = 0 \Leftrightarrow$ Every minimal prime ideal of R which contains $b_1, \ldots, b_t$ $(t \geqslant 1)$ also contains $ra_1 \cdots a_s$ $(s \geqslant 1)$.

We have thus shown

THEOREM 2.2. *Let R be a reduced ring. The following conditions are equivalent:*

(i) *R^B is a universal Baer extension of R.*

(ii) *For every R-compatible homomorphism h: $R \to S$ from R to a Baer ring S, there exists a Baer extension $h^{\#}: R^B \to S$ of h.*

(iii) *For all $r, b_1, \ldots, b_t$ $(t \geqslant 1)$ elements of R, if r belongs to all minimal prime ideals of R containing b_i, $1 \leqslant i \leqslant t$, then $h(r)$ belongs to all minimal prime ideals of S containing $h(b_i)$, $1 \leqslant i \leqslant t$.*

Throughout we shall refer to (iii) as to condition (B).
Let us head to a characterization of such rings R.

PROPOSITION 2.3. *Let R be a ring, t an element of R, and R_t the localization of R at the element t. Then*

(1) *The natural map $\varphi: R \to R_t$ is R-compatible.*

(2) *If R is reduced, then for all $u, v \in R$*

(α) $v/1 \in (u/1)^\perp$ *in* $R_t \Leftrightarrow v \in (tu)^\perp$ *in* R.

(β) $(u/1)^\perp = (v/1)^\perp$ *in* $R_t \Leftrightarrow (tu)^\perp = (tv)^\perp$ *in* R.

(3) *If R is a Baer ring, then φ is a Baer homomorphism.*

Proof. (1) It is enough to recall from [2, Proposition 3.14] that $S^{-1}(\operatorname{Ann} M) = \operatorname{Ann}(S^{-1}M)$ for all finitely generated R-modules M.

(2) $-\alpha$. Let us assume that $v \in (tu)^\perp$ in R, i.e., $v(tu) = 0$. Then $\varphi(v(tu)) = (v/1)(tu/1) = 0/1$ and $t/1$ invertible imply $v/1 \in (u/1)^\perp$. Conversely, let $v/1 \in (u/1)^\perp$ in R_t. Then $t^k(vu) = 0$ in R for some integer $k \geqslant 0$ implies $(tvu)^k = 0$. Therefore $vtu = 0$ as R is a reduced ring, i.e., $v \in (tu)^\perp$ in R. (2) $-\beta$. It follows from (1) and (2) $-\alpha$.

(3) It follows from (1) and Definition 1.3 as it is easy to check that R_t is a Baer ring as well.

COROLLARY 2.4. *An R-compatible homomorphism $h: R \to R'$ between two reduced rings R, R' induces an R-compatible map $h_t: R_t \to R'_{h(t)}$ for all elements t of R.*

Proof. Let us remark that in R_t, $\operatorname{Ann}(b/t) = \operatorname{Ann}(b/1)$ as $1/t$ is invertible. Thus we have to prove that if $\operatorname{Ann}(b/1) = \operatorname{Ann}(c/1)$ in R_t, then $\operatorname{Ann}(h(v)/1) = \operatorname{Ann}(h(c)/1)$ in $S_{h(t)}$, for all $b/1$, $c/1$ in R_t. By Proposition 2.3: (2) $-\beta$, $\operatorname{Ann}(b/1) = \operatorname{Ann}(c/1) \Leftrightarrow (tb)^\perp = (tc)^\perp$ which implies $(h(tb))^\perp = (h(tc))^\perp$ by the R-compatibility of h, and this means $\operatorname{Ann}(h(b)/1) = \operatorname{Ann}(h(c)/1)$ by Proposition 2.3: (2) $-\beta$ already mentioned.

DEFINITION 2.5. An ideal $\mathfrak{i}$ of R is said to be a *B-ideal* if for all elements u, v of R, $u^\perp = v^\perp$ and $u \in \mathfrak{i}$, then $v \in \mathfrak{i}$.

EXAMPLES. The ring itself, the zero ideal, a minimal prime ideal, and, of course, any intersection of them are B-ideals.

DEFINITION 2.6. A B-ideal of a Baer ring is termed a *Baer ideal.*

DEFINITION 2.7. A *dense* ideal $\mathfrak{b}$ of R is an ideal with $\mathfrak{b}^\perp = (0)$.

A few properties of B-ideals, Baer ideals, and dense ideals strictly related to our goal are

PROPOSITION 2.8. (1) *A B-ideal of a reduced ring is radical.*

(1′) *An ideal of a Baer ring is a Baer ideal if and only if it is an intersection of minimal prime ideals of the ring.*

(2) *An ideal of a reduced ring R is a B-ideal if and only if it is the kernel of an R-compatible ring homomorphism having R as a source.*

(2′) *An ideal of a Baer ring S is a Baer ideal if and only if it is the kernel of a Baer homomorphism from S to a Baer ring S′.*

(3) *If* $\mathfrak{b}=(b_1, ..., b_t)$ *is a finitely generated ideal of a reduced ring R, then* $\mathfrak{b}$ *is not dense* (*i.e.*, $\mathfrak{b}^{\perp}\neq(0)$) $\Leftrightarrow \exists a\in R-\{0\}$ *such that* $ab_i=0$, $1\leqslant i\leqslant t \Leftrightarrow V^{c}(\mathfrak{b})=\{\mathfrak{p}\in \mathrm{Min}(R)\mid \mathfrak{p}\supseteq\mathfrak{b}\}\neq\varnothing$.

Proof. (1) For an element x of a reduced ring R we have $(x^n)^{\perp}=x^{\perp}$, hence if $x^n\in\mathfrak{i}$, then $x\in\mathfrak{i}$ since $\mathfrak{i}$ is a B-ideal, i.e., $\mathfrak{i}$ is radical.

(2) Let $\mathfrak{i}$ be a B-ideal of R. Set $\bar{R}=R/\mathfrak{i}$. *Claim*: The natural map $\pi\colon R\twoheadrightarrow\bar{R}$ is R-compatible.

In fact, let $r, u\in R$ be such that $r^{\perp}=u^{-}$. Two cases are possible. 1st *Case*. If r (or u) $\in\mathfrak{i}$, then u (or r) $\in\mathfrak{i}$, hence $\bar{R}=\bar{r}^{\perp}=\bar{u}^{-}$. 2nd *Case*. Assume $r\notin\mathfrak{i}$ and $\bar{r}^{\perp}\neq\bar{u}^{\perp}$. Then there exists an element $\bar{t}\in\bar{R}$ such that $\bar{t}\cdot\bar{r}=\bar{0}$ and $\bar{t}\cdot\bar{u}\neq\bar{0}$; that is, $tr\in\mathfrak{i}$ and $tu\notin\mathfrak{i}$, a contradiction since $r^{\perp}=u^{-}\Rightarrow (tr)^{\perp}=(tu)^{\perp}$ for all $t\in R$, because R is reduced (see Proposition 2.3). Conversely, let $\varphi\colon R\to R'$ be an R-compatible homomorphism. Set $\mathfrak{i}=\mathrm{Ker}\,\varphi$. Let $r, u\in R$ have the property that $r^{-}=u^{\perp}$. If $r\in\mathfrak{i}$, then $R'=(\varphi r)^{\perp}=(\varphi u)^{\perp}$ which implies $\varphi u=0$ hence $u\in\mathfrak{i}$. Note that we do not need R to be reduced in this part.

For the proof of (1′) see [8], for the proof of (3) see [1]. (2′) follows from (2).

DEFINITION 2.9. An R-compatible homomorphism $h\colon R\to S$ from a reduced ring R to a Baer ring S is said to satisfy condition ($\mathrm{B}_{\circ}$) if

($\mathrm{B}_{\circ}$) For all elements $b_1, ..., b_t$ ($t\geqslant 1$) of R, if no minimal prime ideal of R contains b_i, $1\leqslant i\leqslant t$, then no minimal prime ideal of S contains $h(b_i)$, $1\leqslant i\leqslant t$.

Since no minimal prime ideal of R contains b_i, $1\leqslant i\leqslant t\Leftrightarrow$ the ideal $\mathfrak{b}=(b_1, ..., b_t)$ is dense, condition ($\mathrm{B}_{\circ}$) says that under h a finitely generated dense ideal of R expands to a dense ideal of S.

Remark 2.10. Of course, if an R-compatible homomorphism $h\colon R\to S$ from a reduced ring R to a Baer ring S satisfies condition (B), then it satisfies condition ($B_\circ$), since no minimal prime ideal contains b_i, $1 \leqslant i \leqslant t \Leftrightarrow 1$ belongs to all minimal prime ideals containing b_i, $1 \leqslant i \leqslant t$.

THEOREM 2.11. *Let R be a reduced ring, S a Baer ring, and $h: R \to S$ an R-compatible homomorphism.* TFAE

(i) *h satisfies condition* (B).

(ii) *$h_f: R_f \to S_{h(f)}$ satisfies* (B) *for all $f \in R$.*

(iii) *$h_f: R_f \to S_{h(f)}$ satisfies* ($B_\circ$) *for all $f \in R$.*

Proof. (ii) $\Rightarrow$ (iii) for all $f \in R$ by Remark 2.10. (iii) $\Rightarrow$ (i). If (B) fails, we get elements r, $b_1, ..., b_t$ $(t \geqslant 1)$ in R such that r belongs to all minimal prime ideals of R containing $b_1, ..., b_t$ but $h(r)$ does not belong to a minimal prime ideal $\mathfrak{q}$ of S containing $h(b_1), ..., hb_t)$. In the ring R_r, $b_1/1, ..., b_t/1$ do not belong to any minimal prime ideal. By $B_\circ$) for $(R_r, S_{h(r)}, h_r)$ the images $h_r(b_1)/1, ..., h_r(b_t)/1$ are not in any minimal prime ideal of $S_{h(r)}$. But $\mathfrak{q} \cdot S_{h(r)}$ gives a minimal prime which contains $h_r(b_i)/1$, $1 \leqslant i \leqslant t$, a contradiction.

(i) $\Rightarrow$ (ii). Given r/f^m, b_i/f^{m_i}, $1 \leqslant i \leqslant t$, elements of R_f, to show that if r/f^m belongs to all minimal primes containing b_i/f^{m_i}, $1 \leqslant i \leqslant t$, then $h(r)/h(f)^m$ belongs to all minimal primes of $S_{h(f)}$ containing $h(b_i)/h(f)^{m_i}$, $1 \leqslant i \leqslant t$, is equivalent to showing that if $r/1$ belongs to all minimal primes containing $b_i/1$, $1 \leqslant i \leqslant t$, in R_f, then $h(r)/1$ belongs to all minimal primes containing $h(b_i)/1$, $1 \leqslant i \leqslant t$, in $S_{h(f)}$.

If not, choose a minimal prime ideal $\mathfrak{q}$ of $S_{h(f)}$ containing $h(b_i)/1$ $(1 \leqslant i \leqslant t)$ and not containing $h(r)/1$.

Claim. Every minimal prime ideal of R which contains b_i, $1 \leqslant i \leqslant t$, contains rf. Assume not and let $\mathfrak{p}$ be a minimal prime ideal containing b_i, $1 \leqslant i \leqslant t$, and not rf. Then $f \notin \mathfrak{p}$ implies $\mathfrak{p}R_f$ is a minimal prime containing $b_i/1$ $(i = 1, 2, ..., t)$; hence $\mathfrak{p}R_f$ contains $r/1$. This implies $f^k r \in \mathfrak{p} \Rightarrow (fr)^k \in \mathfrak{p} \Rightarrow fr \in \mathfrak{p}$, a contradiction. Therefore, every minimal prime of S which contains $h(b_i)$, $i = 1, ..., t$, contains $h(rf) = h(r)\,h(f)$, a contradiction since $\mathfrak{q}^c$ does not contain $h(f)\,h(r)$ and is a minimal prime containing $h(b_i)$, $i = 1, ..., t$. Q.E.D.

For the next result we need some notation. Let R be a reduced ring. For an element r of R, set $Y = \text{Min}(R_r)$, while $X = \text{Min}(R)$. Let $X_r = \{x \in X / r \notin \mathfrak{p}_x\}$. There is a canonical homomorphism η from X_r to Y. Let $\rho: \prod_{x \in X} R/\mathfrak{p}_x \to \prod_{x \in X_r} R/\mathfrak{p}_x$ be the restriction map.

THEOREM 2.12. *There are natural isomorphisms* $(R^B)_r = (R^B)_{i(r)} \cong (R_r)^B \cong \rho(R^B)_{\rho(i(r))}$.

Proof. Let ρ^B be the restriction of ρ to R^B so that $\rho^B\colon R^B \twoheadrightarrow \rho(R^B)$. Set $\mathfrak{j} = \mathrm{Ker}(\rho^B)$. $\mathfrak{j}$ consists precisely of the elements of R^B vanishing on Y, whence $\mathfrak{j} = i(r)^{\perp} = \bigcup_n (i(r)^n)^{\perp}$. Therefore the induced map $\rho^B_{i(r)}\colon (R^B)_{i(r)} \to (\rho(R^B))_{\rho(i(r))}$ is an isomorphism. Let $j\colon R_r \to \prod_{y \in Y} R_r/\mathfrak{p}_y$ be the map for R_r which corresponds to the map i for R defined earlier. The maps $\rho_{i(r)}$ and ψ in the commutative diagram below

$$\begin{array}{ccc}
R_r & \overset{j}{\hookrightarrow} & \prod_{y \in Y} (R_r/\mathfrak{p}_y) \\
 & \searrow \ (R_r)^B \nearrow & \\
{\scriptstyle i_r}\downarrow & & \uparrow{\scriptstyle \psi} \\
\left(\prod_{x \in X} R/\mathfrak{p}_x\right)_{i(r)} & \xrightarrow{\rho_{i(r)}} & \left(\prod_{x \in X_r} R/\mathfrak{p}_x\right)_{\rho(i(r))}
\end{array}$$

are easily seen to be isomorphisms. (Here, if $g \in \prod_{x \in X_r} R/\mathfrak{p}_x$, $\psi(g/1)(\eta(x)) = g(x)/1$; $R_r/\mathfrak{p}_{\eta(x)}$ is identical with $(R/\mathfrak{p}_x)_r$.) By definition, $(R_r)^B$ is the subring of $\prod_{y \in Y} R_r/\mathfrak{p}_y$ generated by the elements $j(f)$, $f \in R$, $1/j(r)$, and $(j(f)/j(r)^k)^{\circ}$, $f \in R$, or by the elements $j(f)$, $f \in R$, $1/j(r)$, and $(j(f)\,j(r))^{\circ} = (j(fr))^{\circ}$, $f \in R$. The image of this subring under ψ^{-1} in $(\prod_{x \in X_r} R/\mathfrak{p}_x)_{\rho(i(r))}$ is the subring generated by $\rho(j(f))$, $f \in R$, $\rho(j(fr))^{\circ} = \rho[i(f)^{\circ}]$, i.e., $(R_r)^B$ viewed in $(\prod_{x \in X_r} R/\mathfrak{p}_x)_{\rho(i(r))}$ is the subring generated by $\rho(i(f))$, $f \in R$, $1/\rho(i(r))$, and $\rho[i(f)^{\circ}]$, $f \in R$, which is exactly $\rho(R^B)_{\rho(i(r))}$. Therefore we have got the isomorphisms $(R_r)^B \cong \rho(R^B)_{\rho(i(r))} \cong (R^B)_{i(r)} = (R^B)_r$.

PROPOSITION 2.13. *If all R-compatible homomorphisms $h\colon R \to S$ from a reduced ring R to a Baer ring S satisfy condition* (B), *then all R-compatible homomorphisms $k\colon R_r \to T$ from R_r to a Baer ring T satisfy* (B).

Proof. Choose an element r of R and let $k\colon R_r \to T$ be such a homomorphism. Note that $k(r/1)$ is invertible in T. First we get an R-compatible map $h\colon R \to^{\varphi} R_r \to^{k} T$, hence there exists $h^{\#}\colon R^B \to T$ such that $h^{\#} \circ i = k \circ \varphi = h$. By localizing R^B at $i(r)$ we get a map $\varphi^*\colon R_r \to (R^B)_{i(r)}$ by the universality of R_r and also a map $(R^B)_{i(r)} \to T$ since $h(r)$ is invertible in T. Hence by the isomorphism $(R^B)_{i(r)} \cong (R_r)^B$ established earlier we obtain a map $(R_r)^B \to T$ which says that k satisfies (B).

Our task is at end since we can prove

THEOREM 2.14. TFAE *on a reduced ring R.*

(1) *Every R-compatible homomorphism $h\colon R \to S$ from R to a Baer ring S satisfies condition* (B).

(2) *$R \subsetneq R^B$ is a universal R-compatible embedding.*

(3) *A proper B-ideal of R_r has no dense finitely generated subideal, for all r in R.*

(4) *A prime B-ideal of R_r has no dense finitely generated subideal, for all r in R.*

(5) *Every R-compatible map $R_r \to K$ satisfies condition (B_0) for all fields K and r in R.*

Proof. (1)$\Leftrightarrow$(2) by Theorem 2.2. (1)$\Rightarrow$(3), (1)$\Rightarrow$(5) are easy to prove. (3)$\Rightarrow$(4) is trivial.

(4)$\Rightarrow$(1). Let us assume that (4) holds. We want to prove that every R-compatible map $h: R \to S$, S Baer ring, satisfies condition (B). *Claim*: It suffices to show that an R-compatible map $R_r \to S'$ satisfies condition (B_0) for all Baer rings S' and r in R.

Assume not and let $h: R_r \to S'$ fail to satisfy condition (B_0).Then there exists a finitely generated dense ideal of R_r which does not expand to a dense ideal in S'. Say $\mathfrak{b} = (b_1, ..., b_t)$. Choose a minimal prime ideal $\mathfrak{q}$ of S' containing $h(b_i)$, $1 \leqslant i \leqslant t$. *Claim*: $h^{-1}(\mathfrak{q})$ is a B-ideal of R_r. If not, let $x^{\perp} = y^{\perp}$ in R_r and $x \in h^{-1}(\mathfrak{q})$, $y \notin h^{-1}(\mathfrak{q})$. Since h is R-compatible, we have $h(x)^{\perp} = h(y)^{\perp}$, a contradiction because $h(x) \in \mathfrak{q} \Rightarrow h(x)^{\perp} \nsubseteq \mathfrak{q}$, but $h(y) \notin \mathfrak{q} \Rightarrow h(y)^{\perp} = h(x)^{\perp} \subseteq \mathfrak{q}$.

(5)$\Rightarrow$(1). If not, let $h: R \to S$ fail to satisfy condition (B), i.e., there exist elements r, b_1,..., b_t in R such that r belongs to all minimal primes of R containing b_i, $1 \leqslant i \leqslant t$, but $h(r) \notin \mathfrak{q}$ a minimal prime ideal of S which contains $h(b_i)$, $1 \leqslant i \leqslant t$. By localizing at r and $h(r)$ and then taking the fraction field K of $S_{h(r)}/\mathfrak{q}S_{h(r)}$, we obtain an R-compatible map

$$R_r \to S_{h(r)} \to S_{h(r)}/\mathfrak{q}S_{h(r)} \to K$$

which maps the finitely generated dense ideal $(b_1, ..., b_t)$ to (0), a contradiction.

Next is a result, interesting in itself, which implies that for a reduced ring R, the embedding $R[X] \hookrightarrow R[X]^B$ is automatically universal.

THEOREM 2.15. *Let R be a reduced ring. Then in $R[X]_f$, $f \in R[X]$, every finitely generated dense ideal contains a nonzerodivisor.*

Proof. Suppose that $d_0/f^{t_0}, ..., d_r/f^{t_r} \in R[X]_f$ have no common annihilator. Then the elements $d_i/1$, $i = 0, 1, ..., r$, have no common annihilator. *Claim*: If $N > \sup\{\deg d_i,\ 0 \leqslant i \leqslant r\}$, then $\sum_{i=0}^{r} d_i X^{N^i}/1$ is a nonzerodivisor. *Proof.* Say g/f^j kills it. Then $G = f^k g$ kills $D = \sum_{i=0}^{r} d_i X^{N^i}$ in $R[X]$, for some sufficiently large k.

It suffices to show that if G kills $\sum_{i=0}^{r} d_i X^{N^i}$ in $R[X]$ then G kills each $d_i (0 \leqslant i \leqslant r)$, for then $G/1 = 0$ in $R[X]_f$. Let C_G, C_D be the ideals of R generated by the coefficients of G and D respectively. $GD=0 \Rightarrow C_G \cdot C_D=(0)$. (If not, choose a minimal prime ideal $\mathfrak{p}$ of R such that $\mathfrak{p} \not\supseteq C_G \cdot C_D$, that is, $\mathfrak{p} \not\ni c_1 \cdot c_2$ for some coefficient c_1 of G and some coefficient c_2 of D. Then $GD \not\equiv 0 \bmod \mathfrak{p}$, a contradiction.) $C_G \cdot C_D=(0)$, however, implies that $C_G \cdot C_{d_i}=(0)$ since $C_{d_i} \subset C_D$, $0 \leqslant i \leqslant r$. Hence $Gd_i=0$ for $i=0, 1, ..., r$, i.e., $G/1$ kills $d_i/1$ in $R[X]_f$ for all i. Thus $G/1=f^k g/1=0/1$ in $R[X]_f$, i.e., $g/1=0/1$ in $R[X]_f$. Q.E.D.

COROLLARY 2.16. *For a reduced ring R, then embedding $R[X] \subsetneq R[X]^B$ is universal.*

Proof. Assume not and let $h: R[X]_f \to S$ fail to satisfy Theorem 2.14: (4). Let $\mathfrak{d}=(d_0, ..., d_r)$ be a finitely generated dense ideal of $R[X]_f$ which expands to a nondense ideal. There exists a minimal prime ideal $\mathfrak{q}$ of S containing $h(d_0), ..., h(d_r)$, whence $h^{-1}(\mathfrak{q})$ contains $\mathfrak{d}=(d_0, ..., d_r)$ which contains a nonzerodivisor δ by Theorem 2.15. But h is R-compatible and, therefore, $h(\delta)^{\perp}=h(1)^{-}=(0) \subseteq \mathfrak{q}$, a contradiction since $h(\delta) \in \mathfrak{q}$.

SECTION 3

In this section we shall exihibit a ring which fails to satisfy condition (B_0) and hence the conclusion of Theorem 2.12 does not hold for it. Therefore, Theorem 1 as stated in [3] is not correct.

We shall construct a reduced quasilocal ring $(R_\omega, \mathfrak{m}_\omega)$ and elements x, $y \in \mathfrak{m}_\omega$ such that $x^{\perp} \cap y^{\perp}=(x, y)^{-}=(0)$, but every element of $\mathfrak{m}_\omega$ is a zerodivisor. It is then immediate that $R_\omega \twoheadrightarrow K_\omega/\mathfrak{m}_\omega$ is an R-compatible map from R_ω to a field K_ω which does not satisfy (B_0) or (B). Hence $R_\omega \subsetneq R_\omega^B$ does not have the universal mapping property and this is not the universal Baer embedding of R_ω.

LEMMA 3.1. *Let $(R, \mathfrak{m})$ be a quasilocal reduced ring with $x, y \in \mathfrak{m}$ such that*

(1) $\operatorname{Ann} x \cap \operatorname{Ann} y=(0)$.

(2) *If $s \mid x^n$ and $s \mid y^n$, then s is a unit.*

Let $u \in \mathfrak{m}$. Set $R'=R[\![Z]\!]/\mathfrak{j}$ where $\mathfrak{j}=\{w \in R[\![Z]\!]/\exists N$ such that $(xw)^N$, $(yw)^N \in (uZ)\}$. Then

(a) *R' is quasilocal and reduced.*

(b) *$\mathfrak{j} \cap R=(0)$ and hence $R \subsetneq R'$ and $\mathfrak{m}_R \subsetneq \mathfrak{m}_{R'}$.*

(c) *The image of Z in R' is not zero, $uZ=0$ in R', and hence u is a zerodivisor in R'.*

(d) *In R' (1) and (2) hold for the images of x and y.*

Proof. (a) Let $\bar{\alpha} \in R'$ be such that $\bar{\alpha}^i = \bar{0}$, hence $\alpha^i \in \mathfrak{j}$ in $R[\![Z]\!]$ for some i, that is, $(\alpha^i x)^n$, $(\alpha^i y)^m \in (uZ)$. Set $N = \max(in, im)$. Then $(\alpha x)^N$, $(\alpha y)^N \in (uZ)$, whence $\alpha \in \mathfrak{j}$, i.e., $\bar{\alpha} = \bar{0}$. That proves (a) since R' is clearly quasilocal.

(b) We need to check that $\mathfrak{j} \cap R = (0)$. Pick an element $r \in \mathfrak{j} \cap R$. Then $(rx)^N$, $(ry)^N \in (uZ)$. Elements of (uZ) have constant term 0, whence $(rx)^N = (ry)^N = 0$, i.e., $rx = ry = 0$, since R is reduced. Therefore $r \in \operatorname{Ann} x \cap \operatorname{Ann} y = (0)$ hence $r = 0$.

(c) If $Z \in \mathfrak{j}$, then $(Zx)^n = uZ \cdot h_1(Z)$ and $(Zy)^n = uZ \cdot h_2(Z)$ and, therefore, $u \mid x^n$ and $u \mid y^n$, i.e., u is a unit. This is a contradiction since $u \in \mathfrak{m}$. Thus, $Z \notin \mathfrak{j}$ and the image of Z in R' is not zero.

(d) (1) Suppose $\overline{f(Z)} \in \operatorname{Ann} \bar{x} \cap \operatorname{Ann} \bar{y}$ in R'. Then $f(Z) \cdot x \in \mathfrak{j}$ and $f(Z) \cdot y \in \mathfrak{j}$ in $R[\![Z]\!]$, hence $(f(Z) \cdot x)^N \in (uZ)$ and $(f(Z)y \cdot y)^{N'} \in (uZ)$. Set $N'' = \max\{2N, 2N'\}$. Then $(f(Z)\,x)^{N''}$ and $(f(Z)\,y)^{N''}$ belong to (uZ), i.e., $f(Z) \in \mathfrak{j}$ hence $\overline{f(Z)} = \bar{0}$ in R'.

(2) If $\overline{f(Z)} \mid \bar{x}^n$ and $\overline{f(Z)} \mid \bar{y}^n$ in R', then $x^n - f(Z)\,g(Z) \in \mathfrak{j}$ and $y^n - f(Z)\,h(Z) \in \mathfrak{j}$, that is, for sufficiently large N $((x^n - f(Z)\,g(Z)) \cdot x)^N = uZ \cdot k_1(Z)$; $((x^n - f(Z)\,g(Z)) \cdot y)^N = uZ \cdot k_2(Z)$; $((y^n - f[Z]h(Z)) \cdot x)^N = uZ \cdot t_1(Z)$ and $((y^n - f(Z)\,h(Z)) \cdot y)^N = uZ \cdot t_2(Z)$. Substituting 0 for Z we obtain, in R, $((x^n - f(0)\,g(0))\,x)^N = 0$, i.e., $(x^n - f(0)\,g(0))\,x = 0$, and $((x^n - f(0)\,g(0))\,y)^N = 0$, i.e., $(x^n - f(0)\,g(0))\,y = 0$, that is, $(x^n - f(0)\,g(0)) \in \operatorname{Ann} x \cap \operatorname{Ann} y = (0)$, hence $x^n - f(0)\,g(0) = 0$ in R. Therefore $f(0)$ divides x^n.

Similarly $y^n = f(0)\,h(0) \Rightarrow f(0) \mid y^n$. Hence $f(0)$ is a unit in R and, therefore, $f(Z)$ is a unit in $R[\![Z]\!]$ and, of course, $\overline{f(Z)}$ is a unit in R'.

LEMMA 3.2. *Let $(R, \mathfrak{m})$ be a quasilocal, reduced ring. Let $x, y \in \mathfrak{m}$ be such that*

(1) $\operatorname{Ann} x \cap \operatorname{Ann} y = (0)$.

(2) *$s \mid x^n$ and $s \mid y^n \Rightarrow s$ is a unit of R.*

Then $R \subset R_1$, where R_1 is quasilocal, reduced with $\mathfrak{m}_R \subset \mathfrak{m}_{R_1}$, (1) *and* (2) *hold in R_1, and every element of $\mathfrak{m}_R$ is a zerodivisor in R_1.*

Proof. Let Λ be an ordinal with first element 0 such that $\Lambda - \{0\}$ is in 1–1 correspondence with $\mathfrak{m}_R$. Construct a chain of rings R_λ indexed by the ordinal Λ by transfinite induction. Let $R_0 = R$. If $\lambda > 0$, there are two cases. If λ is a limit ordinal, let $S_\lambda = \bigcup_{\mu < \lambda} R_\mu$ and then use Lemma 3.1 to enlarge S_λ to a ring R_λ in which u_λ is a zerodivisor and the conditions specified in the conclusion of the Lemma hold. If λ has an immediate predecessor $\mu \geqslant 0$, use Lemma 3.1 likewise to enlarge R_μ to an R_λ such that u_λ is a zerodivisor in R_λ. Let $R_1 = \bigcup_{\lambda \in \Lambda} R_\lambda$. Q.E.D.

Finally, consider a chain $R \subset R_1 \subset \cdots$ where $R_{n+1} = (R_n)_1$ in the sense of Lemma 3.2, and set $R_\omega = \bigcup_{i \geqslant 0} R_i$ where $R_0 = R$. Then R_ω has the following properties:

(1) It is quasilocal and reduced.

(2) There exist $x, y \in \mathfrak{m}_\omega$ such that $\operatorname{Ann} x \cap \operatorname{Ann} y = (0)$.

(3) Every element of $\mathfrak{m}_\omega$ is a zerodivisor.

As an example of a ring to start with take $R = K[\![X, Y]\!]$, K a field.

For the ring $(R_\omega, \mathfrak{m}_\omega)$, the canonical projection $\pi: R_\omega \to R_\omega/\mathfrak{m}_\omega = K_\omega$ is R-compatible in that $a^- = b^\perp$ in $R_\omega \Rightarrow \bar{a}^- = \bar{b}^\perp$, $\mathfrak{m}_\omega$ is a prime B-ideal containing the finitely generated dense ideal (X, Y), hence by Theorem 2.12 the map $R_\omega \to R_\omega^B$ is not universal.

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