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Algebraic Groups with a Commuting Pair of Involutions and Semisimple Symmetric Spaces

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INTRODUCTION

Let G be a connected reductive algebraic group defined over an algebraically closed field F of characteristic not 2. Denote the Lie algebra of G by $\mathfrak{g}$.

In this paper we shall classify the isomorphism classes of ordered pairs of commuting involutorial automorphisms of G. This is shown to be independent of the characteristic of F and can be applied to describe all semisimple locally symmetric spaces together with their fine structure.

Involutorial automorphisms of $\mathfrak{g}$ occur in several places in the literature. Cartan has already shown that for $F = \mathbb{C}$, the isomorphism classes of involutorial automorphisms of $\mathfrak{g}$ correspond bijectively to the isomorphism classes of real semisimple Lie algebras, which correspond in their turn to the isomorphism classes of Riemannian symmetric spaces (see Helgason [11]). If one lifts this involution to the group G, then the present work gives a characteristic free description of these isomorphism classes. In a similar manner we can show that semisimple locally symmetric spaces correspond to pairs of commuting involutorial automorphisms of $\mathfrak{g}$. Namely let $(\mathfrak{g}_0, \sigma)$ be a semisimple locally symmetric pair; i.e., $\mathfrak{g}_0$ is a real semisimple Lie algebra and $\sigma \in \mathrm{Aut}(\mathfrak{g}_0)$ an involution. Then by a result of Berger [2], there exists a Cartan involution θ of $\mathfrak{g}_0$, such that $\sigma\theta = \theta\sigma$. If we denote the complexification of $\mathfrak{g}_0$ by $\mathfrak{g}$, then σ and θ induce a pair of commuting involutions of $\mathfrak{g}$. Conversely, if σ, $\theta \in \mathrm{Aut}(\mathfrak{g})$ are commuting involutions, then σ and θ determine two locally semisimple symmetric pairs. For if $\mathfrak{u}$ is a σ- and θ-stable compact real form with conjugation τ, then $(\mathfrak{g}_{\theta\tau}, \sigma \mid \mathfrak{g}_{\theta\tau})$ and $(\mathfrak{g}_{\sigma\tau}, \theta \mid \mathfrak{g}_{\theta\tau})$ are semisimple locally symmetric pairs where

$$\mathfrak{g}_{\theta\tau} = \{X \in \mathfrak{g} \mid \theta\tau(X) = X\} \qquad \text{and} \qquad \mathfrak{g}_{\sigma\tau} = \{X \in \mathfrak{g} \mid \sigma\tau(X) = X\}.$$

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These pairs are called *dual*. To get a correspondence with these locally symmetric pairs we consider ordered pairs of involutions and let (θ, σ) correspond to the first and (σ, θ) to the second.

So let $\sigma, \theta \in \mathrm{Aut}(G)$ be commuting involutions and let G_σ resp. G_θ denote the group of fixed points of σ resp. θ. For a σ- and θ-stable torus T of G we write

$$T_\theta^+ = (T \cap G_\theta)^0 \qquad \text{and} \qquad T_\theta^- = \{t \in T \mid \theta(t) = t^{-1}\}^0.$$

The second torus is called a *θ-split* torus of G. Similarly we define T_σ^+ and T_σ^-. The torus $(T_\sigma^-)_\theta^- = \{t \in T \mid \sigma(t) = \theta(t) = t^{-1}\}^0$ is called *(σ, θ)-split* and is denoted by $T_{\sigma,\theta}^-$. Denote the set of characters, the set of roots, and the Weyl group of T with respect to G by, respectively, $X^*(T)$, $\Phi(T)$, and $W(T)$. We use the notation $\mathscr{C}$ for the set of $\mathrm{Int}(G)$-isomorphism classes of ordered pairs of commuting involutions of G and the notation $\mathscr{C}(T)$ for the set of $W(T)$-conjugacy classes of ordered pairs of commuting involutions of $(X^*(T), \Phi(T))$, where T is a maximal torus of G.

To classify these isomorphism classes of ordered pairs of commuting involutions we construct a map from $\mathscr{C}$ to $\mathscr{C}(T)$ (for a fixed maximal torus T) and classify its image and the fibers. In order to construct such a map one could take in any class c of $\mathscr{C}$ a representative (σ, θ) such that T is σ- and θ-stable and consider the $W(T)$-conjugacy class of $(\sigma \mid T, \theta \mid T)$. However, this leaves too much freedom for the choice of (σ, θ). Different representatives of the class c in $\mathscr{C}$, stabilizing T, can induce different classes in $\mathscr{C}(T)$. Hence we have to demand more properties of the representative.

In the case of a single involution (i.e., $\sigma = \theta$) one has two possible choices. Namely one can require of the representative (θ, θ) of c that T_θ^+ is a maximal torus of G_θ or that T_θ^- is a maximal θ-split torus of G. Cartan used the first choice to classify the real semisimple Lie algebras. For Riemannian symmetric spaces however the second choice is more natural, because one obtains also the restricted root system of the symmetric space, which coincides with the non-zero restrictions of $\Phi(T)$ to T_θ^-. Araki [1] followed this method to classify the Riemannian symmetric spaces.

To classify the semisimple locally symmetric pairs, Berger [2] made a choice analogous to that of Cartan, but did not obtain any results concerning the fine structure of those spaces. Our choice is similar to that of Araki. To be more specific, we call a pair (σ, θ) *normally related to* T if T is σ- and θ-stable and if $T_{\sigma,\theta}^-$, T_σ^-, T_θ^- are respectively maximal (σ, θ)-split, σ-split, and θ-split. As in the case of a single involution, $\Phi(T_{\sigma,\theta}^-)$ is the natural root system of the corresponding symmetric pair. Every class in $\mathscr{C}$ contains a pair (σ, θ) which is normally related to T (see (5.13)). Denoting the center of G by $Z(G)$, we have furthermore:

5.16. THEOREM. *Let* (σ_1, θ_1) *and* (σ_2, θ_2) *be pairs of commuting involutorial automorphisms of* G, *normally related to* T. *Then* $(\sigma_1, \theta_1) \mid T$ *and* $(\sigma_2, \theta_2) \mid T$ *are conjugate under* $W(T)$ *if and only if there exists* $\varepsilon \in T^-_{\sigma,\theta}$ *with* $\varepsilon^2 \in Z(G)$ *such that* (σ_2, θ_2) *is isomorphic to* $(\sigma_1, \theta_1 \operatorname{Int}(\varepsilon))$.

The elements $\varepsilon \in T^-_{\sigma,\theta}$ such that $\varepsilon^2 \in Z(G)$ are called *quadratic elements of* $T^-_{\sigma,\theta}$. We can define now a mapping

$$\rho : \mathscr{C} \to \mathscr{C}(T)$$

(see (5.19)). Denote the image of ρ by $\mathcal{A}$ and the fiber above $\rho((\sigma, \theta))$ by $\mathscr{C}(\sigma, \theta)$. The ordered pairs of commuting involutions of $(X^*(T), \Phi(T))$, whose class in $\mathscr{C}(T)$ is contained in $\mathcal{A}$, are called admissible.

The $W(T)$-conjugacy classes of admissible pairs of commuting involutions of $(X^*(T), \Phi(T))$ can be described by a diagram, which can be obtained by gluing together two diagrams of admissible involutions under a combinatorial condition on the simple roots (see (7.11) and (7.16)). From this one obtains all the fine structure of the corresponding semisimple locally symmetric pair.

As to the classification of the classes in $\mathscr{C}(\sigma, \theta)$, it suffices to give a set of quadratic elements of a maximal (σ, θ)-split torus A of G, representing the classes in $\mathscr{C}(\sigma, \theta)$. These quadratic elements can be described by using a basis of $\Phi(A)$. Namely let $\bar{\Delta}$ be a basis of $\Phi(A)$ and $\{\gamma_\lambda\}_{\lambda \in \bar{\Delta}}$ a dual basis in $X_*(A)$, the set of multiplicative one-parameter subgroups of A. If $\varepsilon_\lambda = \gamma_\lambda(-1)$, $\lambda \in \bar{\Delta}$, then $\varepsilon_\lambda^2 = e$. There exists a subset $\Delta_1 \subset \bar{\Delta}$ such that $\{\varepsilon_\lambda\}_{\lambda \in \Delta_1}$ is a set of quadratic elements representing the classes in $\mathscr{C}(\sigma, \theta)$. This subset Δ_1 of $\bar{\Delta}$ is determined by the action of the restricted Weyl group $W(A)$ on the group of quadratic elements of A and the signatures of the roots in $\bar{\Delta}$ (see (8.13) and (8.25)). This completes the classification.

A difference between the above classification of symmetric spaces and the one by Berger [2] is that we give the isomorphism classes under both inner and outer automorphisms, while Berger only classified the semisimple symmetric spaces under the action of the full automorphism group.

Finally we note that every class $\mathscr{C}(\sigma, \theta)$ contains a unique class of standard pairs (see (6.13)). This seems to be the natural class to start with in the analysis on these symmetric spaces. For example, if $\sigma = \theta$, then the standard pair in $\mathscr{C}(\theta, \theta)$ is (θ, θ), which corresponds to a Riemannian symmetric space and the other pairs in $\mathscr{C}(\theta, \theta)$ corespond to the K_ε-spaces described in [18]. Also all the relations between the restricted Weyl groups for the various root systems (see (2.7) and (6.15–6.18)) follow immediately from the properties of this standard pair.

A brief summary of the contents is as follows. After some preliminaries in Section 1. We derive all the properties needed about root systems with involutions in Section 2. The Sections 3 and 4 deal with the classification of single involutorial automorphisms. The method of classification, presented here, simplifies the work of Araki [1] and Sugiura [22, Appendix]. In Section 5 we characterize the isomorphism classes of pairs of commuting involutions on a maximal torus as above. In section 6 we show that for a maximal (σ, θ)-split torus A of G, the set $\Phi(A)$ is a root system and we introduce the standard pair. The classification of admissible pairs of commuting involutions of $(X^*(T), \Phi(T))$ is treated in Section 7, where also all the fine structure is derived. A set of quadratic elements characterizing the classes in $\mathscr{C}(\sigma, \theta)$ is given in Section 8. In Section 9 we deal with the isomorphism classes under the full automorphism group and give also a list of the associated pairs $(\sigma, \sigma\theta)$ and $(\theta, \sigma\theta)$. These will be of importance for the analysis of the corresponding symmetric spaces. Finally the relations between ordered pairs of commuting involutions and semisimple locally symmetric spaces is discussed in Section 10.

Recently Oshima and Sekiguchi [19] also determined the restricted root system of a semisimple locally symmetric pair, based on the classification of Berger. Some of this fine structure of a locally semisimple symmetric pair can be found also in Hoogenboom [13].

1. Preliminaries and Recollections

1.1. Let F denote an algebraically closed field of characteristic $\neq 2$. We use as our basic references for algebraic groups the books of Humphreys [14] and Springer [24] and we shall follow their notations and terminology. Throughout this paper G will denote a connected reductive linear algebraic group, defined over F. For any closed subgroup H of G, denote its Lie algebra by the corresponding (lowercase) German letter $\mathfrak{h}$ and write H^0 for the identity component. The center of H will be denoted by $Z(H)$.

For a subtorus T of H let $X^*(T)$ denote the additively written group of rational characters of T and $X_*(T)$ the group of rational one-parameter multiplicative subgroups of T; i.e., the group of homomorphisms (of algebraic groups): $GL_1 \to T$. The group $X^*(T)$ can be put in duality with $X_*(T)$ by a pairing $\langle \cdot, \cdot \rangle$ defined as follows: if $\chi \in X^*(T)$, $\lambda \in X_*(T)$, then $\chi(\lambda(t)) = t^{\langle \chi, \lambda \rangle}$ for all $t \in F^*$. The torus T acts on the Lie algebra $\mathfrak{h}$ of H by the adjoint representation. For $\alpha \in X^*(T)$ let $\mathfrak{h}_\alpha$ denote the weight space for the character α on $\mathfrak{h}$ and let $\Phi(T, H)$ denote *the set of roots of H with respect to T*; i.e., $\Phi(T, H)$ is the set of non-trivial characters $\alpha \in X^*(T)$ such that $\mathfrak{h}_\alpha \neq 0$. Set $W(T, H) = N_H(T)/Z_H(T)$, where

$$N_H(T) = \{x \in H \mid xTx^{-1} \subset T\},$$

$$Z_H(T) = \{x \in H \mid xt = tx \text{ for all } t \in T\}.$$

If H is connected, $W(T, H)$ is called the *Weyl group of H relative to T.* It is a finite group, which acts on T, $X^*(T)$ and $X_*(T)$. Moreover, the set of roots $\Phi(T, H)$ is stable under the action of $W(T, H)$ on $X^*(T)$. In the case $H=G$ we shall write $\Phi(T)$ for $\Phi(T, G)$ and $W(T)$ for $W(T, G)$.

If T is a torus of G such that $\Phi(T)$ is a root system in the subspace of $X^*(T)\otimes_{\mathbb{Z}}\mathbb{R}$ spanned by $\Phi(T)$ and if $W(T)$ is the corresponding Weyl group, then for each $\alpha\in\Phi(T)$ the subgroup $G_\alpha=Z_G((\operatorname{Ker}\alpha)^0)$ is non-solvable. If we now choose $n_\alpha\in N_{G_\alpha}(T)-Z_{G_\alpha}(T)$ and let s_α be the element of $W(T)$ defined by n_α, then there exists a unique one-parameter subgroup $\alpha^\vee\in X_*(T)$ such that $\langle\alpha,\alpha^\vee\rangle=2$ and $s_\alpha(\chi)=\chi-\langle\chi,\alpha^\vee\rangle\alpha$ $(\chi\in X^*(T))$. We call $\alpha^\vee$ the *coroot* of α and denote the set of these $\alpha^\vee$ in $X_*(T)$ by $\Phi^\vee(T)$. We have a bijection of $\Phi(T)$ onto $\Phi^\vee(T)$.

For $x, y\in G$ denote the commutator $xyx^{-1}y^{-1}$ by (x, y). If A, B are subgroups of G, the subgroup of G generated by all (x, y), $x\in A$, $y\in B$ will be denoted by (A, B).

1.2. *Involutorial automorphisms of G.* Let $\theta\in\operatorname{Aut}(G)$ be an involutorial automorphism of G; i.e., $\theta^2=\mathrm{id}$. We denote the automorphism of $\mathfrak{g}$, induced by θ also by θ and write $K=G_\theta=\{x\in G\mid\theta(x)=x\}$ for the group of fixed points of θ. This is a closed, reductive subgroup of G (see Vust [31, Sect. 1]). If $F=\mathbb{C}$ then G/K is the complexification of a space $G(\mathbb{R})/K(\mathbb{R})$ with $G(\mathbb{R})$-invariant Riemannian structure. Here $G(\mathbb{R})$ (resp. $K(\mathbb{R})$) denotes the set of $\mathbb{R}$-rational points of G (resp. K.)

For a θ-stable subgroup H of G let $S_\theta(H)=\{h\theta(h)^{-1}\mid h\in H\}$. In the case $H=G$, we shall also write S_θ (or S) for $S_\theta(G)$. The group G acts transitively on S_θ by $g*x=gx\theta(g)^{-1}$.

1.3. PROPOSITION. *S_θ is a closed connected subvariety of G and the map $g\to g*e$ induces an isomorphism of affine G-varieties: $G/K\to S_\theta$.*

This is proved in Richardson [20, 2.4]

1.4. *θ-split tori.* Let T be a θ-stable torus of G. (Recall that according to a result of Steinberg [27, 7.5], there exists a θ-stable torus T of G.) If we write $T_\theta^+=(T\cap K)^0$ and $T_\theta^-=\{x\in T\mid\theta(x)=x^{-1}\}^0$, then it is easy to verify that the product map

$$\mu\colon T_\theta^+\times T_\theta^-\to T,\qquad \mu(t_1,t_2)=t_1t_2$$

is a separable isogeny. So in particular $T=T_\theta^+\cdot T_\theta^-$ and $T_\theta^+\cap T_\theta^-$ is a finite group. (In fact it is an elementary abelian 2-group.) If T is a torus in a θ-stable subgroup H of G, then the automorphisms of $\Phi(T, H)$ and $W(T, H)$ induced by $\theta\mid H$ will also be denoted by θ.

A torus A of G is called *θ-split* if $\theta(a)=a^{-1}$ for every $a\in A$. These tori are called θ-anisotropic in Vust [31] and Richardson [20]. We prefer the former terminology, because if $F=\mathbb{C}$, then A is a split torus, defined over $\mathbb{R}$, with respect to the real structure defined by $\theta\tau$, where τ is the complex conjugation with respect to a compact real form of G invariant under θ.

If $\theta \neq \mathrm{id}$, then non-trivial θ-split tori exist (see Vust [31, Sect. 1]), so in particular there are maximal ones. The following result can be found in Vust [31, Sect. 1]:

1.5. PROPOSITION. *Let A be a maximal θ-split torus of G. Then*:

(1) *A is the unique θ-split torus of $Z_G(A)$.*

(2) *$(Z_G(A), Z_G(A)) \subset K^0$ and $Z_G(A)$ is the almost direct product of $Z_K(A)^0$ and A.*

(3) *If T is a maximal torus of G, containing A, then T is θ-stable.*

Moreover, all maximal θ-split tori of G are conjugate under K^0 and so are all maximal tori of G containing a maximal θ-split torus of G.

1.6. PROPOSITION. *Let A be a maximal θ-split torus of G and let E_0 denote the vector subspace of $X^*(A) \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by $\Phi(A)$. Then $\Phi(A)$ is a root system in E_0 and the corresponding Weyl group is given by the restriction of $W(A)$ to E_0. Moreover, every element of $W(A)$ has a representative in $N_{K^0}(A)$.*

For a proof, see Richardson [20, 4.7].

Note that if T is a maximal torus of G containing A, then $\Phi(A)$ coincides with the set of restrictions of the elements of $\Phi(T)$ to A.

2. INVOLUTIONS OF ROOT DATA

To deal with the notion root system in reductive groups it is quite useful to work with the notion of root datum (see Springer [23, Sect. 1]).

2.1. *Root data.* A *root datum* is a quadruple $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$, where X and $X^\vee$ are free abelian groups of finite rank, in duality by a pairing $X \times X^\vee \to \mathbb{Z}$, denoted by $\langle \cdot, \cdot \rangle$, Φ and $\Phi^\vee$ are finite subsets of X and $X^\vee$ with a bijection $\alpha \to \alpha^\vee$ of Φ onto $\Phi^\vee$. If $\alpha \in \Phi$ we define endomorphisms s_α and $s_{\alpha^\vee}$ of X and $X^\vee$, respectively, by

$$s_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee \rangle \alpha, \qquad s_{\alpha^\vee}(\lambda) = \lambda - \langle \alpha, \lambda \rangle \alpha^\vee.$$

The following two axioms are imposed:

(1) If $\alpha \in \Phi$, then $\langle \alpha, \alpha^\vee \rangle = 2$.

(2) If $\alpha \in \Phi$, then $s_\alpha(\Phi) \subset \Phi$, $s_{\alpha^\vee}(\Phi^\vee) \subset \Phi^\vee$.

It follows from (1), that $s_\alpha^2=1$, $s_\alpha(\alpha)=-\alpha$ and similarly for $s_{\alpha^\vee}$. Let Q be the subgroup of X generated by Φ and put $V=Q\otimes_{\mathbb{Z}}\mathbb{R}$, $E=X\otimes_{\mathbb{Z}}\mathbb{R}$. Consider V as a linear subspace of E. Define similarly the subgroup $Q^\vee$ of $X^\vee$ and the vector space $V^\vee$. If $\Phi\neq\emptyset$, then Φ is a not necessarily reduced root system in V in the sense of Bourbaki [5, Chap. VI, No. 1]. The rank of Φ is by definition the dimension of V. The root datum Ψ is called semisimple if $X\subset V$. We observe that $s_{\alpha^\vee}={}^t s_\alpha$ and $s_\alpha(\beta)^\vee=s_{\alpha^\vee}(\beta^\vee)$ as follows by an easy computation (cf. Springer [23, 1.4]). Let $(\cdot,\cdot)$ be a positive definite symmetric bilinear form on E, which is $\mathrm{Aut}(\Phi)$ invariant. Now the s_α $(\alpha\in\Phi)$ are Euclidean reflections, so we have

$$\langle\chi,\alpha^\vee\rangle=2(\alpha,\alpha)^{-1}\cdot(\chi,\alpha)\qquad(\chi\in E,\alpha\in\Phi).$$

Consequently, we can identify $\Phi^\vee$ with the set $\{2(\alpha,\alpha)^{-1}\alpha\mid\alpha\in\Phi\}$ and $\alpha^\vee$ with $2(\alpha,\alpha)^{-1}\alpha$. If $\phi\in\mathrm{Aut}(X,\Phi)$, then its transpose ${}^t\phi$ induces an automorphism of $\Phi^\vee$, so ϕ induces a unique automorphism in $\mathrm{Aut}(\Psi)$, the set of automorphisms of the root datum Ψ. We shall frequently identify $\mathrm{Aut}(X,\Phi)$ and $\mathrm{Aut}(\Psi)$.

For any closed subsystem Φ_1 of Φ let $W(\Phi_1)$ denote the finite group generated by the s_α for $\alpha\in\Phi_1$.

2.1.1. Example. If T is a torus in a reductive group G, such that $\Phi(T)$ is a root system with Weyl group $W(T)$, then the root datum associated to the pair (G,T) is: $(X^*(T),\Phi(T),X_*(T),\Phi^\vee(T))$, where $X^*(T)$, $\Phi(T)$, $X_*(T)$, and $\Phi^\vee(T)$ are as defined in (1.1). In particular, if T is a maximal torus of G or $T=A$ a maximal θ-split torus of G, as in (1.6), then the above root datum exists.

2.2. *Involutions.* Let Ψ be a root datum with $\Phi\neq\emptyset$, as in (2.1), and let $\sigma,\theta\in\mathrm{Aut}(\Psi)$ be commuting involutions; i.e., $\sigma^2=\theta^2=\mathrm{id}$, $\sigma\theta=\theta\sigma$. We now derive some properties of the set of restrictions of Φ to the common (-1)-eigenspace of σ and θ, which will play an important role in our classification.

Let $X_0(\sigma,\theta)=\{\chi\in X\mid\chi-\sigma(\chi)-\theta(\chi)+\sigma\theta(\chi)=0\}$ and let $\Phi_0(\sigma,\theta)=\Phi\cap X_0(\sigma,\theta)$. Clearly $X_0(\sigma,\theta)$ and $\Phi_0(\sigma,\theta)$ are σ- and θ-stable and $\Phi_0(\sigma,\theta)$ is a closed subsystem of Φ. We denote the Weyl group of $\Phi_0(\sigma,\theta)$ by $W_0(\sigma,\theta)$ and identify it with the subgroup $W(\Phi_0(\sigma,\theta))$ of $W(\Phi)$. Put $W_1(\sigma,\theta)=\{w\in W(\Phi)\mid w(X_0(\sigma,\theta))=X_0(\sigma,\theta)\}$, $\bar X_{\sigma,\theta}=X/X_0(\sigma,\theta)$ and let π be the natural projection from X to $\bar X_{\sigma,\theta}$. We frequently identify $\bar X_{\sigma,\theta}$ with $\{\chi\in X\mid\sigma(\chi)=\theta(\chi)=-\chi\}$, such that $\pi(\chi)$ corresponds to $\frac14(\chi-\sigma(\chi)-\theta(\chi)+\sigma\theta(\chi))$. Every $w\in W_1(\sigma,\theta)$ induces an automorphism $\pi(w)$ of $\bar X_{\sigma,\theta}$ and $\pi(w\chi)=\pi(w)\,\pi(\chi)$ $(\chi\in X)$. If $\bar W_{\sigma,\theta}=\{\pi(w)\mid w\in W_1(\sigma,\theta)\}$, then $\bar W_{\sigma,\theta}\cong W_1(\sigma,\theta)/W_0(\sigma,\theta)$ (see Satake [22, 2.1.3]). We call this the *restricted Weyl* *group*, with respect to the action of (σ, θ) on X. It is not necessarily a Weyl group in the sense of Bourbaki [5, Chap. VI, No. 1].

Let $\bar{\Phi}_{\sigma,\theta} = \pi(\Phi - \Phi_0(\sigma, \theta))$ denote the set of *restricted roots of Φ relative to* (σ, θ). We shall mainly be concerned with the case that $\bar{\Phi}_{\sigma,\theta}$ is a root system with Weyl group $\bar{W}_{\sigma,\theta}$ (see, e.g., (1.6), where $\sigma = \theta$).

2.3. DEFINITION. An order $\succ$ on X is called a (σ, θ)-*order* if it has the following property:

$$\text{if } \chi \in X,\ \chi \succ 0, \text{ and } \chi \notin X_0(\sigma, \theta) \qquad \text{then } \sigma(\chi) \prec 0 \text{ and } \theta(\chi) \prec 0.$$

If $\succ$ is a (σ, θ)-order on X, then for $\chi \in X$ we have

$$\chi \succ 0 \Leftrightarrow \text{either } \chi - \sigma(\chi) - \theta(\chi) + \sigma\theta(\chi) \succ 0 \text{ or } \pi(\chi) = 0 \text{ and } \chi \succ 0.$$

So a (σ, θ)-order on X induces orders on $X_0(\sigma, \theta)$ and $\bar{X}_{\sigma,\theta}$ and vice versa.

A basis Δ of Φ with respect to a (σ, θ)-order on X will be called a (σ, θ)-*basis* of Φ. We then write $\Delta_0(\sigma, \theta) = \Delta \cap \Phi_0(\sigma, \theta)$ and $\bar{\Delta}_{\sigma,\theta} = \pi(\Delta - \Delta_0(\sigma, \theta))$. (We call $\bar{\Delta}_{\sigma,\theta}$ *a restricted basis of $\bar{\Phi}_{\sigma,\theta}$* with respect to Δ.) It is not hard to see that $\Delta_0(\sigma, \theta)$ is a basis of $\Phi_0(\sigma, \theta)$ and that a similar property holds for $\bar{\Delta}_{\sigma,\theta}$.

2.4. LEMMA. *The elements of $\bar{\Delta}_{\sigma,\theta}$ are linearly independent. Moreover, every $\lambda \in \bar{\Phi}_{\sigma,\theta}$ can be expressed uniquely in the form*

$$\lambda = \pm \sum_{\mu \in \bar{\Delta}_{\sigma,\theta}} m_\mu \mu \qquad \text{with} \quad m_\mu \in \mathbb{Z},\ m_\mu \geqslant 0.$$

For a proof see Satake [22, 2.1.6].

Note that $W_1(\sigma, \theta)$ permutes the (σ, θ)-bases of Φ; i.e., if $w \in W_0(\sigma, \theta)$ and Δ is a (σ, θ)-basis of Φ, then $w(\Delta)$ is also a (σ, θ)-basis of Φ. Moreover, $w \in W_0(\sigma, \theta)$ if and only if $\pi(w) = \mathrm{id}$. This is again equivalent to $\pi(w)(\bar{\Delta}_{\sigma,\theta}) = \bar{\Delta}_{\sigma,\theta}$ as is easily seen from the following useful result:

2.5. LEMMA. *Let Δ, Δ' be (σ, θ)-bases of Φ such that $\bar{\Delta}_{\sigma,\theta} = \bar{\Delta}'_{\sigma,\theta}$.* Then *$\Delta' = w_0(\Delta)$, where $w_0 \in W_0(\sigma, \theta)$ is the unique element such that $w_0(\Delta_0(\sigma, \theta)) = \Delta_0(\sigma, \theta)'$.*

For a proof see Satake [22, 2.1.2]. The proof follows also immediately from the observation that a (σ, θ)-basis of Φ is completely determined by bases $\Delta_0(\sigma, \theta)$ resp. $\bar{\Delta}_{\sigma,\theta}$ of $\Phi_0(\sigma, \theta)$ resp. $\bar{\Phi}_{\sigma,\theta}$.

2.6. In case of a single involution we take $\sigma = \theta$ and we use the results stated above. Moreover, we omit σ in the notations; i.e., we write $X_0(\theta)$,

$\bar{X}_\theta$, $\Phi_0(\theta)$, $\bar{\Phi}_\theta$, $W_0(\theta)$, $W_1(\theta)$, $\bar{W}_\theta$, $\Delta_0(\theta)$, $\bar{\Delta}_\theta$ instead of, respectively, $X_0(\theta, \theta)$, $\bar{X}_{\theta,\theta}$, $\Phi_0(\theta, \theta)$, $\bar{\Phi}_{\theta,\theta}$, $W_0(\theta, \theta)$, $W_1(\theta, \theta)$, $\bar{W}_{\theta,\theta}$, $\Delta_0(\theta, \theta)$, $\bar{\Delta}_{\theta,\theta}$. A (θ, θ)-order on X will be called a *θ-order* on X and a (θ, θ)-basis of Φ a *θ-basis* of Φ.

2.7. *Relations between (restricted) Weyl groups.* Assume that (σ, θ) is a pair of commuting involutions of Φ such that $\bar{\Phi}_\theta$ is a root system with Weyl group $\bar{W}_\theta$. Then $\sigma \mid \bar{X}_\theta$ is an involution of $(\bar{X}_\theta, \bar{\Phi}_\theta)$, so we can also view $\bar{\Phi}_{\sigma,\theta}$ as the set of restricted roots of $\bar{\Phi}_\theta$ with respect to $\sigma \mid \bar{X}_\theta$. Denote the restriction of σ to $\bar{X}_\theta$ also by σ and let $W_1^\theta(\sigma, \theta) = \{w \in W_1(\sigma, \theta) \mid w(X_0(\theta)) = X_0(\theta)\} = \{w \in W_1(\theta) \mid w\theta = \theta w\}$. Put $\bar{W}_\theta^\sigma = W_1^\theta(\sigma, \theta)/W_0(\sigma, \theta)$. It is not hard to show that $\bar{W}_\theta^\sigma$ is isomorphic to the restricted Weyl group of $\bar{\Phi}_{\sigma,\theta}$ with respect to the action of σ on $\bar{X}_\theta$ (see (2.2)). However, this will not be needed in the sequel.

In case $\bar{\Phi}_\sigma$ is a root system with Weyl group $\bar{W}_\sigma$ we define $W_1^\sigma(\sigma, \theta)$ and $\bar{W}_\sigma^\theta$ similarly. In Section 6 we shall encounter the situation that $\bar{W}_\theta^\sigma$, $\bar{W}_\sigma^\theta$, and $\bar{W}_{\sigma,\theta}$ coincide and are equal to the Weyl group of $\bar{\Phi}_{\sigma,\theta}$.

2.8. *A characterization of θ on a θ-basis of Φ.* In the remaining part of this section we restrict ourselves to the situation of a single involution $\theta \in \mathrm{Aut}(X, \Phi)$. Let Δ be a θ-basis of Φ. Then $\theta(-\Delta)$ is also a θ-basis of Φ with the same restricted basis, so by (2.5) there is $w_0(\theta) \in W_0(\theta)$ such that $w_0(\theta)\,\theta(\Delta) = -\Delta$. Here $w_0(\theta)$ is the longest element of $W_0(\theta)$ with respect to $\Delta_0(\theta)$. Put $\theta^* = \theta^*(\Delta) = -w_0(\theta) \cdot \theta$. Then $\theta^*(\Delta) \in \mathrm{Aut}(X, \Phi, \Delta) = \{\phi \in \mathrm{Aut}(X, \Phi) \mid \phi(\Delta) = \Delta\}$, $\theta^*(\Delta)^2 = \mathrm{id}$, and $\theta^*(\Delta_0(\theta)) = \Delta_0(\theta)$.

2.9. *Remarks.* (1) θ^* can be described by its action on the Dynkin diagram of Δ. Note that

(a) if Φ is irreducible, then θ^* is either the identity or a diagram automorphism of order 2;

(b) if $\Phi = \Phi_1 \amalg \Phi_2$ with Φ_1, Φ_2 irreducible and $\theta(\Phi_1) = \Phi_2$, then θ^* exchanges the Dynkin diagrams of Φ_1 and Φ_2. In particular $\Phi_0(\theta) = \varnothing$, so $w_0(\theta) = \mathrm{id}$ and $\theta = -\theta^*$.

(2) If $\theta = \mathrm{id}$ and Δ is a basis of Φ, then $\theta^*(\Delta) = -w_0(\mathrm{id})$ is called the *opposition involution* of Δ. In this case we shall also write $\mathrm{id}^*(\Delta)$ for $\theta^*(\Delta)$.

(3) If Φ is irreducible and Δ a basis of Φ, then the opposition involution is non-trivial if and only if Φ is either of type A_l $(l \geqslant 2)$, $D_{2l+1}(l \geqslant 2)$ or E_6.

(4) The action of θ^* on $\Delta_0(\theta)$ is determined by $\Delta_0(\theta)$, because $\theta^* \mid \Delta_0(\theta) = -w_0(\theta)$ is the opposition involution of $\Delta_0(\theta)$, which is uniquely determined on each irreducible component of $\Phi_0(\theta)$.

(5) For Φ irreducible, the action of θ^* can only be non-trivial if Φ is of type A_l $(l \geqslant 2)$, D_l $(l \geqslant 4)$ or E_6.

The diagram automorphism θ^* relates the simple roots in Δ, which are lying above a restricted root in $\bar{\Delta}_\theta$:

2.10. LEMMA. *Let Δ be a θ-basis of Φ and $\alpha, \beta \in \Delta$, $\alpha \neq \beta$ such that $\pi(\alpha) = \pi(\beta) \neq 0$. Then $\alpha = \theta^*(\beta)$.*

Proof. Working in V, we have $\pi(\alpha) = \frac{1}{2}(\alpha - \theta(\alpha)) = \frac{1}{2}(\beta - \theta(\beta))$, so

$$\alpha - \beta = \theta(\alpha - \beta) = -\theta^*(w_0(\theta)(\alpha - \beta)) = \theta^*(\beta - \alpha - \delta)$$

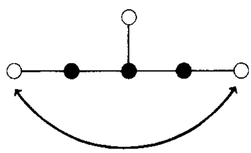
for some $\delta \in \mathrm{Span}(\Delta_0(\theta))$. Since Δ is a basis of V and α, β, $\theta^*(\alpha)$, $\theta^*(\beta) \in (\Delta - \Delta_0(\theta))$, it follows that $\alpha = \theta^*(\beta)$, $\beta = \theta^*(\alpha)$, and $\delta = 0$.

2.11. *The index of θ.* Assume that the root datum Ψ is semisimple. If $\theta \in \mathrm{Aut}(\Psi)$ is an involution and Δ a θ-basis of Φ, then θ is determined by the quadruple $(X, \Delta, \Delta_0(\theta), \theta^*(\Delta))$, because $\theta = -\theta^*(\Delta)\, w_0(\theta)$. We call such a quadruple $(X, \Delta, \Delta_0(\theta), \theta^*(\Delta))$ an *index of θ*.

Two indices $(X, \Delta, \Delta_0(\theta_1), \theta_1^*(\Delta))$ and $(X, \Delta', \Delta_0'(\theta_2), \theta_2^*(\Delta'))$ are said to be *isomorphic* if there is a $w \in W(\Phi)$, which maps $(\Delta, \Delta_0(\theta_1))$ onto $(\Delta', \Delta_0'(\theta_2))$ and which satisfies $w\theta_1^*(\Delta)\, w^{-1} = \theta_2^*(\Delta')$.

2.12. *Remarks.* (1) The above index of θ is the same as the Satake diagram corresponding to an action of the finite group $\Gamma_\theta = \{\mathrm{id}, -\theta\}$ on (X, Φ) (See Satake [22, 2.4]). Our terminology follows Tits [29].

(2) As in [29] we make a diagrammatic representation of the index of θ by colouring black those vertices of the ordinary Dynkin diagram of θ, which represent roots in $\Delta_0(\theta)$, and by indicating the action of θ^* on $\Delta - \Delta_0(\theta)$ by arrows. An example in type E_6 is:



θ^*

We omit the action of θ^* on $\Delta_0(\theta)$ because $\theta^* | \Delta_0(\theta) = -w_0(\theta)$ is completely determined by the type of $\Phi_0(\theta)$ (see (2.9.4)).

(3) An index of θ may depend on the choice of the θ-basis of Φ; i.e., for two θ-bases Δ, Δ', the corresponding indices $(X, \Delta, \Delta_0(\theta), \theta^*(\Delta))$ and $(X, \Delta', \Delta_0'(\theta), \theta^*(\Delta'))$ need not be isomorphic. However, this cannot happen if $\bar{\Phi}_\theta$ is a root system with Weyl group $\bar{W}_\theta$:

2.13. LEMMA. *Let Ψ be semisimple and $\theta \in \mathrm{Aut}(\Psi)$ an involution such that $\bar{\Phi}_\theta$ is a root system with Weyl group $\bar{W}_\theta$. Let Δ, Δ' be two θ-bases of Φ. Then $(X, \Delta, \Delta_0(\theta), \theta^*(\Delta))$ and $(X, \Delta', \Delta_0'(\theta), \theta^*(\Delta'))$ are isomorphic.*

Proof. Since $\bar{W}_\theta = W_1(\theta)/W_0(\theta)$ is the Weyl group of $\bar{\Phi}_\theta$, there is by (2.5) a unique element $w \in W_1(\theta)$ such that $w(\Delta) = \Delta'$. Then also $w(\Delta_0(\theta)) = \Delta'_0(\theta)$, so it suffices to show that $\theta^*(\Delta') = w\theta^*(\Delta)\, w^{-1}$.

Since $w_0(\theta)' = ww_0(\theta)\,\theta(w^{-1})$, where $w_0(\theta)'$, resp. $w_0(\theta) \in W_0(\theta)$, are as in (2.8), we get $w_0(\theta)' = \theta(w)(w_0(\theta)\theta)(\theta(w)^{-1})$, which implies the desired relation.

To classify the indices of involutions we note:

2.14. LEMMA. *Let Δ be a basis of Φ, $\Delta_0 \subset \Delta$ a subset and $\theta^* \in \mathrm{Aut}(X, \Phi, \Delta)$ such that $\theta^*(\Delta_0) = \Delta_0$, $(\theta^*)^2 = \mathrm{id}$. Let X_0 be the $\mathbb{Z}$-span of Δ_0 in X and $\Phi(\Delta_0) = \Phi \cap X_0$. Then there is an involution $\theta \in \mathrm{Aut}(X, \Phi)$ with index $(X, \Delta, \Delta_0, \theta^*)$ if and only if $\theta^* \mid \Delta_0 = \mathrm{id}^*(\Delta_0)$ (the opposition involution of Δ_0 with respect to $\Phi(\Delta_0)$).*

Proof. "Only if" being clear, assume $\theta^* \mid \Delta_0 = \mathrm{id}^*(\Delta_0)$. Let w_0 be the longest element of $W(\Phi(\Delta_0))$ with respect to Δ_0 and let $\theta = -\theta^* w_0 \in \mathrm{Aut}(X, \Phi)$. Since $\theta \mid X_0 = \mathrm{id}$ it follows that θ^* and w_0 commute, so θ is an involution. On the other hand, since $\theta^* \mid \Delta_0 = \mathrm{id}^*(\Delta_0)$ it follows that $\Delta_0 = \Delta_0(\theta)$, so $(X, \Delta, \Delta_0, \theta^*)$ is an index of θ. This proves the result.

2.15. *θ-normal root systems.* Let X, Φ, and θ be as in (2.8) and let $\Phi' = \{\alpha \in \Phi \mid \frac{1}{2}\alpha \notin \Phi\}$ be the set of indivisible roots.

2.15.1. DEFINITION. Φ is called *θ-normal* if for all $\alpha \in \Phi'$ with $\theta(\alpha) \neq \alpha$, we have $\theta(\alpha) + \alpha \notin \Phi$. This definition is a generalization of the known definition of normality to non-reduced root systems (see Warner [32, 1.1.3]).

2.16. *Remark.* If Φ is θ-normal, then $\bar{\Phi}_\theta$ is a root system with Weyl group $\bar{W}_\theta$ (see Warner [32, 1.1.3.1]).

In the sequel we shall need the following results:

2.17. LEMMA. *Assume Φ to be irreducible and let Δ be a θ-basis of Φ. Let* $\mathrm{id}^*(\Delta) \in \mathrm{Aut}(X, \Phi, \Delta)$ *be the opposition involution, as in* (2.9.2). *Then the following statements are equivalent*:

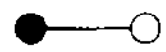
(1) $\mathrm{id}^*(\Delta)$ *and* $w_0(\theta)$ *commute.*

(2) $\Phi_0(\theta)$ *is stable under* $\mathrm{id}^*(\Delta)$.

Proof. The proof follows from the following equivalences:

$(1) \Leftrightarrow \mathrm{id}^*(\Delta)$ and θ commute $\Leftrightarrow$ 1 and (-1)-eigenspaces of θ are $\mathrm{id}^*(\Delta)$-stable $\Leftrightarrow (2)$.

2.18. Note that in general $w_0(\theta)$ and id* need not commute. For example, if Φ is of type A_2, then



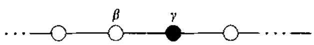
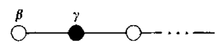
is the index of an involution $\theta \in \mathrm{Aut}(X, \Phi)$, but clearly $\Phi_0(\theta)$ is not stable under id*. However, when Φ is θ-normal, then the condition is satisfied.

2.19. Lemma. *Let* Φ, θ, Δ, *and* id* *be as in* (2.17). *If* Φ *is a* θ*-normal, then* $\Phi_0(\theta)$ *is stable under* id*.

Proof. We first note that we may assume that $\mathrm{id}^* \neq \mathrm{id}$. Then Φ is of type A_l, $D_{2l+1} (l \geqslant 2)$ or E_6. We may also assume $\theta^* = \mathrm{id}$ (if not, we would have $\theta^* = \mathrm{id}^*$ and we are done). Now $\Phi_0(\theta)$ must be a union of irreducible components, whose Weyl groups contain $-\mathrm{id}$. From the preceding remarks, it follows that $\Phi_0(\theta)$ is a union of a number of irreducible components of type A_1 and at most one component of type D_{2l} $(l \geqslant 2)$.

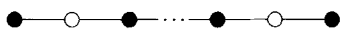
If $\Phi_0(\theta)$ has an irreducible component of type D_{2l} $(l \geqslant 2)$, then Φ is of type D_{2l+1} $(l \geqslant 2)$ or E_6 and in both cases $\Phi_0(\theta)$ is stable under id*. So we may assume that $\Phi_0(\theta)$ is of type $A_1 \times \cdots \times A_1$. Say $\Delta_0(\theta) = \{\alpha_1, ..., \alpha_r\}$. Then $w_0(\theta) = s_{\alpha_1} \cdots s_{\alpha_r}$.

If the index of θ would contain a subdiagram of the form

 or 

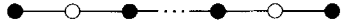
then Φ is not θ-normal; namely since $\theta = -w_0(\theta)$ we have $\theta(\beta+\gamma) = s_\gamma(-\beta-\gamma) = -\beta$, hence $\beta + \gamma + \theta(\beta+\gamma) = \gamma \in \Phi$.

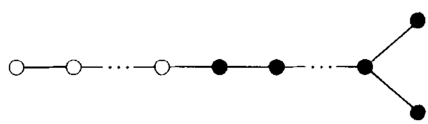
It follows that the only possible indices of θ, with $\Phi_0(\theta)$ of type $A_1 \times \cdots \times A_1$ are

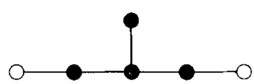
A_{2l+1}: 

In this case $\Phi_0(\theta)$ is obviously stable under id*, which proves the result.

2.20. From this proof it also follows that the indices of involutions θ with Φ irreducible and θ-normal and $\mathrm{id}^* \neq \mathrm{id}$, $\theta^* = \mathrm{id}$ are

A_{2l+1}: 

D_{2l+1}: 

E_6: 

3. A Characterization of the Conjugacy Classes of Involutorial Automorphisms of G

3.1. *A realization of $\Phi(T)$ in G.* Let T be a maximal torus of G. If $\alpha \in \Phi(T)$, let x_α be the corresponding one-parameter additive subgroup of G defined by α. This is an isomorphism of the additive subgroup onto a closed subgroup U_α of G, normalized by T, such that

$$t x_\alpha(\xi)\, t^{-1} = x_\alpha(\alpha(t)\xi) \qquad (t \in T, \xi \in F).$$

The x_α may be chosen such that

$$n_\alpha = x_\alpha(1)\, x_{-\alpha}(-1)\, x_\alpha(1)$$

lies in $N_G(T)$ for all $\alpha \in \Phi(T)$, as can be derived using a SL_2-computation. In that case we have

$$x_\alpha(\xi)\, x_\alpha(-\xi^{-1})\, x_\alpha(\xi) = \alpha^\vee(\xi)\, n_\alpha \qquad (\xi \in F),$$

where $\alpha^\vee \in X_*(T)$ is the coroot of α. Moreover, $n_\alpha \cdot T$ is the reflection $s_\alpha \in W(T)$ defined by α and $n_\alpha^2 = \alpha^\vee(-1) = t_\alpha$, $n_{-\alpha} = t_\alpha n_\alpha$, $t_{-\alpha} = t_\alpha$.

A family $\{x_\alpha\}_{\alpha \in \Phi(T)}$ with the above properties (3.1.1), (3.1.2) is called a *realization* of $\Phi(T)$ in G. Similarly the set of root vectors $X_\alpha = dx_\alpha(1) \in \mathfrak{g}_\alpha$ is called a *realization* of $\Phi(T)$ in $\mathfrak{g}$. We then have $\mathrm{Ad}(t)\, X_\alpha = \alpha(t)\, X_\alpha$ $(t \in T)$. For these facts see Springer [24, 11.2]. If $\alpha, \beta \in \Phi(T)$ are linearly independent (i.e., $\alpha \neq \pm\beta$) we have a formula:

$$(x_\alpha(\xi), x_\beta(\eta)) = \prod_{\substack{i\alpha + j\beta \in \Phi(T) \\ i, j > 0}} x_{i\alpha + j\beta}(c_{\alpha,\beta;i,j}\xi^i\eta^j) \qquad (\xi, \eta \in F),$$

the product being taken in a preassigned order. The elements $c_{\alpha,\beta;i,j}$ are called the *structure constants* of G for the given realization $\{x_\alpha\}_{\alpha \in \Phi(T)}$.

3.2. Let Δ be a basis of $\Phi(T)$. If $w \in W(T)$ and $w = s_{\alpha_1} \cdots s_{\alpha_k}$ is a shortest expression of w, the α_i being simple roots, then $\phi(w) = n_{\alpha_1} \cdots n_{\alpha_k} \in N_G(T)$ is a representative of $w \in W(T)$ in $N_G(T)$, depending only on w and not on the choice of the shortest expression (see Springer [24, 11.2.9]). There exists a realization $\{x_\alpha\}_{\alpha \in \Phi(T)}$ such that

$$\phi(w)\, x_\alpha(\xi)\, \phi(w)^{-1} = x_{w(\alpha)}(\pm\xi)$$

for $\alpha \in \Phi(T)$, $w \in W(T)$, $\xi \in F$, and ϕ as above. Moreover all the structure constants are of the form $n \cdot 1$ with $n \in \mathbb{Z}$. In particular, if $\alpha, \beta \in \Phi$, $\alpha + \beta \in \Phi$, $\alpha - c\beta \in \Phi$, $\alpha - (c+1)\beta \notin \Phi$, then $c_{\alpha,\beta;1,1} = \pm(c+1)$ and $c_{\alpha,\beta;1,1}\, c_{-\alpha,-\beta;1,1} = -(c+1)^2$. For more details see Springer [24, 11.3.6].

3.3. *θ-singular roots.* Let T be a maximal torus of G and $\{x_\alpha\}_{\alpha\in\Phi(T)}$ a realization of $\Phi(T)$ in G. If $\phi\in\mathrm{Aut}(G)$ such that $\phi(T)=T$, then there exists $c_{\alpha,\phi}\in F^*$ such that for $\xi\in F$

$$\phi(x_\alpha(\xi))=x_{\phi(\alpha)}(c_{\alpha,\phi}\xi).$$

Now ϕ is an involution if and only if

$$(\phi\mid T)^2=\mathrm{id}_T \text{ and } c_{\alpha,\phi}c_{\phi(\alpha),\phi}=1 \qquad \text{for all} \quad \alpha\in\Phi(T).$$

Let $\theta\in\mathrm{Aut}(G)$ be an involution stabilizing T. Then a root $\alpha\in\Phi(T)$ is called *θ-singular* if $\theta(\alpha)=\pm\alpha$ and $\theta\mid Z_G((\mathrm{Ker}\,\alpha)^0)\neq\mathrm{id}$. If $\theta(\alpha)=-\alpha$ we say that α is *real* with respect to θ. If $\theta(\alpha)=\alpha$ and α is θ-singular, then α is also called *noncompact imaginary* with respect to θ. In that case $c_{\alpha,\theta}=-1$, as follows also by a simple computation in SL_2. If $\theta(\alpha)=\alpha$ and α is not θ-singular, then $c_{\alpha,\theta}=1$. These roots are called *compact imaginary* with respect to θ.

3.4. LEMMA. *Let T be a θ-stable maximal torus of G. Then T_θ^- is a maximal θ-split torus of G if and only if $\Phi(T)$ has no roots, which are noncompact imaginary with respect to θ, i.e., if and only if $c_{\alpha,\theta}=1$ for all $\alpha\in\Phi_0(\theta)$.*

For a proof see [12] or [25].

3.5. LEMMA. *Let T be a θ-stable maximal torus of G, Δ a θ-basis of $\Phi(T)$ and write $\theta=-\theta^*w_0(\theta)$ as in* (2.8). *Then for all $t\in\bigcap_{\beta\in\Delta_0(\theta)}\mathrm{Ker}(\beta)$ such that $\theta(t)\cdot t\in Z(G)$ we have $\theta^*(\alpha)(t)=\alpha(t)$ for all $\alpha\in\Phi(T)$.*

Proof. If $t\in\bigcap_{\beta\in\Delta_0(\theta)}\mathrm{Ker}(\beta)$ such that $\theta(t)t\in Z(G)$, then $\theta^*(\alpha)(t)=w_0(\theta)\,\alpha(\theta(t)^{-1})=w_0(\theta)(\alpha)(t)=\alpha(t)\gamma(t)$ for some $\gamma\in\mathrm{Span}(\Delta_0(\theta))$. Since for all $\beta\in\Phi_0(\theta)$ we have $\beta(t)=1$, it follows that $\theta^*(\alpha)(t)=\alpha(t)$.

Note that among others all elements of T_θ^- satisfy the above conditions.

3.6. DEFINITION. Let T be a maximal torus of G. An automorphism θ of G of order $\leqslant 2$ is said to be *normally related* to T if $\theta(T)=T$ and T_θ^- is a maximal θ-split torus of G.

3.7. THEOREM. *Let $\theta_1,\theta_2\in\mathrm{Aut}(G)$ be such that $\theta_1^2=\theta_2^2=\mathrm{id}$ and assume θ_1, θ_2 are normally related to T. Then θ_1 and θ_2 are conjugate under* $\mathrm{Int}(G)$ *if and only if $\theta_1\mid T$ and $\theta_2\mid T$ are conjugate under $W(T)$.*

Proof. If $\theta_2=\mathrm{Int}(g)\,\theta_1\,\mathrm{Int}(g^{-1})$ for some $g\in G$, then since all maximal θ_2-split tori are conjugate under $G_{\theta_2}^0$ and also all maximal tori containing them (see (1.5)), we may assume $g\in N_G(T)$. But then $\theta_1\mid T$ and $\theta_2\mid T$ are conjugate under $W(T)$, which proves the "only if" statement.

Assuming that $\theta_1 | T$ and $\theta_2 | T$ are conjugate under $W(T)$, it then suffices to consider the case that $\theta_1 | T = \theta_2 | T$. Henceforth we assume this and write θ for $\theta_1 | T$. By the isomorphism theorem (see Springer [24, 11.4.3]), there is a $t \in T$ such that $\theta_1 = \theta_2 \operatorname{Int}(t)$.

Since $\theta_1^2 = \theta_2^2 = \mathrm{id}$, we get $\operatorname{Int}(\theta(t)t) = \mathrm{id}$, so $\theta(t)t \in Z(G)$. If $\alpha \in \Phi_0(\theta)$, then by (3.4) α is a compact imaginary root with respect to θ_1 as well as θ_2, so in particular $c_{\alpha,\theta_1} = c_{\alpha,\theta_2} = 1$, which implies $\alpha(t) = 1$.

Let Δ be a θ-basis of $\Phi(T)$ and let $\Delta_0(\theta)$, $\bar{\Delta}_\theta$ be as in (2.6). If $\gamma \in \bar{\Delta}_\theta$ and $\alpha, \beta \in \Delta$, $\alpha \neq \beta$, such that $\pi(\alpha) = \pi(\beta)$, then by (2.10) $\beta = \theta^*(\alpha)$. So by (3.5) we have $\alpha(t) = \theta^*(\alpha)(t)$.

For each $\gamma \in \bar{\Delta}_\theta$, now take $\alpha \in \Delta$ such that $\gamma = \pi(\alpha) = \alpha | T_\theta^-$ and choose $u_\gamma \in T_\theta^-$ such that $\lambda(u_\gamma) = 1$ for $\lambda \in \bar{\Delta}_\theta$, $\lambda \neq \gamma$ and $\gamma(u_\gamma^2) = \alpha(t)$. Let $u = \prod_{\gamma \in \bar{\Delta}_\theta} u_\gamma$. Then by (2.10) and (3.5) we find $\alpha(tu^2) = 1$ for all $\alpha \in \Delta$. So $tu^2 \in Z(G)$ and it follows that $\operatorname{Int}(u)\,\theta_1 \operatorname{Int}(u^{-1}) = \theta_2$. This proves the result.

3.8. Corollary. *Let $\theta_1, \theta_2 \in \operatorname{Aut}(G)$ be as above. If $\theta_1 | T = \theta_2 | T$, then there is $t \in T_\theta^-$ such that $\theta_1 = \theta_2 \operatorname{Int}(t)$.*

This follows from the proof of (3.7).

3.9. Definition. Let $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ be a root datum with Φ a reduced root system and let $\theta \in \operatorname{Aut}(\Psi)$ be an involution. Then θ is called *admissible* if there exists a reductive algebraic group G with maximal torus T and an involution $\tilde{\theta} \in \operatorname{Aut}(G, T)$ such that Ψ is isomorphic to $(X^*(T), \Phi(T), X_*(T), \Phi^\vee(T))$, $\tilde{\theta}$ induces θ on Ψ and such that T_θ^- is a maximal $\tilde{\theta}$-split torus of G. If X is semisimple, then the indices of admissible involutions of Ψ shall be called *admissible indices.*

3.10. *Remark.* Let G, T be as in (3.1). If $\theta \in \operatorname{Aut}(X^*(T), \Phi(T))$ is an admissible involution, then by (1.6) $\bar{\Phi}_\theta = \Phi(T_\theta^-)$ is a root system with Weyl group $W_1(\theta)/W_0(\theta) \cong W(T_\theta^-)$. So if G is semisimple, then by (2.13) the $W(\Phi)$-conjugacy class of θ corresponds bijectively with the isomorphism class of the index of θ. We have obtained the following result.

3.11. Theorem. *Assume that G is semisimple and T is a maximal torus of G. Then there is a bijection of the set of* $\operatorname{Int}(G)$*-conjugacy classes of involutorial automorphisms of G and the isomorphism classes of indices of admissible involutions of* $(X^*(T), \Phi(T))$.

Proof. Since all maximal tori of G are conjugate under $\operatorname{Int}(G)$, every involutorial automorphism of G is conjugate to one which is normally related to T. The result follows now from Theorem 3.7, Lemma 2.13, and Remark 3.10.

3.12. *θ-normality of $\Phi(T)$.* For later use it is useful to note that an admissible involution $\theta \in \operatorname{Aut}(X^*(T), \Phi(T))$ implies θ-normality of the root system:

LEMMA. *Let G, T be as in* (3.1). *If $\theta \in \operatorname{Aut}(X^*(T), \Phi(T))$ is an admissible involution, then $\Phi(T)$ is θ-normal.*

Proof. By (3.4) it suffices to show that if $\alpha \in \Phi(T)$ such that $\theta(\alpha) \neq \alpha$ and $\alpha + \theta(\alpha) \in \Phi(T)$, that then $\alpha + \theta(\alpha)$ must be non-compact imaginary. This last statement follows immediately by choosing a realization $\{X_\alpha\}_{\alpha \in \Phi(T)}$ of $\Phi(T)$ in $\mathfrak{g}$ such that $\theta(X_\alpha) = X_{\theta(\alpha)}$ and $[X_\alpha, X_{\theta(\alpha)}] = X_{\alpha+\theta(\alpha)}$. Then

$$\theta(X_{\alpha+\theta(\alpha)}) = [X_{\theta(\alpha)}, X_\alpha] = -X_{\alpha+\theta(\alpha)},$$

so $\alpha + \theta(\alpha)$ is non-compact imaginary.

For this result see also Springer [25, 2.6].

4. Classification of Admissible Involutions

We discuss here the classification of involutorial automorphisms of G. It is quite similar to the classification of real forms of a complex semisimple Lie algebra, as is carried out by Araki [1]. See also Section 10.

4.1. *Lifting involutions of (X, Φ).* In this section we assume G to be semisimple. Let T be a fixed maximal torus of G and write Φ for $\Phi(T)$, X for $X^*(T)$, W for $W(T)$. Choose a realization of Φ in G as in (3.2). To determine whether an involution $\theta \in \operatorname{Aut}(X, \Phi)$ is admissible we need to determine first whether it can be lifted; i.e.,

DEFINITION. An involution $\theta \in \operatorname{Aut}(X, \Phi)$ can be *lifted* if there is an involutorial automorphism $\phi \in \operatorname{Aut}(G, T)$ inducing θ on (X, Φ).

Note that by the isomorphism theorem there always exists a possibly non-involutorial $\phi \in \operatorname{Aut}(G, T)$, inducing θ on (X, Φ). So by (3.3) ϕ is involutorial if and only if $c_{\alpha,\phi} c_{\theta(\alpha),\phi} = 1$ for all $\alpha \in \Phi$. Moreover, θ is admissible if and only if its can be lifted to $\phi \in \operatorname{Aut}(G, T)$ satisfying $c_{\alpha,\phi} = 1$ for all $\alpha \in \Phi_0(\theta)$ (cf. (3.4)). On the other hand, it follows from the isomorphism theorem (see Springer [24, 11.4.3]) that it also suffices to restrict to a basis of Φ:

4.2. LEMMA. *Let Δ be a basis of Φ, $\theta \in \operatorname{Aut}(X, \Phi)$ an involution and $\phi \in \operatorname{Aut}(G, T)$ such that $\phi \mid T = \theta$. Then ϕ is uniquely determined by the tuple $(c_{\alpha,\phi})_{\alpha \in \Delta}$.*

This result is discussed in Séminaire C. Chevalley [7, 17-08, 17-09].

4.3. Definition. Let Δ be a fixed basis of Φ. For any involution $\theta \in \mathrm{Aut}(X, \Phi)$ let $\theta_\Delta \in \mathrm{Aut}(G, T)$ denote the unique automorphism of G such that

$$\theta_\Delta(x_\alpha(\xi)) = x_{\theta(\alpha)}(\xi) \qquad \text{for all} \quad \alpha \in \Delta,\ \xi \in F.$$

It follows now from a result of Steinberg [26, Th. 29] that $c_{\alpha, \theta_\Delta} = \pm 1$ for all $\alpha \in \Phi$ and moreover, the constants $c_{\alpha, \theta_\Delta}$ do not depend on the characteristic of the field of definition F.

Summarizing, involutions of (X, Φ) which can be lifted, can be characterized as follows:

4.4. Proposition. *Let $\theta \in \mathrm{Aut}(X, \Phi)$ be an involution and Δ a basis of Φ. Then the following are equivalent:*

(i) *θ can be lifted.*

(ii) *There is a $t \in T$ such that $\theta_\Delta \operatorname{Int}(t)$ is an involution.*

(iii) *There is a $t \in T$ such that $c_{\theta(\alpha), \theta_\Delta} = \alpha(\theta(t)t)$ for all $\alpha \in \Delta$.*

(iv) *There is a $t \in T_\theta^+$ such that $c_{\theta(\alpha), \theta_\Delta} = \alpha(t)$ for all $\alpha \in \Delta$.*

This result follows immediately from the definition of θ_Δ, (4.2) and (3.3).

Note that if $t \in T_\theta^+$ such that $\theta_\Delta \operatorname{Int}(t)$ is an involution, then, since $c_{\alpha, \theta_\Delta} = \pm 1$ for all $\alpha \in \Phi$, we have by (iv) that $\alpha(t^4) = 1$ for all $\alpha \in \Phi$, hence $t^4 \in Z(G)$.

4.5 Corollary. *Let $\theta \in \mathrm{Aut}(X, \Phi)$ be an involution and let Δ be a θ-basis of Φ. Then θ is admissible if and only if there is a $t \in T$ such that*

(i) $c_{\theta(\alpha), \theta_\Delta} = \alpha(\theta(t)t)$ *for all* $\alpha \in \Delta - \Delta_0(\theta)$,

(ii) $\alpha(t) = 1$ *for all* $\alpha \in \Delta_0(\theta)$.

This follows from (4.4) and (3.4).

4.6. Proposition. *Assume that G, T, X, and Φ are as in* (4.1). *Whether an involution $\theta \in \mathrm{Aut}(X, \Phi)$ is admissible or not is independent of the field of definition F of G, if only* $\mathrm{char}(F) \neq 2$.

Proof. An involution $\theta \in \mathrm{Aut}(X, \Phi)$ is admissible if for a fixed θ-basis Δ of Φ, there is a $t \in T_\theta^+$ such that the conditions (i) and (ii) of (4.5) are satisfied. But these conditions imply that $t^4 \in Z(G)$, so this can be verified independently of F, if only $\mathrm{char}(F) \neq 2$.

4.7. The classification of conjugacy classes of involutorial automorphisms of G coincides now with the known classification over $\mathbb{C}$. For G of adjoint type this comes down to the classification of real forms of a semisimple Lie algebra over $\mathbb{C}$, as is carried out by Araki [1]. See also Sugiura [22, Appendix] for a simplification of this method. Different treatments of the classification of real semisimple Lie algebras can be found for instance in Cartan [6], Gantmacher [9] (simplified by Murakami [17]), Helgason [11], and Freudenthal and de Vries [8].

On the other hand, with the above results it is possible to give a simplification of Araki's classification (see [1]). We will sketch this in the remainder of this section.

4.8. *Reduction to restricted rank one.* Let G, T, X, Φ be as in (4.1).

The *restricted rank* of an involution $\theta \in \mathrm{Aut}(X, \Phi)$ is defined as the rank of the set of restricted roots $\bar{\Phi}_\theta$. If Δ is a θ-basis of Φ, then the restricted rank of θ is equal to $|\bar{\Delta}_\theta|$.

For each $\lambda \in \bar{\Phi}_\theta$ such that $\frac{1}{2}\lambda \notin \bar{\Phi}_\theta$ (i.e., $\lambda \in \bar{\Phi}'_\theta$, see (2.15)), let $\Phi(\lambda)$ denote the set of all roots $\beta \in \Phi$ such that the restriction of β to $\bar{X}_\theta$ is an integral multiple of λ. Then $\Phi(\lambda)$ is a θ-stable closed symmetric subsystem of Φ (See Borel and Tits [3, p. 71]). Let $X(\lambda)$ denote the projection of X on the subspace of $E = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by $\Phi(\lambda)$.

4.9. Proposition. *Let $\theta \in \mathrm{Aut}(X, \Phi)$ be an involution and Δ a θ-basis of Φ. Then θ is admissible if and only if $\theta \mid X(\lambda) \in \mathrm{Aut}(X(\lambda), \Phi(\lambda))$ is admissible for all $\lambda \in \bar{\Delta}_\theta$.*

This result is derived immediately from (4.5) (see also Satake [22]).

4.10. *Classification of involutions of restricted rank one.* To determine the indices of involutions of restricted rank one we need a notion of irreducibility:

Definition. Let $\theta \in \mathrm{Aut}(X, \Phi)$ be an involution and Δ a θ-basis of Φ. An index $S = (X, \Delta, \bar{\Delta}_\theta, \theta^*)$ of θ is called *irreducible* if Δ is not the union of two mutually orthogonal θ^*-stable non-empty subsystems Δ_1 and Δ_2. The index is called *absolutely irreducible* if Δ is connected.

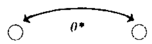
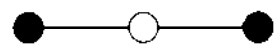
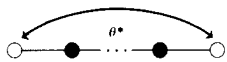
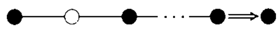
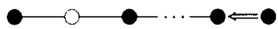
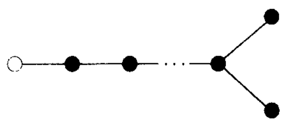
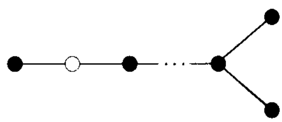
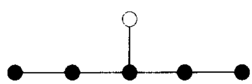
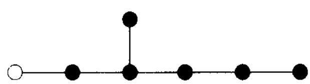
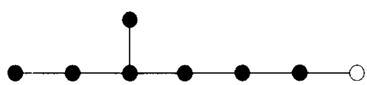
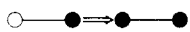
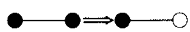
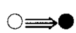
Clearly an absolutely irreducible index is irreducible. From (2.14) and (2.9) one easily now deduces:

4.11. Proposition. *Let X be of adjoint type. Then there exist* 17 *types of absolutely irreducible indices of non-trivial involutions of* (X, Φ) *of restricted rank one and one type of restricted rank one, which is ireducible but not absolutely irreducible* (*see Table* I).

This result can also be found in Sugiura [22, Appendix, Prop. 4].

4.12. To restrict this set of rank one indices Araki [1] and Sugiura [22, Appendix] used the θ-normality of Φ (see (3.12)). One can also exclude these indices with Φ not θ-normal, using the following results:

TABLE I

No.	Φ	θ-index	Normal	Admissible
1	$A_1 \times A_1$		+	+
2	A_1		+	+
3	A_2		−	−
4	A_3		+	+
5	A_l		+	+
6	B_l		+	+
7	$(l \geqslant 2)$		−	−
8	C_l		−	−
9	$(l \geqslant 3)$		+	+
10	D_l $(l \geqslant 4)$		+	+
11			+	−
12	E_6		+	−
13	E_7		+	−
14	E_8		+	−
15	F_4		−	−
16			+	+
17	G_2		−	−
18			−	−

LEMMA. *Let X be of adjoint type, $\theta \in \mathrm{Aut}(X, \Phi)$ an involution of restricted rank one, Δ a θ-basis of Φ and G, T, θ_Δ as in* (4.3). *If $|\Delta - \Delta_0(\theta)| = 1$, then θ is admissible if and only if θ_Δ is an involution* (*i.e.*, $c_{\theta(\alpha),\theta_\Delta} = 1$ *for* $\alpha \in \Delta - \Delta_0(\theta)$).

Proof. The "if" statement being obvious, assume θ is admissible. By (4.5) there exists $t \in T_\theta^+$ such that $c_{\theta(\alpha),\theta_\Delta} = \alpha(t^2)$ for all $\alpha \in \Delta - \Delta_0(\theta)$ and $\alpha(t) = 1$ for all $\alpha \in \Delta_0(\theta)$. So let $\alpha \in \Delta - \Delta_0(\theta)$. It suffices to show that $c_{\theta(\alpha),\theta_\Delta} = 1$ or equivalently $\alpha(t^2) = 1$. Since $|\Delta - \Delta_0(\theta)| = 1$ we have

$$\theta(\alpha) = -w_0(\theta)(\alpha) = -\left(\alpha + \sum_{\beta \in \Delta_0(\theta)} m_\beta \beta\right) \qquad (m_\beta \in \mathbb{N}),$$

so it follows from (4.5) that $\theta(\alpha)(t) = \alpha(t)^{-1}$. On the other hand, since $t \in T_\theta^+$ we have $\theta(\alpha)(t) = \alpha(\theta(t)) = \alpha(t)$, hence $\alpha(t^2) = 1$. This proves the result.

4.13. Whether θ_Δ is an involution or not is a matter determined by structure constants. This can be seen as follows.

Assume $\theta \in \mathrm{Aut}(X, \Phi)$ an involution of restricted rank one, Δ a θ-basis and $|\Delta - \Delta_0(\theta)| = 1$. Let $\alpha \in \Delta - \Delta_0(\theta)$. Then $\theta(\alpha) = -w_0(\theta)(\alpha)$. Since $w_0(\theta)$ is an involution in $W_0(\theta)$ we can write $w_0(\theta) = s_{\alpha_1} \cdots s_{\alpha_r}$, where $\alpha_1, ..., \alpha_r \in \Phi_0(\theta)$ are strongly orthogonal roots (i.e., for all $i, j = 1, ..., r$ we have $\alpha_i \pm \alpha_j \notin \Phi_0(\theta)$) (see, e.g., Helminck [12]). To determine $\theta(\alpha)$ we need to consider only those α_i such that $(\alpha, \alpha_i) \neq 0$. Note that if $\Phi_0(\theta) \neq \emptyset$, then we can choose α_i such that $(\alpha, \alpha_i) \neq 0$ and $\alpha_1 \in \Delta_0(\theta)$. Moreover, there are at most 4 strongly orthogonal roots α_i such that $(\alpha, \alpha_i) \neq 0$ (see Helminck [12] or Kostant [15]).

Choose a realization of Φ in $\mathfrak{g}$ as in (3.1) and for $\alpha, \beta \in \Phi$ let $N_{\alpha,\beta} \in F$ denote the corresponding structure constant (i.e., $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$). Let $\beta_1, ..., \beta_k$ be the set of those α_i for which $(\alpha, \alpha_i) \neq 0$. We can determine $c_{\theta(\alpha),\theta_\Delta}$ now by applying θ_Δ on the identity:

$$\begin{aligned}&[\dots[[X_{-\alpha}, X_{\beta_1}], X_{\beta_2}], ..., X_{\beta_k}]\\ &\quad = N_{-\alpha,\beta_1} \cdot N_{s_{\beta_1}(-\alpha),\beta_2} \cdots N_{s_{\beta_i} \cdots s_{\beta_1}(-\alpha),\beta_{i+1}} \cdots N_{s_{\beta_{k-1}} \cdots s_{\beta_1}(-\alpha),\beta_k} X_{\theta(\alpha)}.\end{aligned}$$

Note that it also follows from this identity that $c_{\theta(\alpha),\theta_\Delta}$ depends only on the structure constants. We can characterize these restricted rank one indices now as follows:

4.14. LEMMA. *Let θ, Δ, θ_Δ, α and $w_0(\theta) = s_{\alpha_1} \cdots s_{\alpha_r}$ be as in* (4.13). *Then θ is admissible if and only if $\sum_{i=1}^r \langle \alpha, \alpha_i^\vee \rangle$ is even.*

We give a proof for $\sum_{i=1}^r \langle \alpha, \alpha_i^\vee \rangle = 1$. The other cases are left to the reader. So assume $\sum_{i=1}^r \langle \alpha, \alpha_i^\vee \rangle = 1$. Say $\langle \alpha, \alpha_1^\vee \rangle = 1$. Then

$$[X_{-\alpha}, X_{\alpha_1}] = N_{-\alpha,\alpha_1} X_{\theta(\alpha)}.$$

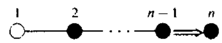
Applying θ_Δ on this identity gives:

$$N_{-\alpha,\alpha_1}\cdot c_{\theta(\alpha),\theta_\Delta}=N_{\theta(-\alpha),\alpha_1}=N_{\alpha-\alpha_1,\alpha_1}.$$

Since $N_{\alpha-\alpha_1,\alpha_1}=N_{\alpha,-\alpha_1}$, it follows that $c_{\theta(\alpha),\theta_\Delta}=-1$, hence θ is not admissible.

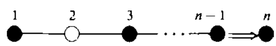
It is not hard to determine $w_0(\theta)$ as a product of strongly orthogonal roots (see, e.g., Helminck [12] or Kostant [15]). Here are two examples:

4.15. EXAMPLES. (1) Assume θ is of type



In this case θ is admissible. One sees this as follows. $\Phi_0(\theta)$ is of type B_{n-1} and $\alpha=\alpha_1$, where $\Delta=\{\alpha_1,\dots,\alpha_n\}$ is a basis of Φ corresponding to the above diagram. So if $n=2$, then $w_0(\theta)=s_{\alpha_2}$ and $\langle\alpha_1,\alpha_2\rangle=2$. If $n>2$, then let β_1 be the longest root of $\Phi_0(\theta)$ with respect to $\Delta_0(\theta)$ and let $\beta_2,\dots,\beta_{n-2}\in\Phi_0(\theta)$ be such that $\alpha,\beta_1,\dots,\beta_{n-2}$ are strongly orthogonal. Now $w_0(\theta)=s_{\alpha_2}s_{\beta_1}\cdots s_{\beta_{n-2}}$, so $\sum_{i=1}^r\langle\alpha,\alpha_i^\vee\rangle=2$ and θ is admissible by (4.14).

(2) Similarly as in (1) one shows that the involution θ with index



is not admissible, because $w_0(\theta)$ contains s_{α_1} additional to the factors in (1).

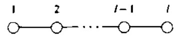
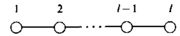
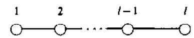
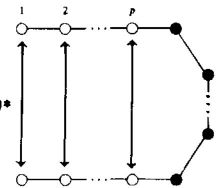
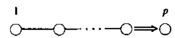
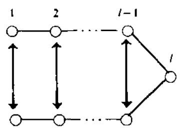
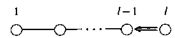
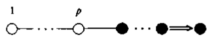
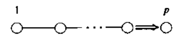
4.16. There remain still 2 indices in Table I, which do not satisfy the conditions in (4.13) and (4.14). However, in these cases one easily shows directly that the index is admissible. In summarizing, we have obtained the following result:

THEOREM. *Let X be of adjoint type. The absolutely irreducible indices of non-trivial admissible involutions of (X,Φ) are the ones given in Table* II. *The irreducible, but not absolutely irreducible indices are the ones given in Table* III.

We added in these tables some extra information which will be explained and used in Section 7, 8.

4.17. *Passage to arbitrary G.* The classification for arbitrary groups G now follows easily from the above results. It is only a matter of checking whether a lattice X is θ-stable. Namely let Φ be a reduced root system and let Q, resp. P, denote the root lattice, resp. weight lattice, of Φ. If $\theta\in\operatorname{Aut}(Q,\Phi)$ is an admissible involution, then θ induces a (unique) involution $\tilde\theta\in\operatorname{Aut}(P,\Phi)$, which is also admissible by a result of Steinberg [27, 9.16]. Now if X is any lattice such that $Q\subset X\subset P$, then θ may be lifted to an admissible involution of (X,Φ) if and only if X is $\tilde\theta$-stable.

TABLE II

Type θ Cartan not.	Type $(\theta, \theta\,\mathrm{Int}(\varepsilon_i))$	(θ, θ)-index	$\bar{\Delta}_\theta$	$m(\lambda)$	$m(2\lambda)$	Quadratic elements	$\mathfrak{g}_\theta^\theta(\varepsilon_i)$
AI	$A_l^l(\mathrm{I}, \varepsilon_i)$			1	0	ε_i $(0 \leqslant 2i \leqslant l+1)$	$\mathfrak{so}(l+1-i, i)$
AII	$A_{2l+1}^l(\mathrm{II}, \varepsilon_i)$			4	0	ε_i $(0 \leqslant 2i \leqslant l+1)$	$\mathfrak{sp}(l+1-i, i)$
$A\mathrm{III}_a$ $(A\mathrm{IV}\ (p=1))$	$A_l^p(\mathrm{III}_a, \varepsilon_i)$ $(1 \leqslant 2p \leqslant l)$	θ^* 		2 $(i<p)$ $2(l-2p+1)$ $(i=p)$	0 1	ε_i $(0 \leqslant i \leqslant p)$	$\mathfrak{su}(l-p+1-i, i)$ $+\,\mathfrak{su}(p-i, i)$ $+\,\mathfrak{so}(2)$
$A\mathrm{III}_b$	$A_{2l-1}^l(\mathrm{III}_b, \varepsilon_i)$ $(l \geqslant 2)$	θ^* 		2 $(i<l)$ 1 $(i=l)$	0 0	ε_i $(0 \leqslant 2i < l)$ ε_l	$\mathfrak{su}(l-i, i)$ $+\,\mathfrak{su}(l-i, i)+\mathfrak{so}(2)$ $\mathfrak{sl}(l, \mathbb{C})+\mathbb{R}$
BI $(B\mathrm{II}\ (p=1))$	$B_l^p(\mathrm{I}, \varepsilon_i)$ $(l \geqslant 2, 1 \leqslant p \leqslant l)$			1 $(i<p)$ $2(l-p)+1$ $(i=p)$	0 0	ε_i $(0 \leqslant i \leqslant p)$	$\mathfrak{so}(2l+1-p-i, l)$ $+\,\mathfrak{so}(p-i, i)$

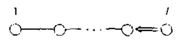
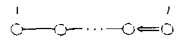
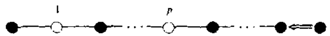
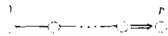
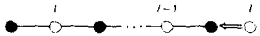
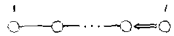
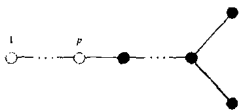
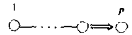
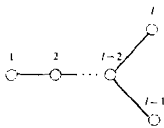
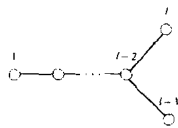
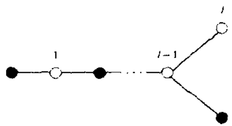
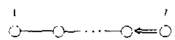
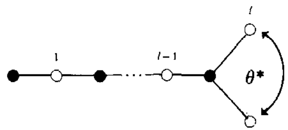
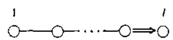
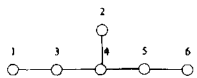
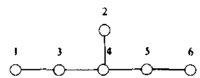
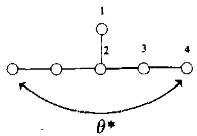
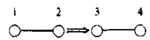
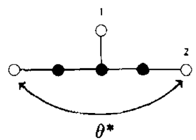
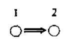
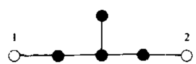
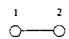
CI	$C_l^l(\mathrm{I}, \varepsilon_i)$			1	0	ε_i $(0 \leqslant 2i \leqslant l)$ ε_l	$\mathfrak{u}(l-i, i)$ $\mathfrak{gl}(l, \mathbb{R})$
CII$_a$	$C_l^p(\mathrm{II}_a, \varepsilon_i)$ $(l \geqslant 3)$ $(1 \leqslant p \leqslant \frac{1}{2}(l-1))$			4 $(i < p)$ $4(l-2p)$ $(i = p)$	0 3	ε_i $(0 \leqslant i \leqslant p)$	$\mathfrak{sp}(l-p-i, i)$ $+\mathfrak{sp}(p-i, i)$
CII$_b$	$C_{2l}^l(\mathrm{II}_b, \varepsilon_i)$ $(l \geqslant 2)$			4 $(i < l)$ 3 $(i = l)$	0 0	ε_i $(0 \leqslant 2i < l)$ ε_l	$\mathfrak{sp}(l-i, i) + \mathfrak{sp}(l-i, i)$ $\mathfrak{sp}(l, \mathbb{C})$
DI$_a$ (DII $(p = 1)$)	$D_l^p(\mathrm{I}_a, \varepsilon_i)$ $(l \geqslant 4)$ $(1 \leqslant p \leqslant l-1)$			1 $(i < p)$ $2(l-p)$ $(i = p)$	0 0	ε_i $(0 \leqslant i \leqslant p)$	$\mathfrak{so}(2l-p-i, i)$ $+\mathfrak{so}(p-i, i)$
DI$_b$	$D_{l}^{l}(\mathrm{I}_b, \varepsilon_i)$ $(l \geqslant 4)$			1	0	ε_i $(0 \leqslant 2i \leqslant l)$ $\varepsilon_{l-1}, \varepsilon_l$	$\mathfrak{so}(l-i, i)$ $+\mathfrak{so}(l-i, i)$ $\mathfrak{so}(l, \mathbb{C})$
DIII$_a$	$D_{2l}^l(\mathrm{III}_a, \varepsilon_i)$ $(l \geqslant 2)$			4 $(i < l)$ 1 $(i = l)$	0 0	ε_i $(0 \leqslant 2i \leqslant l)$ ε_l	$\mathfrak{u}(2l-2i, 2i)$ $\mathfrak{su}^*(2l) + \mathfrak{so}(l, l)$

Table continued

TABLE II — *Continued*

Type θ Cartan not.	Type $(\theta, \theta\,\mathrm{Int}(\varepsilon_i))$	(θ, θ)-index	$\bar{\Delta}_\theta$	$m(\lambda)$	$m(2\lambda)$	Quadratic elements	$\mathfrak{g}_0^\theta(\varepsilon_i)$
$DIII_b$	$D^l_{2l+1}(\mathrm{III}_b, \varepsilon_i)$ $(l \geqslant 2)$			4 $(i<l)$ 4 $(i=l)$	0 1	ε_i $(0 \leqslant i \leqslant l)$	$\mathfrak{u}(2l+1-2i, 2i)$
EI	$E_6^6(\mathrm{I}, \varepsilon_i)$			1	0	ε_1 ε_2	$\mathfrak{sp}(2, 2)$ $\mathfrak{sp}(4, \mathbb{R})$
EII	$E_6^2(\mathrm{II}, \varepsilon_i)$			1 $(i=1, 2)$ 2 $(i=3, 4)$	0 0	ε_1 ε_4	$\mathfrak{su}(3, 3)+\mathfrak{sl}(2, \mathbb{R})$ $\mathfrak{su}(4, 2)+\mathfrak{su}(2)$
$EIII$	$E_6^2(\mathrm{III}, \varepsilon_i)$			6 $(i=1)$ 8 $(i=2)$	0 0	ε_1 ε_2	$\mathfrak{so}^*(10)+\mathfrak{so}(2)$ $\mathfrak{so}(8, 2)+\mathfrak{so}(2)$
EIV	$E_6^2(\mathrm{IV}, \varepsilon_i)$			8	0	ε_1	FII

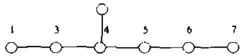
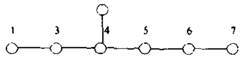
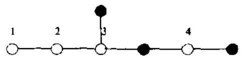
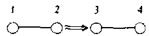
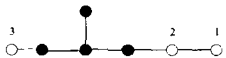
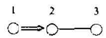
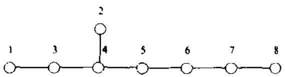
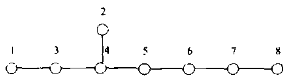
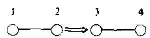
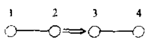
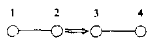
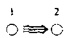
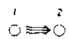
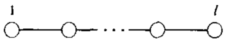
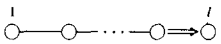
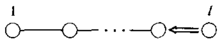
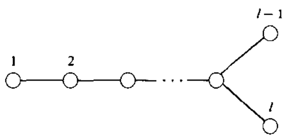
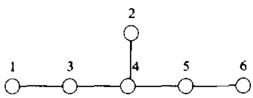
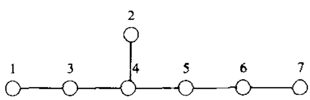
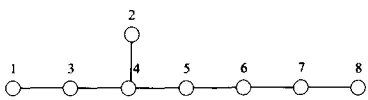
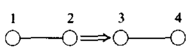
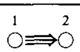
EV	$E_7^7(\mathrm{V}, \varepsilon_i)$			1	0	ε_1 ε_2 ε_7	$\mathfrak{su}(4,4)$ $\mathfrak{sl}(8, \mathbb{R})$ $\mathfrak{su}^*(8)$
EVI	$E_7^4(\mathrm{VI}, \varepsilon_i)$			1 $(i=1, 2)$ 4 $(i=3, 4)$	0 0	ε_1 ε_4	$\mathfrak{so}^*(12)+\mathfrak{sl}(2, \mathbb{R})$ $\mathfrak{so}(8, 2)+\mathfrak{su}(2)$
$EVII$	$E_7^3(\mathrm{VII}, \varepsilon_i)$			1 $(i=1)$ 8 $(i=2, 3)$	0 0	ε_1 ε_3	$EIV+\mathfrak{so}(1, 1)$ $EIII+\mathfrak{so}(2)$
$EVIII$	$E_8^8(\mathrm{VIII}, \varepsilon_i)$			1	0	ε_1 ε_8	$\mathfrak{so}(8, 8)$ $\mathfrak{so}^*(16)$
EIX	$E_8^4(\mathrm{IX}, \varepsilon_i)$			1 $(i=1, 2)$ 8 $(i=3, 4)$	0 0	ε_1 ε_4	$EVII+\mathfrak{sl}(2, \mathbb{R})$ $EVI+\mathfrak{su}(2)$
FI	$F_4^4(\mathrm{I}, \varepsilon_i)$			1	0	ε_1 ε_4	$\mathfrak{sp}(3, \mathbb{R})+\mathfrak{sl}(2, \mathbb{R})$ $\mathfrak{sp}(2, 1)+\mathfrak{su}(2)$
FII	$F_4^1(\mathrm{II}, \varepsilon_i)$			8	7	ε_1	$\mathfrak{so}(8, 1)$
G	$G_2^2(\varepsilon_i)$			1	0	ε_1	$\mathfrak{sl}(2, \mathbb{R})+\mathfrak{sl}(2, \mathbb{R})$

TABLE III

Type(θ, $\theta\,\mathrm{Int}(\varepsilon_j)$)	$\bar{\Delta}_0$	$m(\lambda)$	Quadratic elements
$(A_l \times A_l)(\varepsilon_j)$ $(l \geqslant 1)$		2	ε_j $(2j \leqslant l+1)$
$(B_l \times B_l)(\varepsilon_j)$ $(l \geqslant 2)$		2	ε_j $(j \leqslant l)$
$(C_l \times C_l)(\varepsilon_j)$ $(l \geqslant 3)$		2	ε_j $(2j \leqslant l)$ ε_l
$(D_l \times D_l)(\varepsilon_j)$ $(l \geqslant 4)$		2	ε_j $(2j \leqslant l)$ ε_{l-1} ε_l
$(E_6 \times E_6)(\varepsilon_j)$		2	ε_1 ε_2
$(E_7 \times E_7)(\varepsilon_j)$		2	ε_1 ε_2 ε_7
$(E_8 \times E_8)(\varepsilon_j)$		2	ε_1 ε_8
$(F_4 \times F_4)(\varepsilon_j)$		2	ε_1 ε_4
$(G_2 \times G_2)(\varepsilon_j)$		2	ε_1

5. Conjugacy Classes of Pairs of Commuting Involutorial Automorphisms of G

In this section we characterize conjugacy classes of pairs of commuting involutorial automorphisms of G in a manner similar to that of Section 3.

5.1. Let $\sigma, \theta \in \mathrm{Aut}(G)$ be such that $\sigma^2 = \theta^2 = \mathrm{id}$ and $\sigma\theta = \theta\sigma$. Let $\mathfrak{g}_\sigma$, $\mathfrak{g}_\theta$, $\mathfrak{g}_{\sigma\theta}$ denote the Lie algebras of G^0_σ, G^0_θ, $G^0_{\sigma\theta}$, respectively. Write (for $\xi, \eta = \pm 1$) $\mathfrak{g}(\xi, \eta) = \{X \in \mathfrak{g} \mid \sigma(X) = \xi X,\ \theta(X) = \eta X\}$. Then,

$$\mathfrak{g}_\sigma = \mathfrak{g}(1, 1) \oplus \mathfrak{g}(1, -1);$$
$$\mathfrak{g}_\theta = \mathfrak{g}(1, 1) \oplus \mathfrak{g}(-1, 1);$$
$$\mathfrak{g}_{\sigma\theta} = \mathfrak{g}(1, 1) \oplus \mathfrak{g}(-1, -1).$$

Note that $\mathfrak{g}$ is the direct sum of the $\mathfrak{g}(\xi, \eta)$ ($\xi, \eta = \pm 1$).

5.2. DEFINITION. A torus A of G is called *(σ, θ)-split* if A is σ- and θ-split. A torus T of G, which is σ- and θ-stable shall be called *(σ, θ)-stable.* We then put $T^-_{\sigma,\theta} = \{t \in T \mid \sigma(t) = \theta(t) = t^{-1}\}^0$.

If G is an arbitrary reductive connected algebraic group and σ, $\theta \neq \mathrm{id}$, then non-trivial (σ, θ)-split tori of G need not exist. One sees this in the example of a direct product $G = G_1 \times G_2$, with G_1, G_2 (reductive) groups and $\sigma(G_i) = G_i$, $\theta(G_i) = G_i (i = 1, 2)$, $\theta|_{G_1} = \mathrm{id}$, $\sigma|_{G_2} = \mathrm{id}$.

In (5.10) we shall see that if G is simple and σ, $\theta \neq \mathrm{id}$, then non-trivial (σ, θ)-split tori exist. In fact we shall show an equivalent statement that if G has no (σ, θ)-split tori, that then on each irreducible component of $\Phi(T)$ we have $\sigma = \mathrm{id}$ or $\theta = \mathrm{id}$. Here T is a (σ, θ)-stable maximal torus of G. To do so we first prove some results on (σ, θ)-stable tori.

5.3. LEMMA. *The following statements are equivalent*:

(a) *G contains no nontrivial (σ, θ)-split tori.*

(b) *$G^0_{\sigma\theta}$ contains no non-trivial σ-split tori.*

(c) $G^0_{\sigma\theta} = G^0_\sigma \cap G^0_\theta$.

(d) $\mathfrak{g}(-1, -1) = 0$.

Proof. (a) $\Leftrightarrow$ (b) is clear from the observation that the (σ, θ)-split tori of G are precisely the σ-split (or θ-split) tori of $G^0_{\sigma\theta}$.

(b) $\Rightarrow$ (c) follows immediately from (1.4) and (c) $\Rightarrow$ (d) follows from (5.1). Finally (d) $\Rightarrow$ (a) is immediate from the observation that the Lie algebra of a (σ, θ)-split torus is contained in $\mathfrak{g}(-1, -1)$.

5.4. PROPOSITION. *Let $\sigma, \theta \in \mathrm{Aut}(G)$ be a pair of commuting involutorial automorphisms of G. If $\theta \neq \mathrm{id}$, then there exists a maximal θ-split torus of G, which is σ-stable.*

Proof. Let A be a maximal (σ, θ)-split torus of G. It suffices to show that $Z_G(A)/A$ contains a σ-stable maximal θ-split torus.

If A is already maximal θ-split, we are done, so assume A is not maximal θ-split. Then passing to $Z_G(A)/A$, we may assume that G has no (σ, θ)-split tori and $\theta \neq \mathrm{id}$. Now $\theta \mid G^0_\sigma \neq \mathrm{id}$, because if $G^0_\sigma \subset G^0_\theta$ then using (5.3) we get $\mathfrak{g}(1, -1) = \mathfrak{g}(-1, -1) = 0$, whence $\mathfrak{g}_\theta = \mathfrak{g}$, contradicting $\theta \neq \mathrm{id}$. Let S be a maximal θ-split torus of G^0_σ. Then, since G has no non-trivial (σ, θ)-split tori, the same holds for $Z_G(S)/S$. In other words, S is a σ-stable maximal θ-split torus of G. This proves the result.

5.5. COROLLARY. *There exists a maximal torus of G, which is (σ, θ)-stable.*

Proof. Let T be a σ-stable maximal torus of $Z_G(A)$, where A is a σ-stable maximal θ-split torus of G. Then by (1.5) T is also θ-stable, hence the result.

5.6. Let T be a (σ, θ)-stable maximal torus of G, denote by $\Psi = (X^*(T), \Phi(T), X_*(T), \Phi^\vee(T))$ the corresponding root datum and write $A = T^-_{\sigma,\theta}$ (For the moment we do not yet assume that A is a maximal (σ, θ)-split torus of G). Using the notations of (2.2) we have the following identifications:

LEMMA. *Let T, Ψ, σ, θ and A be as above. Then*

(i) $X_0(\sigma, \theta) = \{\chi \in X^*(T) \mid \chi(A) = 1\}$;

(ii) $\bar{\Phi}_{\sigma,\theta} = \Phi(A)$;

(iii) $W_1(\sigma, \theta) = \{w \in W(T) \mid w(A) = A\}$ *and* $W_0(\sigma, \theta) = \{w \in W(T) \mid w \mid A = \mathrm{id}\}$;

(iv) $W(A) \cong W_1(\sigma, \theta)/W_0(\sigma, \theta) \cong \bar{W}_{\sigma,\theta}$.

Proof. (i) Note first that $X_0(\sigma) = \{\chi \in X^*(T) \mid \chi(T^-_\sigma) = 1\}$. Let $\chi \in X^*(T)$ be such that $\chi(A) = 1$. If $t \in T^-_\sigma$, then writing $t = t_1 \cdot t_2$, where $t_1 \in A$, $t_2 \in (T^-_\sigma)^+_\theta$, it follows that $\chi(t) = \chi(\theta(t))$, whence $\chi - \theta(\chi) \in X_0(\sigma)$. But then $\chi - \theta(\chi) = \sigma(\chi - \theta(\chi))$; in other words, $\chi \in X_0(\sigma, \theta)$. On the other hand, if $\chi \in X_0(\sigma, \theta)$, then $\chi - \theta(\chi) \in X_0(\sigma)$, so for all $t \in A$:

$$\chi(t) = \theta(\chi)(t) = \chi(t^{-1}),$$

hence $\chi(A) = 1$. This proves (i).

As for (ii), we only note that the roots of G with respect to the adjoint action of A on $\mathfrak{g}$ are exactly the restrictions of $\Phi(T)$ to A.

(iii) follows from the fact that $A = \{t \in T \mid \chi(t) = 1 \text{ for all } \chi \in X_0(\sigma, \theta)\}$.

Finally, using (iii), the proof of (iv) is as in Richardson [20, 4.1].

5.7. Assume G does not contain a non-trivial (σ, θ)-split torus and let T be a (σ, θ)-stable maximal torus of G.

LEMMA. *$X^*(T) = X_0(\sigma, \theta)$, $\Phi(T) = \Phi_0(\sigma, \theta)$. In particular if $\chi \in X^*(T)$, $\sigma(\chi) = -\chi$ and $\theta(\chi) = -\chi$, then $\chi = 0$.*

Proof. Since G has no (σ, θ)-split tori $T^-_{\sigma,\theta} = \{e\}$, hence by (5.6),

$X^*(T) = X_0(\sigma, \theta)$ and $\Phi(T) = \Phi_0(\sigma, \theta)$. But for $X_0(\sigma, \theta)$ the second assertion follows immediately.

5.8. PROPOSITION. *Let (G, T) be as in (5.7). If $\alpha \in \Phi(T)$, then $\sigma(\alpha) = \alpha$, $\mathfrak{g}_\alpha \subset \mathfrak{g}_\sigma$ or $\theta(\alpha) = \alpha$, $\mathfrak{g}_\alpha \subset \mathfrak{g}_\theta$.*

Proof. Let $\alpha \in \Phi(T)$ and let $0 \neq X_\alpha \in \mathfrak{g}_\alpha$ denote a root vector. Since G has no (σ, θ)-split tori, we have $\mathfrak{g}(-1, -1) = 0$, by (5.3). So $(1 - \sigma)(1 - \theta)\, X_\alpha = 0$, whence

$$X_\alpha - \sigma(X_\alpha) - \theta(X_\alpha) + \sigma\theta(X_\alpha) = 0.$$

Now $\theta(X_\alpha) \in \mathfrak{g}_{\theta(\alpha)}$, $\sigma(X_\alpha) \in \mathfrak{g}_{\sigma(\alpha)}$ and $\sigma\theta(X_\alpha) \in \mathfrak{g}_{\sigma\theta(\alpha)}$. It follows that if $\theta(\alpha) \neq \alpha$, $\sigma(\alpha) \neq \alpha$, we must have $\sigma\theta(\alpha) = \alpha$. Since $X^*(T) = X_0(\sigma, \theta)$ we have $\chi = \alpha - \sigma(\alpha) = \alpha - \theta(\alpha)$. But then $\sigma(\chi) = -\chi = \theta(\chi)$, so, by (5.7), $\chi = 0$. This is a contradiction, hence the assertion has been shown.

5.9. PROPOSITION. *Let (G, T) be as in (5.7) and let $\Phi_1 \subset \Phi(T) = \Phi_0(\sigma, \theta)$ be an irreducible component. Then $\sigma \mid \Phi_1 = \mathrm{id}$ or $\theta \mid \Phi_1 = \mathrm{id}$.*

Proof. Let Δ be a basis of Φ_1. Assume $\sigma \mid \Phi_1 \neq \mathrm{id}$ and $\theta \mid \Phi_1 \neq \mathrm{id}$. Then there are $\alpha, \beta \in \Delta$ such that $\sigma(\alpha) = \alpha$, $\sigma(\beta) \neq \beta$, $\theta(\alpha) \neq \alpha$, $\theta(\beta) = \beta$. Since Φ_1 is irreducible, there is a string of simple roots $\alpha_1 = \alpha$, $\alpha_2, ..., \alpha_r = \beta$ connecting α and β. Moreover, we can choose $\alpha, \beta \in \Delta$ such that $\sigma(\alpha_i) = \theta(\alpha_i) = \alpha_i$ for $i = 2, ..., r-1$. Now $\gamma = \alpha_1 + \cdots + \alpha_r \in \Phi_1$, while $\sigma(\gamma) \neq \gamma$, $\theta(\gamma) \neq \gamma$, which contradicts (5.8).

From (5.8) and (5.9) we conclude:

5.10. COROLLARY. *If $\Phi(G, T)$ is irreducible and $\sigma \neq \mathrm{id}$, $\theta \neq \mathrm{id}$, then non-trivial (σ, θ)-split tori exist.*

5.11. COROLLARY. *Let A be a maximal (σ, θ)-split torus of G and A_1 resp. A_2 maximal σ-split resp. θ-split tori of $Z_G(A)$. Then A_1 and A_2 commute.*

Proof. We may assume $G = Z_G(A)$. If $\bar{A}_i = (A_i \cap (G, G))^0$ $(i = 1, 2)$, then it suffices to show that $\bar{A}_1$ and $\bar{A}_2$ commute in (G, G). But this follows from (5.9).

5.12. LEMMA. *All maximal (σ, θ)-split tori of G are conjugate under $(G_0 \cap G_\theta)^0$.*

Proof. Let A_1, A_2 be maximal (σ, θ)-split of G. Since A_1 and A_2 are also maximal σ-split tori of $G^0_{\sigma\theta}$, they are conjugate under $(G^0_{\sigma\theta})^0_\sigma = (G_0 \cap G_\theta)^0$.

5.13. Proposition. *There exist (σ, θ)-stable maximal tori of G such that $T^-_{\sigma,\theta}$ is a maximal (σ, θ)-split torus of G, T^-_σ is a maximal σ-split torus of G, and T^-_θ is a maximal θ-split torus of G. Moreover, all such maximal tori of G are conjugate under $(G_\sigma \cap G_\theta)^0$.*

Proof. Let A be a maximal (σ, θ)-split torus of G and A_1 (resp. A_2) a maximal σ-split (resp. θ-split) torus of $Z_G(A)$. Since A_1 and A_2 commute (see (5.11)), the first assertion follows by taking a (σ, θ)-stable maximal torus T of $Z_G(A_1 A_2)$.

If $\tilde{T}$ is another maximal torus of G satisfying the above condtions, then by (5.12) we may assume that $A = T^-_{\sigma,\theta} = \tilde{T}^-_{\sigma,\theta}$. Moreover, passing to $Z_G(A)/A$, we may also assume that G has no nontrivial (σ, θ)-split tori. But then $T^-_{\sigma\theta}$ and $\tilde{T}^-_{\sigma\theta}$ are maximal $\sigma\theta$-split tori of G, hence by (1.5) there exists $g \in G^0_{\sigma\theta}$ such that $gTg^{-1} = \tilde{T}$. Since $G^0_{\sigma\theta} = (G_\sigma \cap G_\theta)^0$ (see (5.3)) the result follows.

The notion "normally related" is defined as in the case of one involution (see (3.6)):

5.14. Definition. If (σ, θ) is a pair of commuting involutorial automorphisms of G and T is a maximal torus of G, then (σ, θ) is said to be *normally related* to T if $\sigma(T) = \theta(T) = T$ and $T^-_{\sigma,\theta}$, T^-_σ, T^-_θ are maximal (σ, θ)-split, σ-split, θ-split tori of G, respectively.

Note that in this case both σ and θ are also normally related to T. Moreover, using (5.9) on $Z_G(T^-_{\sigma,\theta})$ it follows that $\Phi(T)$ has an order which is simultaneously a σ- and θ-order. This will be used to represent such a pair of commuting involutions of $(X^*(T), \Phi(T))$ by a diagram (see Section 7).

5.15. Definition. Two pairs of involutorial automorphisms (σ_1, θ_1) and (σ_2, θ_2) of G are *isomorphic* if there exists a $g \in G$ such that $\mathrm{Int}(g)\, \sigma_1\, \mathrm{Int}(g^{-1}) = \sigma_2$ and $\mathrm{Int}(g)\, \theta_1\, \mathrm{Int}(g^{-1}) = \theta_2$. The family of all pairs of commuting involutorial automorphisms of G will be denoted by $\mathscr{F}$ and the set of isomorphism classes in $\mathscr{F}$ by $\mathscr{C}$.

Note that we only consider isomorphisms of *ordered* pairs of commuting involutions of G. We could also allow isomorphisms which map σ_1 onto θ_2 and θ_1 onto σ_2. Such an isomorphism identifies the isomorphism classes of (σ_1, θ_1) and (θ_1, σ_1). However, when passing from pairs of commuting involutions to symmetric spaces (see Section 9) it is more convenient to work with ordered pairs, because the pairs (θ, σ) and (σ, θ) will correspond to dual symmetric spaces.

An identification of the isomorphism classes in $\mathscr{F}$ under the action of the group of outer automorphisms of G will be discussed in Section 9.

5.16. THEOREM. *Let* (σ_1, θ_1) *and* (σ_2, θ_2) *be pairs of commuting involutorial automorphisms of* G, *normally related to* T. *Then* $(\sigma_1, \theta_1)\,|\,T$ *and* $(\sigma_2, \theta_2)\,|\,T$ *are conjugate under* $W(T)$ *if and only if there exists* $\varepsilon \in T^-_{\sigma,\theta}$ *with* $\varepsilon^2 \in Z(G)$ *such that* (σ_2, θ_2) *is isomorphic to* $(\sigma_1, \theta_1 \operatorname{Int}(\varepsilon))$.

Proof. We may assume that $\sigma_1\,|\,T=\sigma_2\,|\,T=\sigma$ and $\theta_1\,|\,T=\theta_2\,|\,T=\theta$. By the proof of (3.7) we see that after conjugation with a suitable element of T, we may assume that $\sigma_1=\sigma_2$. Since $\theta_1\,|\,T=\theta_2\,|\,T$, there exists $t\in T^-_\theta$ such that $\theta_2=\theta_1 \operatorname{Int}(t)$ (see (3.8)). Write $t=t_1t_2$ where $t\in (T^-_\theta)^+_\sigma$ and $t_2\in T^-_{\sigma,\theta}$. Taking $c\in (T^-_\theta)^+_\sigma$ such that $c^2=t_1$, we obtain $\operatorname{Int}(c)\,\theta_2 \operatorname{Int}(c)^{-1}=\theta_2 \operatorname{Int}(c^{-2})=\theta_1 \operatorname{Int}(t_2)$ and $\operatorname{Int}(c)\,\sigma_1 \operatorname{Int}(c)^{-1}=\sigma_1$. Since $t_2\in T^-_{\sigma,\theta}$ and $t_2^2\in Z(G)$ we are done.

5.17. For a torus S of G we call the elements $s\in S$ for which $s^2\in Z(G)$ *quadratic elements of* S. We can again define a notion of admissibility:

5.18. DEFINITION. Let T be a maximal torus of G. A pair of commuting involutorial automorphisms (σ, θ) of $(X^*(T), \Phi(T))$ is said to be *admissible* (with respect to G) if there exists a pair of commuting involutorial automorphisms $(\tilde\sigma, \tilde\theta)$ of G, normally related to T and such that $\tilde\sigma\,|\,T=\sigma$, $\tilde\theta\,|\,T=\theta$.

5.19. Let $\mathscr{C}(T)$ denote the set of $W(T)$-conjugacy classes of ordered pairs of commuting involutions of $(X^*(T), \Phi(T))$.

By (5.13) and the conjugacy of the maximal tori of G it follows that every pair of commuting involutorial automorphisms of G is isomorphic to one normally related to T, so we have a natural map

$$\rho : \mathscr{C} \to \mathscr{C}(T).$$

Denote the image of ρ by $\mathscr{A}(T)$ and the fiber above $\rho((\sigma, \theta))$ by $\mathscr{C}(\sigma, \theta)$. We note that $\mathscr{A}(T)$ is nothing other than the set of $W(T)$-isomorphism classes of admissible pairs of commuting involutions of $(X^*(T), \Phi(T))$.

We now have the following result:

5.20. THEOREM. *Let* T *be a maximal torus of* G. *There is a bijection between the* $W(T)$-*conjugacy classes of admissible pairs of commuting involutions of* $(X^*(T), \Phi(T))$ *and the sets* $\mathscr{C}(\sigma, \theta)$ *in* $\mathscr{C}$.

5.21. *Remarks.* (i) As in the case of single involutorial automorphisms, the classes in $\mathscr{A}(T)$ will be represented by diagrams ((σ, θ)-indices; see Section 7). To show that these diagrams are independent of the choice of the (σ, θ)-basis we will need some properties of the restricted root system and Weyl group of a maximal (σ, θ)-split torus A of G. This will be treated in the next section.

(ii) We denote the subset of $\mathscr{F}$ consisting of all pairs of commuting involutions, whose isomorphism classes are contained in $\mathscr{C}(\sigma, \theta)$ by $\mathscr{F}(\sigma, \theta)$. For $(\sigma, \theta) \in \mathscr{F}$ and a maximal (σ, θ)-split torus A of G let $\mathscr{F}_A(\sigma, \theta) = \{(\sigma, \theta \operatorname{Int}(\varepsilon)) \mid \varepsilon \in A, \varepsilon^2 \in Z(G)\}$. It follows from (5.16) that any pair in $\mathscr{F}(\sigma, \theta)$ is isomorphic to a pair $(\sigma, \theta \operatorname{Int}(a)) \in \mathscr{F}_A(\sigma, \theta)$. So the classes in $\mathscr{C}(\sigma, \theta)$ can be represented by a set of quadratic elements of A. We will show in (8.2) that we can restrict ourselves to the action of $N_G(A)$ on $\mathscr{F}_A(\sigma, \theta)$. In order to show that this action of $N_G(A)$ on $\mathscr{F}_A(\sigma, \theta)$ can be split in an action of $W(A)$ on $\mathscr{F}_A(\sigma, \theta)$ and an action of $Z_G(A)$ on $\mathscr{F}_A(\sigma, \theta)$, we will need to have a kind of standard pair $(\sigma, \theta) \in \mathscr{F}_A(\sigma, \theta)$ with the property that every $w \in W(A)$ has a representative in $(G_\sigma \in G_\theta)^0$. This will be defined in (6.11).

(iii) If $(\sigma, \theta) \in \mathscr{F}$, A a (σ, θ)-split torus of G and $\varepsilon \in A$, $\varepsilon^2 \in Z(G)$, then the pairs $(\sigma, \theta \operatorname{Int}(\varepsilon))$ and $(\sigma \operatorname{Int}(\varepsilon), \theta)$ are isomorphic. Namely, take $c \in A$ such that $c^2 = \varepsilon$. Then conjugating by $\operatorname{Int}(c)$ gives the desired isomorphism.

6. The Restricted Root System of (σ, θ) and Standard Pairs

6.1. Let (σ, θ) be a pair of commuting involutorial automorphisms of G and A a non-trivial maximal (σ, θ)-split torus of G.

In this section we shall prove that $\Phi(A)$ is a root system in the vector space $X^*(A) \otimes_{\mathbb{Z}} \mathbb{R}$ and that the corresponding Weyl group is $W(A)$. Since A is also a maximal σ-split torus of $G^0_{\sigma\theta}$ (see (5.3)), we already know, by (1.6), that $\Phi(A, G^0_{\sigma\theta})$ is a root system. The relations between $\Phi(A)$ and $\Phi(A, G^0_{\sigma\theta})$ will be treated and moreover, we shall show that there exists a pair $(\sigma, \theta) \in \mathscr{F}_A(\sigma, \theta)$ for which the Weyl groups of $\Phi(A)$ and $\Phi(A, G^0_{\sigma\theta})$ coincide. In particular in this case every $w \in W(A)$ has a representative in $(G_\sigma \cap G_\theta)^0$. This will be used for the classification of those quadratic elements of A, which represent an isomorphism class in $\mathscr{C}(\sigma, \theta)$.

6.2. Let T be a (σ, θ)-stable maximal torus of G and let $A = T^-_{\sigma, \theta}$. For the moment we do not yet assume that A is a maximal (σ, θ)-split torus of G. This will only be needed to obtain all the reflections in $W(A)$ (see (6.11)).

For $\lambda \in \Phi(A)$ let $\mathfrak{g}(A, \lambda)$ be the corresponding root space. Since $\sigma\theta(\lambda) = \lambda$, we have $\sigma\theta(\mathfrak{g}(A, \lambda)) = \mathfrak{g}(A, \lambda)$. Put

$$\mathfrak{g}(A, \lambda)^{\pm}_{\sigma\theta} = \{X \in \mathfrak{g}(A, \lambda) \mid \sigma\theta(X) = \pm X\},$$

$$m^{\pm}(\lambda, \sigma\theta) = \dim_F \mathfrak{g}(A, \lambda)^{\pm}_{\sigma\theta},$$

$$\Phi(T, \lambda) = \{\alpha \in \Phi(T) \mid \alpha \mid A = \lambda\},$$

and

$$m(\lambda) = \dim_F \mathfrak{g}(A, \lambda) = m^+(\lambda, \sigma\theta) + m^-(\lambda, \sigma\theta) = |\Phi(T, \lambda)|.$$

6.3. DEFINITION. For $\lambda \in \Phi(A)$ call $m(\lambda)$ the *multiplicity* of λ and $(m^+(\lambda, \sigma\theta), m^-(\lambda, \sigma\theta))$ the *signature* of λ.

6.4. *Remark.* If $a \in A$ is a quadratic element and $\lambda \in \Phi(A)$ is such that $\lambda(a) = -1$, then $m^-(\lambda, \sigma\theta) = m^+(\lambda, \sigma\theta \operatorname{Int}(a))$ and $m^+(\lambda, \sigma\theta) = m^-(\lambda, \sigma\theta \operatorname{Int}(a))$.

Whether a root of $\Phi(A)$ is contained in $\Phi(A, G^0_{\sigma\theta})$ can be detected from its signature:

6.5. LEMMA. *Let $\lambda \in \Phi(A)$, then $\lambda \in \Phi(A, G^0_{\sigma\theta})$ if and only if $m^+(\lambda, \sigma\theta) > 0$.*

6.6. *Quadratic elements of A with respect to a basis of $\Phi(A)$.* Let Δ be a (σ, θ)-basis of $\Phi(T)$ and let $\bar{\Delta}_{\sigma,\theta}$ denote the restricted basis of (2.2). Since the elements of $\bar{\Delta}_{\sigma,\theta}$ are linearly independent (see (2.3)) and they generate $X^*(\operatorname{Ad}(A))$, it follows that for every $\lambda \in \bar{\Delta}_{\sigma,\theta}$ there exists $\gamma_\lambda \in X_*(A)$ such that $\langle \lambda, \gamma_{\lambda'} \rangle = \delta_{\lambda,\lambda'}$ for $\lambda, \lambda' \in \bar{\Delta}_{\sigma,\theta}$.

For $\lambda \in \bar{\Delta}_{\sigma,\theta}$ put $\varepsilon_\lambda = \gamma_\lambda(-1)$. Then $\varepsilon_\lambda^2 = \gamma_\lambda(-1)\gamma_\lambda(-1) = \gamma_\lambda(+1) = e$, hence ε_λ is a quadratic element of A. If $\Phi(T)$ has a (σ, θ)-basis, which is simultaneously a σ- and θ-basis, then we can describe ε_λ also in terms of one-parameter subgroups of $X_*(T)$ (for this see Section 8).

6.7. LEMMA. *Let Δ be a (σ, θ)-basis of $\Phi(T)$. There exists $\varepsilon \in A$ with $\varepsilon^2 = e$ such that for $\lambda \in \bar{\Delta}_{\sigma,\theta}$*

$$m^+(\lambda, \sigma\theta \operatorname{Int}(\varepsilon)) \geqslant m^-(\lambda, \sigma\theta \operatorname{Int}(\varepsilon)).$$

In particular we then have: $\bar{\Delta}_{\sigma,\theta} \subset \Phi(A, G^0_{\sigma\theta \operatorname{Int}(\varepsilon)})$.

Proof. Taking ε to be the product of those ε_λ, $\lambda \in \bar{\Delta}_{\sigma,\theta}$, for which $m^+(\lambda, \sigma\theta) < m^-(\lambda, \sigma\theta)$, the result follows from (6.4) and (6.6).

Let $E = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ and let $E^-_{\sigma,\theta}$ be the common (-1)-eigenspace of σ and θ in E. Take a positive definite σ, θ and $W(T)$-invariant inner product $(\cdot, \cdot)$ in E. We identify $W(A)$ with its image in $GL(E^-_{\sigma,\theta})$ and, for $\lambda \in \Phi(A)$, let $s_\lambda \in GL(E^-_{\sigma,\theta})$ denote the reflection in the hyperplane $E^-_{\sigma,\theta}(\lambda) = \{x \in E^-_{\sigma,\theta} \mid (x, \lambda) = 0\}$. So $s_\lambda(x) = x - 2(\lambda, x)(\lambda, \lambda)^{-1}\lambda$. If A is a maximal (σ, θ)-split torus of G, then, by (1.6), $\Phi(A, G^0_{\sigma\theta})$ is a root system. Hence for every $\lambda \in \Phi(A)$ with $m^+(\lambda, \sigma\theta) \neq 0$, there exists a reflection s_λ in $W(A)$. Combining this with (6.7) we obtain:

6.8. LEMMA. *Let A be a maximal (σ, θ)-split torus of G. If A is not* *central, then $N_G(A) \neq Z_G(A)$. In particular for every $\lambda \in \Phi(A)$ there exists $n \in N_G(A)$ whose image in $W(A)$ is s_λ.*

Proof. The first statement readily follows from the above remark and the second statement follows from this by considering $Z_G((\operatorname{Ker}\lambda)^0)$, in which A is not central. Then any $n \in N_G(A) \cap Z_G((\operatorname{Ker}\lambda)^0)$ such that $n \notin Z_G(A)$, represents the reflection s_λ in $W(A)$.

Now that we have constructed the reflections in $W(A)$, we can follow the proof of Springer [24, 9.1.9] to show:

6.9. Lemma. (i) *$W(A)$ is generated by the reflections s_λ, $\lambda \in \Phi(A)$;*
(ii) *If $\lambda \in \Phi(A)$ and $\chi \in X^*(A)$ then $2(\lambda, \lambda^{-1})(\lambda, \chi) \in \mathbb{Z}$.*

(See also Richardson [20, 4.5].)

For $\lambda \in \Phi(A)$ now define the dual root as the unique $\lambda^\vee \in X_*(A)$ such that $\langle \chi, \lambda^\vee \rangle = 2(\lambda, \lambda^{-1})(\lambda, \chi) \in \mathbb{Z}$ for all $\chi \in X^*(A)$ (i.e., $s_\lambda(\chi) = \chi - \langle \chi, \lambda^\vee \rangle \lambda$). Then denoting the set of dual roots by $\Phi^\vee(A)$ we have proved:

6.10. Proposition. *Let A be a non-central maximal (σ, θ)-split torus of G. Then the quadruple $(X^*(A), \Phi(A), X_*(A), \Phi^\vee(A))$ is a root datum in the sense of* (2.1). *In particular $\Phi(A)$* is a *root system in the subspace E' of $X^*(A) \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by $\Phi(A)$ and its Weyl group is given by the restriction of $W(A)$ to E'.*

Put $\Phi(A)' = \{\lambda \in \Phi(A) \mid \frac{1}{2}\lambda \notin \Phi(A)\}$.

6.11. *Standard pairs.* For the remaining part of this section we assume A to be maximal (σ, θ)-split and non-central. In order to have the Weyl group of A acting on the quadratic elements in a family $\mathscr{F}_A(\sigma, \theta)$ we need representatives in $(G_\sigma \cap G_\theta)^0$. In case the Weyl groups of $\Phi(A)$ and $\Phi(A, G^0_{\sigma\theta})$ coincide, this condition is satisfied, because every element of $W(A, G^0_{\sigma\theta})$ has a representative in $(G^0_{\sigma\theta})^0_\sigma = (G_\sigma \cap G_\theta)^0$. This leads to the following definition:

Definition. A pair of commuting involutorial automorphisms (σ, θ) of G is called a *standard pair* if $m^+(\lambda, \sigma\theta) \geqslant m^-(\lambda, \sigma\theta)$ for any maximal (σ, θ)-split torus A of G and any $\lambda \in \Phi(A)'$.

6.12. Lemma. *Let σ, θ and A be as in* (6.11) *and let $\bar{\Delta}(A)$ be a basis of $\Phi(A)$. If $m^+(\lambda, \sigma\theta) \geqslant m^-(\lambda, \sigma\theta)$ for any $\lambda \in \bar{\Delta}(A)$, then $m^\pm(\lambda, \sigma\theta) = m^\pm(w(\lambda), \sigma\theta)$ for any $\lambda \in \Phi(A)$, $w \in W(A)$. In particular (σ, θ) is a standard pair.*

Proof. Since $\bar{\Delta}(A)$ is a basis of $\Phi(A)$ and $\Phi(A, G^0_{\sigma\theta})$, their Weyl groups coincide. By (1.6) every $w \in W(A, G^0_{\sigma\theta})$ has a representative in $(G^0_{\sigma\theta})^0_\sigma = (G_\sigma \cap G_\theta)^0$, so we get $m^+(w(\lambda), \sigma\theta) = m^+(\lambda, \sigma\theta)$ for any $\lambda \in \Phi(A)$, $w \in W(A)$. However, since also $m(w(\lambda)) = m(\lambda)$ for any $\lambda \in \Phi(A)$, $w \in W(A)$, we have $m^-(w(\lambda), \sigma\theta) = m^-(\lambda, \sigma\theta)$, which proves the first statement. Finally, observing that for any $\lambda \in \Phi(A)'$, there exists $w \in W(A)$ such that $w(\lambda) \in \bar{\Delta}(A)$, the result is a consequence of (5.12).

Using (6.7) this lemma implies immediately:

6.13. THEOREM. *Every family $\mathscr{F}(\sigma, \theta)$ contains a standard pair.*

We shall see later, as a consequence of the classification, that the standard pair in $\mathscr{F}(\sigma,)$ is unique up to isomorphism.

6.14. Note that if G is of adjoint type and (σ, θ), $(\sigma, \theta \operatorname{Int}(\varepsilon))$ ($\varepsilon \in A$, $\varepsilon^2 = e$) are standard pairs in $\mathscr{F}_A(\sigma, \theta)$, then ε is a product of a number of the ε_λ ($\lambda \in \bar{\Delta}(A)$), where $\bar{\Delta}(A)$ is a basis of $\Phi(A)$ and ε_λ is as defined in (6.6) (see also (8.11)). But since both pairs are standard we must have $m^+(\lambda, \sigma\theta \operatorname{Int}(\varepsilon)) \geqslant m^-(\lambda, \sigma\theta \operatorname{Int}(\varepsilon))$ and $m^+(\lambda, \sigma\theta) \geqslant m^-(\lambda, \sigma\theta)$ for all $\lambda \in \bar{\Delta}(A)$. It follows that ε is a product of those ε_λ for which $m^+(\lambda, \sigma\theta) = m^-(\lambda, \sigma\theta)$ (see also (6.4)). Thus, in order to show that the standard pair (σ, θ) is unique up to isomorphism we need to show that $(\sigma, \theta \operatorname{Int}(\varepsilon_\lambda))$ and (σ, θ) are isomorphic if $m^+(\lambda, \sigma\theta) = m^-(\lambda, \sigma\theta)$. This will be proved in (8.14).

6.15. COROLLARY. *Let (σ, θ) be a standad pair. Then any $w \in W(A)$ has a representative in $(G_\sigma \cap G_\theta)^0$.*

6.16. COROLLARY. *Let (σ, θ) be a pair of commuting involutorial automorphisms of G (not necessarily standard) and A a maximal (σ, θ)-split torus of G. Then any $w \in W(A)$ has a representative in $N_{G^0_\sigma}(A)$ as well as in $N_{G^0_\theta}(A)$.*

Proof. By (6.7) there exists an $\varepsilon \in A$, $\varepsilon^2 = e$ such that $(\sigma, \theta \operatorname{Int}(\varepsilon))$ is a standard pair. Since $(\sigma, \theta \operatorname{Int}(\varepsilon))$ is isomorphic to $(\sigma \operatorname{Int}(\varepsilon), \theta)$ (see (5.21(i))) the result follows from (6.15).

6.17. If (σ, θ) is a pair of commuting involutorial automorphisms of G, normally related to a maximal torus T, then by (1.6) both $\bar{\Phi}_\theta$ and $\bar{\Phi}_\sigma$ are root systems with Weyl groups $\bar{W}_\theta = W(T^-_\theta)$ and $\bar{W}_\sigma = W(T^-_\sigma)$, respectively. Now if $A = T^-_{\sigma,\theta}$, then we can see $\Phi(A) = \bar{\Phi}_{\sigma,\theta}$ also as the set of restricted roots of $\bar{\Phi}_\theta$ with respect to σ (or of $\bar{\Phi}_\sigma$ with respect to θ). Now (6.16) implies that we can choose representatives in $W(T)$, commuting with θ (resp. σ). So together with (2.7) we have obtained:

6.18. PROPOSITION, *Let $(\sigma, \theta) \in \mathscr{F}$ be normally related to T and identify $W(T^-_{\sigma,\theta})$, $W(T^-_\theta)$ and $W(T^-_\sigma)$ with $\bar{W}_{\sigma,\theta}$, $\bar{W}_\theta$, and $\bar{W}_\sigma$, respectively. Then $\bar{W}_{\sigma,\theta} \cong \bar{W}^\sigma_\theta \cong \bar{W}^\theta_\sigma$, where $\bar{W}^\sigma_\theta$ and $\bar{W}^\theta_\sigma$ are as defined in* (2.7).

7. CLASSIFICATION OF ADMISSIBLE PAIRS OF COMMUTING INVOLUTIONS

In this section we shall classify the isomorphism of admissible pairs of commuting involutions. To do this we shall first show that this classification can be obtained from the classification of single admissible involutions (see Section 4), by use of a simple (combinatorial) condition on a (σ, θ)-basis of Φ. Moreover, the pair of isomorphism classes (σ, θ) and (θ, σ) can be represented by a diagram.

We fix a maximal torus T of G and write Φ for $\Phi(T)$, X for $X^*(T)$ and W for $W(T)$. Let $(\sigma, \theta) \in \operatorname{Aut}(X, \Phi)$ be a pair of commuting involutions.

7.1. *A strong (σ, θ)-order on Φ.*

DEFINITION. A (σ, θ)-order $\succ$ on Φ is called a *strong (σ, θ)-order* if it is simultaneously a σ- and θ-order of Φ. A basis of Φ with respect to a strong (σ, θ)-order will be called a *strong (σ, θ)-basis.*

A strong (σ, θ)-order does not always exist. Another way to characterize such an order is given in the following result:

7.2. PROPOSITION. *Let (σ, θ) be a pair of commuting involutions of (X, Φ). The following are equivalent*:

(1) *Φ has a strong (σ, θ)-order*;

(2) *$\Phi_0(\sigma, \theta) = \Phi_0(\sigma) \cup \Phi_0(\theta)$*;

(3) *for each irreducible component Φ_1 of $\Phi_0(\sigma, \theta)$ we have $\sigma \mid \Phi_1 = \mathrm{id}$ or $\theta \mid \Phi_1 = \mathrm{id}$.*

Proof. (2) $\Rightarrow$ (1) is obvious, namely if $\Phi_0(\sigma, \theta)$ satisfies this condition, then $\Phi_0(\sigma, \theta)$ has a strong (σ, θ)-order, which we can extend to a strong (σ, θ)-order on Φ by choosing an arbitrary order on $\bar{\Phi}_{\sigma,\theta}$. Assume $\succ$ is a strong (σ, θ)-order on Φ and let Φ^+ be the set of positive roots with respect to this order. Now the induced order on $\Phi_0(\sigma, \theta)$ is also a strong (σ, θ)-order of $\Phi_0(\sigma, \theta)$. Suppose that there is $\alpha \in \Phi^+ \cap \Phi_0(\sigma, \theta)$ such that $\sigma(\alpha) \neq \alpha \neq \theta(\alpha)$. Then $\alpha \succ 0$, $-\sigma(\alpha) \succ 0$, $-\theta(\alpha) \succ 0$, $\sigma\theta(\alpha) \succ 0$. Hence $0 = \alpha - \sigma(\alpha) - \theta(\alpha) + \sigma\theta(\alpha) \succ 0$, a contradiction.

The equivalence of (2) and (3) is proved in the same way as in (5.9).

7.3. *Remark.* If (σ, θ) is an admissible pair of commuting involutions, then it follows from (5.9) that Φ has a strong (σ, θ)-basis. These involutions satisfy even a stronger condition, as follows from the next results:

7.4. LEMMA. *Let (σ, θ) be an admissible pair of commuting involutions of (X, Φ) and Δ a strong (σ, θ)-basis of Φ. Write $\theta = -\theta^* w_0(\theta)$ as in* (2.8). *Then $\Phi_0(\sigma) \cap \Phi_0(\theta)$ is invariant under $w_0(\theta)$.*

Proof. Since $\Phi_0(\theta)$ is σ-stable and σ is admissible it follows by (3.12) that $\Phi_0(\theta)$ is σ-normal. The result follows now from Lemma 2.19 and Remark 2.9.

7.5. LEMMA. *Let (σ, θ), Δ be as in* (7.4). *Then σ, $w_0(\theta)$ and θ^* commute.*

Proof. Since θ^* and $w_0(\theta)$ commute it suffices to show that σ and $\omega_0(\theta)$ commute. For this we show that $\sigma w_0(\theta)\, \sigma(\Phi_0(\theta)^+) = \Phi_0(\theta)^-$. Let $\alpha \in \Phi_0(\theta)^+$. If $\alpha \in \Phi_0(\theta) \cap \Phi_0(\sigma)$ then $\sigma w_0(\theta)\, \sigma(\alpha) = \sigma(w_0(\theta)(\alpha)) = w_0(\theta)(\alpha) \in \Phi_0(\theta)^-$.

If $\alpha \in \Phi_0(\theta) - (\Phi_0(\theta) \cap \Phi_0(\sigma))$, then $\sigma(\alpha) \in \Phi_0(\theta)^-$, $\sigma(\alpha) \notin \Phi_0(\theta) \cap \Phi_0(\sigma)$. On the other hand, by (7.4) we also have $w_0(\theta)\, \sigma(\alpha) \in \Phi_0(\theta)^+$, $w_0(\theta)\, \sigma(\alpha) \notin \Phi_0(\theta) \cap \Phi_0(\sigma)$, which implies $\sigma w_0(\theta)\, \sigma(\alpha) \in \Phi_0(\theta)^-$. It follows that σ and $w_0(\theta)$ commute, hence we are done.

7.6. LEMMA. *Let (σ, θ), Δ be as in* (7.4). *Then $w_0(\sigma)$ and θ^* commute.*

Proof. Since, by (7.5), $\Phi_0(\sigma)$ is θ^*-stable, we have $\theta^* w_0(\sigma)\, \theta^*(\Phi_0(\sigma)^+) = \Phi_0(\sigma)^-$, hence $\theta^* w_0(\sigma)\, \theta^* = w_0(\sigma)$.

Summarizing (7.4), (7.5), (7.6) we have obtained the following result:

7.7. THEOREM. *Let (σ, θ) be an admissible pair of commuting involutions of (X, Φ) and Δ a strong (σ, θ)-basis of Φ. Then $w_0(\theta)$, $w_0(\sigma)$, θ^* and σ^* mutually commute.*

7.8. *Remark.* Note that for the proof of this result it is only needed that Φ has a strong (σ, θ)-order and that Φ is both σ- and θ-normal. Under these conditions it is also possible to prove that $\bar{\Phi}_{\sigma, \theta}$ is a root system with Weyl group $\bar{W}_{\sigma, \theta}$.

7.9. DEFINITION. A pair of commuting involutions (σ, θ) of (X, Φ) is called *basic* if Φ has a strong (σ, θ)-basis Δ for which $w_0(\theta)$, $w_0(\sigma)$, σ^* and θ^* mutually commute.

These basic pairs of commuting involutions suffice to obtain all the admissible pairs of commuting involutions of (X, Φ). We still need a characterization of the roots in Δ lying above a restricted root in $\bar{\Delta}_{\sigma,\theta}$.

7.10. LEMMA. *Let (σ, θ) be a basic pair of commuting involutions of (X, Φ) and let Δ be a strong (σ, θ)-basis of Φ. If $\alpha, \beta \in \Delta$ such that $\alpha \neq \beta$ and $\pi(\alpha) = \pi(\beta) \neq 0$, then α equals $\theta^*(\beta)$ or $\sigma^*(\beta)$ or $\sigma^*\theta^*(\beta)$.*

Proof. Let V be the subspace of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by Φ (see (2.1)). Arguing as in (2.10) we obtain

$$\alpha + \theta^*(\alpha) + \sigma^*(\alpha) + \sigma^*\theta^*(\alpha) = \beta + \theta^*(\beta) + \sigma^*(\beta) + \sigma^*\theta^*(\beta) + \delta$$

with $\delta \in \operatorname{Span} \Delta_0(\sigma, \theta)$. From this we deduce, as in (2.10), that $\delta = 0$ and α equals $\theta^*(\beta)$ or $\sigma^*(\beta)$ or $\sigma^*\theta^*(\beta)$.

7.11. THEOREM. *Let (σ, θ) be a pair of commuting involutions of (X, Φ). Then (σ, θ) is admissible if and only if (σ, θ) is basic and both σ and θ are admissible.*

Proof. This result is proved by using more or less the same arguments as in (3.7). If (σ, θ) is admissible, then both σ and θ are admissible involutions and also (σ, θ) is basic by (7.7). So it suffices to show the "if" statement.

Assume (σ, θ) is basic and σ, θ are admissible involutions of (X, Φ). Let $\{x_\alpha\}_{\alpha \in \Phi(T)}$ be a realization of Φ in G as in (3.1) and let $\tilde{\sigma}, \tilde{\theta} \in \operatorname{Aut}(G, T)$ be involutions inducing σ resp. θ on (X, Φ). Since both $\overline{\sigma\theta}$ and $\overline{\theta\sigma}$ induce $\sigma\theta$ on (X, Φ) it follows from the isomorphism theorem (see Springer [24, 11.4.3]) that there is a $t \in T$ such that $\overline{\sigma\theta} = \overline{\theta\sigma} \operatorname{In}(t)$. If $\alpha \in \Phi_0(\sigma, \theta)$ then, since $\Phi_0(\sigma, \theta) = \Phi_0(\theta) \cup \Phi_0(\sigma)$, we have by (3.4) $c_{\alpha,\tilde{\theta}} = c_{\sigma(\alpha),\tilde{\theta}} = 1$ or $c_{\alpha,\tilde{\sigma}} = c_{\theta(\alpha),\tilde{\sigma}} = 1$. But then

$$c_{\alpha,\tilde{\theta}} c_{\theta(\alpha),\tilde{\sigma}} = c_{\alpha,\tilde{\sigma}} c_{\sigma(\alpha),\tilde{\theta}},$$

which implies $\alpha(t) = 1$.

Let Δ be a strong (σ, θ)-basis of Φ and write $\sigma = -\sigma^* w_0(\sigma)$, $\theta = -\theta^* w_0(\theta)$ with respect to Δ (see (2.8)). Since $\tilde{\theta}\tilde{\sigma}\tilde{\theta} = \tilde{\sigma} \operatorname{Int}(t)$ is an involution, we get $\operatorname{Int}(\sigma(t)t) = \mathrm{id}$, hence $\sigma(t)t \in Z(G)$. Similarly we get $\theta(t)t \in Z(G)$. It follows now from (3.5) that for any $\alpha \in \Phi$ we have

$$\alpha(t) = \theta^*(\alpha)(t) = \sigma^*(\alpha)(t) = \sigma^*\theta^*(\alpha)(t).$$

If $\gamma \in \bar{\Delta}_{\sigma,\theta}$ and $\alpha, \beta \in \Delta$, $\alpha \neq \beta$ such that $\pi(\alpha) = \pi(\beta) = \gamma$, then it follows from (7.10) that $\beta = \sigma^*(\alpha)$ or $\theta^*(\alpha)$ or $\sigma^*\theta^*(\alpha)$. Similarly as in (3.7), now take, for each $\gamma \in \bar{\Delta}_{\sigma,\theta}$, an $\alpha \in \Delta$ such that $\gamma = \pi(\alpha)$ and choose $u_\gamma \in T^-_{\sigma,\theta}$ such that $\lambda(u_\gamma) = 1$ for $\lambda \in \bar{\Delta}_\theta$, $\lambda \neq \gamma$ and $\gamma(u_\gamma^4) = \alpha(t)$. Take $u = \prod_{\gamma \in \bar{\Delta}_{\sigma,\theta}} u_\gamma$. Then by

(7.10) and (3.5) we find $\alpha(tu^4)=1$ for all $\alpha \in \Delta$. So $tu^4 \in Z(G)$ and it follows that $\mathrm{Int}(u)^{-1}\,\bar{\sigma}\,\mathrm{Int}(u)\bar{\theta}=\bar{\theta}\,\mathrm{Int}(u)^{-1}\,\bar{\sigma}\,\mathrm{Int}(u)$.

It remains to show that $T^{-}_{\sigma,\theta}$ is a maximal (σ,θ)-split torus of G. This follows immediately from the fact that Φ has a strong (σ,θ)-order. Namely $\Phi_0(\sigma,\theta)$ is the root system of $Z_G(T^{-}_{\sigma,\theta})$ and from (7.2.(3)) it follows that $Z_G(T^{-}_{\sigma,\theta})/T^{-}_{\sigma,\theta}$ contains no non-trivial (σ,θ)-split torus, what proves the result.

7.12. *Related involutions of* (X,Φ). Whether two involutions σ and θ of (X,Φ) are basic or not can be detected from their indices. To show this we need besides conditions to assure that σ^*, θ^*, $w_0(\theta)$ and $w_0(\sigma)$ commute, also an order on Φ, which is simultaneously a σ- and θ-order.

Definition. Two involutions σ, θ of (X,Φ) (not necessarily commuting) are said to be *related* if Φ has a basis Δ, which is simultaneously a σ- and θ-basis of Φ. In this case Δ is called the *relating* basis of Φ (relative to (σ,θ)).

Note that if σ, θ are related involutions, for which σ^*, θ^*, $w_0(\theta)$ and $w_0(\sigma)$ commute, then also σ and θ commute, so (σ,θ) is basic. Analogously to (2.11) we can define an index for a related pair of involutions of (X,Φ):

7.13. (σ,θ)-*indices*. Assume that X is semisimple.

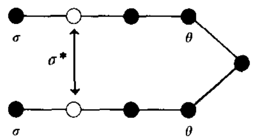
For a pair of related involutions (σ,θ) of (X,Φ) and a relating basis Δ of Φ, call the sextuple $(X,\Delta,\Delta_0(\sigma),\Delta_0(\theta),\sigma^*(\Delta),\theta^*(\Delta))$ an *index* of (σ,θ) (or (σ,θ)-index). This (σ,θ)-index determines both σ and θ. If (σ,θ) is basic (resp. admissible) then we call this also a *basic* (resp. *admissible*) (σ,θ)-index. Two indices $(X,\Delta,\Delta_0(\sigma_1),\Delta_0(\theta_1),\sigma_1^*(\Delta),\theta_1^*(\Delta))$ and $(X,\Delta',\Delta_0'(\sigma_2),\Delta_0'(\theta_2),\sigma_2^*(\Delta'),\theta_2^*(\Delta'))$ are said to be *isomorphic* if there is a $w\in W(\Phi)$, which maps $(X,\Delta,\Delta_0(\sigma_1),\Delta_0(\theta_1))$ onto $(X,\Delta',\Delta_0'(\sigma_2),\Delta_0'(\theta_2))$ and which satisfies

$$w\theta_1^*(\Delta)\,w^{-1}=\theta_2^*(\Delta') \qquad \text{and} \qquad w\sigma_1^*(\Delta)\,w^{-1}=\sigma_2^*(\Delta').$$

7.14. *Remarks*. (1) The above index of (σ,θ) determines the indices of both σ and θ and vice versa. When σ and θ commute, then this definition of (σ,θ)-index is an extension of the definition of the Satake diagram corresponding to the action of $\Gamma=\{\mathrm{id},-\sigma,-\theta,\sigma\theta\}$ on (X,Φ). In our situation we have additional actions of σ and θ on $\Phi_0(\sigma,\theta)$.

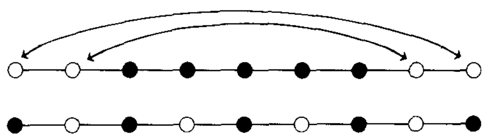
(2) We can make a diagrammatic representation of the (σ,θ)-index by colouring black those vertices of the ordinary Dynkin diagram of Φ, which represent roots in $\Delta_0(\sigma)\cup\Delta_0(\theta)$, and by giving the vertices of $\Delta_0(\sigma)\cup\Delta_0(\theta)$ which are not in $\Delta_0(\sigma)\cap\Delta_0(\theta)$ a label σ or θ if $\sigma(\alpha)\neq\alpha$ or $\theta(\alpha)\neq\alpha$, respectively. The actions of σ^* and θ^* are indicated by arrows. As in (2.12) we again omit the actions of σ^*, θ^* on $X_0(\sigma)$, $X_0(\theta)$, respectively.

Here is an example with Φ of type A_9:



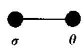
This (σ, θ)-index is obtained by gluing together the indices

σ^*



of σ resp. θ with the above recipe. Note that such a diagram represents the indices of both (σ, θ) and (θ, σ).

(3) If σ, θ are related involutions of (X, Φ), then they need not commute. One can easily see this in the following example of a (σ, θ)-index, where Φ is of type A_2:



From (7.2) we see that σ and θ cannot commute.

(4) An index of (σ, θ) may depend again on the choice of the (σ, θ)-basis of Φ. Similarly to (2.13) we can prove:

7.15. Proposition. *Assume X is semisimple and let (σ, θ) be an admissible pair of commuting involutions of (X, Φ). Let Δ, Δ' be strong (σ, θ)-bases of Φ. Then $(X, \Delta, \Delta_0(\sigma), \Delta_0(\theta), \sigma^*(\Delta), \theta^*(\Delta))$ and $(X, \Delta', \Delta'_0(\sigma), \Delta'_0(\theta), \sigma^*(\Delta'), \theta^*(\Delta'))$ are isomorphic. In particular there is a bijection between the W-isomorphism classes of admissible pairs of commuting involutions of (X, Φ) and the isomorphism classes of indices of basic pairs of admissible involutions of (X, Φ).*

Proof. Since $\bar{W}^{\sigma}_{\theta}$ corresponds to the Weyl group of $\bar{\Phi}_{\sigma,\theta}$ (see (6.18)), there is by (2.5) a unique element $w \in W^{\theta}_1(\sigma, \theta)$ such that $w(\Delta) = \Delta'$. Since $w \in W^{\theta}_1(\sigma, \theta)$ we have $w(\Delta_0(\sigma, \theta)) = \Delta'_0(\sigma, \theta)$ and $w(\Delta_0(\theta)) = \Delta'_0(\theta)$. But by (7.2) $\Delta_0(\sigma, \theta) = \Delta_0(\sigma) \cup \Delta_0(\theta)$, so w maps $(X, \Delta, \Delta_0(\sigma), \Delta_0(\theta))$ onto $(X, \Delta', \Delta'_0(\sigma), \Delta'_0(\theta))$.

Similarly, as in the proof of (2.13), one shows now that w satisfies $w\theta^*(\Delta)\, w^{-1} = \theta^*(\Delta')$ and $w\sigma^*(\Delta)\, w^{-1} = \sigma^*(\Delta')$, which proves the first statement. The second statement follows immediately from this and Theorem 7.11.

Whether two related involutions of (X, Φ) are basic or not can be detected now directly from their (σ, θ)-index:

7.16. THEOREM. *Let σ, θ be related involutions of (X, Φ) and Δ a relating basis of Φ with respect to (σ, θ). Then (σ, θ) is basic if and only if*

(1) *σ^* and θ^* commute,*

(2) *$\Delta_0(\theta)$ is σ^*-stable and $\Delta_0(\sigma)$ is θ^*-stable,*

(3) *for every connected component Δ_1 of $\Delta_0(\theta) \cup \Delta_0(\sigma)$ we have $\Delta_1 \subset \Delta_0(\sigma)$ or $\Delta_1 \subset \Delta_0(\theta)$.*

Proof. If (σ, θ) is basic, then (1) and (2) are clear and (3) follows from (7.2), using the same arguments as in (5.9). So assume (1), (2), and (3) hold.

Then (2) implies that $w_0(\theta)$ and σ^* (resp. $w_0(\sigma)$ and θ^*) commute, because $\sigma^* w_0(\theta)\, \sigma^*(\Delta_0(\theta)) = -\Delta_0(\theta)$ (resp. $\theta^* w_0(\sigma)\ \theta^*(\Delta_0(\sigma)) = -\Delta_0(\sigma)$). So it suffices to show that $w_0(\sigma)$ and $w_0(\theta)$ commute or equivalently: $w_0(\sigma)\, w_0(\theta)\, w_0(\sigma)(\Delta_0(\theta)) = -\Delta_0(\theta)$.

If $\alpha \in \Delta_0(\sigma) \cap \Delta_0(\theta)$, then $w_0(\sigma)(\alpha) = -\sigma^*(\alpha) \in -(\Delta_0(\sigma) \cap \Delta_0(\theta))$ by (2). Similarly, since $\Delta_0(\sigma)$ is θ^*-stable, we have $w_0(\theta)\, w_0(\sigma)(\alpha) = -\theta^* w_0(\sigma)(\alpha) \in \Delta_0(\sigma) \cap \Delta_0(\theta)$. Hence $w_0(\sigma)\, w_0(\theta)\, w_0(\sigma)(\alpha) \in -(\Delta_0(\sigma) \cap \Delta_0(\theta))$.

If $\alpha \in \Delta_0(\theta) - (\Delta_0(\sigma) \cap \Delta_0(\theta))$, then let $\Delta_1 \subset \Delta_0(\sigma) \cup \Delta_0(\theta)$ be the connected component such that $\alpha \in \Delta_1$. By (3) we have $\Delta_1 \subset \Delta_0(\theta)$ and $w_0(\sigma)(\alpha) = \alpha + \sum_{\beta \in \Delta_0(\sigma) \cap \Delta_1} m_\beta \beta$ with $m_\beta \in \mathbb{Z}$, $m_\beta \geqslant 0$. Since, by (2), we have

$$-\theta^*(\alpha) = w_0(\theta)(\alpha) \in -(\Delta_0(\theta) - (\Delta_0(\sigma) \cap \Delta_0(\theta))),$$

it follows that $w_0(\sigma)\, w_0(\theta)\, w_0(\sigma)(\alpha) \in \Phi_0(\theta)^-$, which proves the result.

With the above result and (7.15) it becomes an easy exercise to obtain all the indices of basic pairs of admissible involutions of (X, Φ). Before we describe them, we need again a notion of irreducibility.

7.17. DEFINITION. A (σ, θ)-index $S = (X, \Delta, \Delta_0(\sigma), \Delta_0(\theta), \sigma^*, \theta^*)$ is called *irreducible* if Δ is not the union of two mutually orthogonal σ^*- and θ^*-stable non-empty subsets Δ_1, Δ_2. S is called *absolutely irreducible* if Δ is connected.

Note that S is irreducible if and only if $\bar{\Delta}_{\sigma,\theta}$ is connected.

7.18. *Classification of irreducible admissible (σ, θ)-indices.* Assume that X is semisimple and of adjoint type. Let (σ, θ) be a basic pair of non-trivial admissible involutions of (X, Φ) with (σ, θ)-index $S=(X, \Delta, \Delta_0(\sigma), \Delta_0(\theta), \sigma^*, \theta^*)$. We assume that S is irreducible. The standard pair in $\mathscr{F}(\sigma, \theta)$ (see (6.11)) will also be denoted by (σ, θ). We shall use the Cartan notation to describe involutions, whose index is absolutely irreducible (see Table II).

If S is absolutely irreducible, then we denote the pair (σ, θ) by $X_l^{p,q}$ (type σ, type θ), where X denotes the type of Φ, i.e., one of $A, B, ..., G$ and $l=\operatorname{rank} \Phi$, $p=\operatorname{rank} \bar{\Delta}_\sigma$, $q=\operatorname{rank} \bar{\Delta}_\theta$. For example, $A_{2l-1}^{2l-1,l}(\mathrm{I}, \mathrm{III}_b)$ means that Φ is of type A_{2l-1}, σ is of type AI, θ is of type AIII_b and $\operatorname{rank} \bar{\Delta}_\sigma=2l-1$, $\operatorname{rank} \bar{\Delta}_\theta=l$. We shall use the same notation for the isomorphism class of the standard pair within a family $\mathscr{F}(\sigma, \theta)$. To describe the other isomorphism classes in $\mathscr{F}(\sigma, \theta)$ we add the representing quadratic element in $T_{\sigma,\theta}^-$. So if we write $\varepsilon_1, ..., \varepsilon_p$ for the quadratic elements in $T_{\sigma,\theta}^-$ with respect to $\bar{\Delta}_{\sigma,\theta}=\{\lambda_1, ..., \lambda_p\}$ (see (6.5)), then in the above example $A_{2l-1}^{2l-1,l}(\mathrm{I}, \mathrm{III}_b)$ denotes the standard pair (σ, θ) in $\mathscr{F}(\sigma, \theta)$ and $A_{2l-1}^{2l-1,l}(\mathrm{I}, \mathrm{III}_b, \varepsilon_i)$ denotes the pair $(\sigma, \theta \operatorname{Int}(\varepsilon_i))$ $(i=1, ..., p)$. For a classification of these quadratic elements, see Section 8. We denote the pairs $(\theta, \theta \operatorname{Int}(\varepsilon_i))$ by X_l^p (type θ, ε_i).

To make identification with Berger's classification of affine symmetric spaces, it is sometimes useful to take $\varepsilon_0=e$ and to denote the standard pair by $(\sigma, \theta \operatorname{Int}(\varepsilon_0))$ (see Table II). In the classification of admissible irreducible (σ, θ)-indices with both σ and θ non-trivial, we have six cases:

7.18.1. *Φ is irreducible and $\sigma=\theta$.* In this case, the (σ, θ)-index equals the index of θ (and σ). If $F=\mathbb{C}$, then the standard pair corresponds to the complexification of a Riemannian symmetric pair and the quadratic elements give the K_ε-spaces as described in Oshima and Sekiguchi [18]. See also Section 10. As for the signatures of the standard pair, we note that $m^-(\lambda, \sigma\theta)=0$ for all $\lambda \in \bar{\Delta}_\theta$, so we have $m^+(\lambda, \sigma\theta)=m(\lambda)$.

In Table II we list the (θ, θ)-index, the diagram of $\bar{\Delta}_\theta$, the multiplicities of the restricted roots in $\bar{\Delta}_\theta$, and the quadratic elements in $T_{\sigma,\theta}^-$ representing the classes in $\mathscr{C}(\sigma, \theta)$ (see Section 8). We have added also some information to identify these pairs with Berger's classification [2]. This will be explained in Section 10.

7.18.2. *Φ is irreducible and $\sigma \neq \theta$.* The diagrams representing the indices of (σ, θ) and (θ, σ) are listed in Table IV. We also give the type of $(\sigma, \theta \operatorname{Int}(\varepsilon_i))$ as explained above, the diagram of $\bar{\Delta}_{\sigma,\theta}$ together with the signatures of the standard pair and the quadratic elements in $T_{\sigma,\theta}^-$ representing the classes in $\mathscr{C}(\sigma, \theta)$ (see Section 8).

7.18.3. *$\Phi=\Phi_1 \amalg \Phi_2$ with Φ_1, Φ_2 irreducible, $\sigma=\theta$ and $\sigma(\Phi_1)=\Phi_2$.* In this case the (σ, θ)-index equals again the indices of θ (and σ). Here σ^* and θ^* exchange the Dynkin diagrams of Φ_1 and Φ_2. We denote (σ, θ)

by $(X_l \times X_l)$, where X_l denotes the type of Φ_1 (i.e., one of $A, ..., G$). If $F = \mathbb{C}$ these pairs correspond to the symmetric pairs $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g})$, where $\mathfrak{g}$ is a real semisimple Lie algebra of inner type (i.e., $\mathfrak{g}$ contains a compact Cartan subalgebra) and $\mathfrak{g}_{\mathbb{C}}$ its complexification.

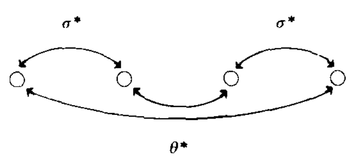
In Table III we give the type of $(\theta, \theta \operatorname{Int}(\varepsilon_i))$, the diagram of $\bar{\Delta}_\theta$, the multiplicities of the roots in $\bar{\Delta}_\theta$ and the quadratic elements representing a class in $\mathscr{C}(\sigma, \theta)$.

7.18.4. *$\Phi = \Phi_1 \amalg \Phi_2$ with Φ_1, Φ_2 irreducible, $\sigma(\Phi_1) = \Phi_2$, $\theta(\Phi_1) = \Phi_2$, $\sigma^* \neq \theta^*$.* Since both $\sigma = -\sigma^*$ and $\theta = -\theta^*$, this can only occur if $\operatorname{Aut}(\Phi_i)$ $(i = 1, 2)$ contains a non-trivial diagram automorphism of order 2; i.e., Φ_i $(i = 1, 2)$ is one of A_l $(l \geqslant 2)$, D_l $(l \geqslant 4)$ or E_6. In this case (σ, θ) and (θ, σ) are isomorphic and we denote (σ, θ) by $({}^2X_l \times {}^2X_l)$, where ${}^2X_l$ denotes the twisted Dynkin diagram of Φ of type X_l. If $F = \mathbb{C}$ these pairs correspond to the symmetric pairs $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g})$, where $\mathfrak{g}$ is of outer type.

In Table V we list the type of (σ, θ), the (σ, θ)-index, the diagram of $\bar{\Delta}_{\sigma,\theta}$ together with the signatures of the standard pair, and the quadratic elements representing the isomorphism classes in $\mathscr{C}(\sigma, \theta)$.

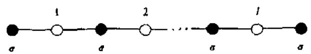
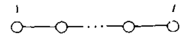
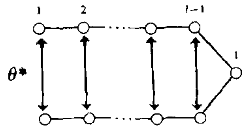
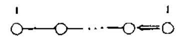
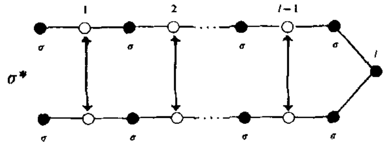
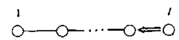
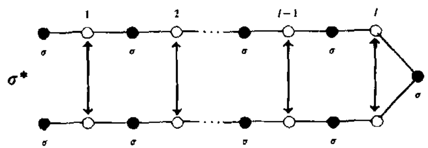
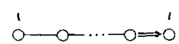
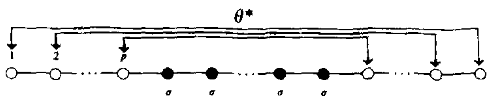
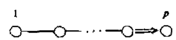
7.18.5. *$\Phi = \Phi_1 \amalg \Phi_2$ with Φ_1, Φ_2 irreducible, $\sigma(\Phi_1) = \Phi_2$, $\theta(\Phi_i) = \Phi_i$ $(i = 1, 2)$.* The diagram representing the indices of (σ, θ) and (θ, σ) is a double copy of the index of $\theta \mid \Phi_1$ and the action of σ^* is described by arrows connecting both diagrams. Moreover, $\bar{\Phi}_{\sigma,\theta} \cong \bar{\Phi}_{\theta \mid \Phi_1}$ and for $\lambda \in \bar{\Phi}_{\sigma,\theta}$ we have $m^+(\lambda, \sigma\theta) = m^-(\lambda, \sigma\theta)$, which equals again the multiplicity of the corresponding root in $\bar{\Phi}_{\theta \mid \Phi_1}$. All pairs in $\mathscr{F}(\sigma, \theta)$ are isomorphic (see Section 8). In Berger these pairs are denoted by $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$ and $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g})$, where g is a real semisimple Lie algebra and $\mathfrak{k}$ a maximal compact subalgebra of $\mathfrak{g}$. The pairs $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$ are associated to the ones in (7.18.3) and (7.18.4) (see also (10.4)).

7.18.6. *$\Phi = \Phi_1 \amalg \Phi_2 \amalg \Phi_3 \amalg \Phi_4$ with Φ_i $(i = 1, 2, 3, 4)$ irreducible, $\sigma(\Phi_1) = \Phi_2$, $\sigma(\Phi_3) = \Phi_4$, $\theta(\Phi_1) = \Phi_4$, $\theta(\Phi_2) = \Phi_3$.* The diagram representing the index of (σ, θ) and (θ, σ) consists of four copies of the Dynkin diagram of Φ_1 and the actions of σ^* and θ^* are described by arrows. Here is an example of Φ_1 of type A_1:



In this case is $\bar{\Phi}_{\sigma,\theta}$ isomorphic to Φ_1 and $m^+(\lambda, \sigma\theta) = m^-(\lambda, \sigma\theta) = 2$ for all $\lambda \in \bar{\Phi}_{\sigma,\theta}$. All pairs in $\mathscr{F}(\sigma, \theta)$ are isomorphic (see Section 8).

TABLE IV

Type$(\sigma, \theta\,\mathrm{Int}(\varepsilon_i))$	(σ, θ)-index	$\bar{\Delta}_{\sigma,\theta}$	$m^+(\lambda, \sigma\theta)$ $m^-(\lambda, \sigma\theta)$	$m^+(2\lambda, \sigma\theta)$ $m^-(2\lambda, \sigma\theta)$	Quadratic elements
$A_{2l+1}^{2l+1,l}(\mathrm{I, II})$ $(l \geq 1)$			2 0 2 0		
$A_{2l-1}^{2l-1,l}(\mathrm{I, III}_b, \varepsilon_i)$ $(l \geq 2)$			1 0 1 0 $(i<l)$	1 0 0 0 $(i=l)$	ε_l
$A_{4l-1}^{2l,2l-1}(\mathrm{III}_b, \mathrm{II}, \varepsilon_i)$ $(l \geq 1)$			4 0 4 0 $(i<l)$	3 0 1 0 $(i=l)$	ε_l
$A_{4l+1}^{2l+1,2l}(\mathrm{III}_b, \mathrm{II})$ $(l \geq 1)$			4 0 4 0 $(i<l)$	4 1 4 3 $(i=l)$	
$A_l^{l,p}(\mathrm{I, III}_a)$ $(1 \leq 2p \leq l+1)$			1 0 1 0 $(i<p)$	$(l-2p)$ 0 $(l-2p)$ 1 $(i=p)$	

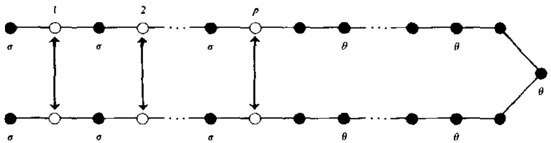
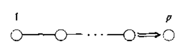
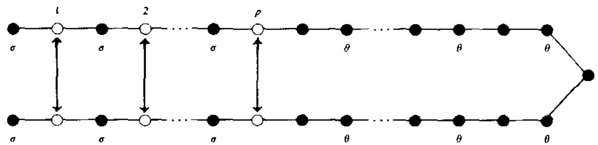
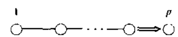
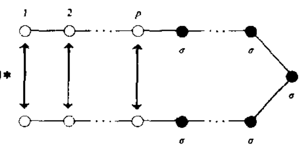
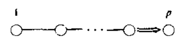
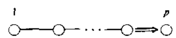
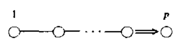
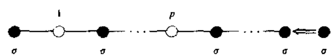
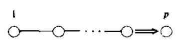
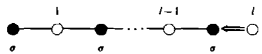
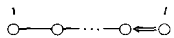
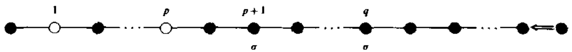
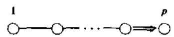
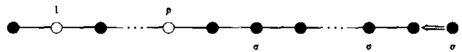
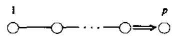
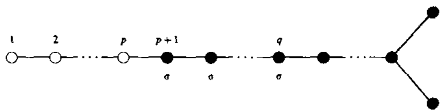
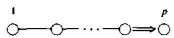
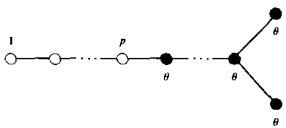
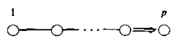
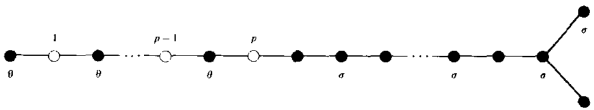
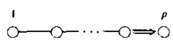
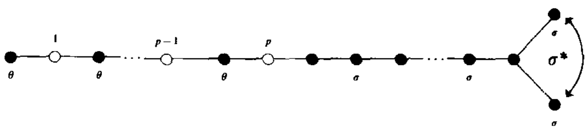
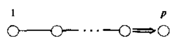
$A_{4l-1}^{2p,2l-1}(\mathrm{III}_a, \mathrm{II})$ $(1 \leqslant 2p < 4l)$	σ^* 		4 0 4 0 $(i<p)$	$4(2l-2p)$ 3 $4(2l-2p)$ 1 $(i=p)$	
$A_{4l+1}^{2p,2l}(\mathrm{III}_a, \mathrm{II})$ $(1 \leqslant 2p < 4l+2)$	σ^* 		4 0 4 0 $(i<p)$	$4(2l-2p+1)$ 3 $4(2l-2p+1)$ 1 $(i=p)$	
$A_{2l-1}^{l,p}(\mathrm{III}_b, \mathrm{III}_a, \varepsilon_i)$ $(1 \leqslant p \leqslant l)$	σ^* & θ^* 		2 0 0 0 $(i<p)$	$2(l-p)$ 1 $2(l-p)$ 0 $(i=p)$	ε_i $(1 \leqslant i \leqslant p-1)$
$A_l^{q,p}(\mathrm{III}_a, \mathrm{III}_a, \varepsilon_i)$ $(1 \leqslant p < q \leqslant \frac{1}{2}(l+1))$	σ^* & θ^* 		2 0 0 0 $(i<p)$	$2(l-q-p+1)$ 1 $2(q-p)$ 0 $(i=p)$	ε_i $(1 \leqslant i \leqslant p)$
$B_l^{q,p}(\mathrm{I}, \mathrm{I}, \varepsilon_i)$ $(1 \leqslant p < q \leqslant l)$ $(l \geqslant 2)$			1 0 0 0 $(i<p)$	$(2l-q-p+1)$ 0 $(q-p)$ 0 $(i=p)$	ε_i $(1 \leqslant i \leqslant p)$
$C_l^{l,p}(\mathrm{I}, \mathrm{III}_a)$ $(1 \leqslant p \leqslant \frac{1}{2}l)$ $(l \geqslant 3)$			2 0 2 0 $(i<p)$	$2(l-2p)$ 1 $2(l-2p)$ 2 $(i=p)$	

Table continued

TABLE IV — *Continued*

Type(σ, θ Int(ε_i))	(σ, θ)-index	$\bar{\Delta}_{\sigma,\theta}$	$m^+(\lambda, \sigma\theta)$ $m^-(\lambda, \sigma\theta)$	$m^+(2\lambda, \sigma\theta)$ $m^-(2\lambda, \sigma\theta)$	Quadratic elements
$C_{2l}^{2l,l}(\mathrm{I}, \mathrm{II}_b, \varepsilon_i)$ $(l \geqslant 2)$			2 0 2 0 $(i<p)$	2 0 1 0 $(i=l)$	
$C_l^{q,p}(\mathrm{II}_a, \mathrm{II}_a, \varepsilon_i)$ $(1 \leqslant p < q \leqslant \frac{1}{2}l)$ $(l \geqslant 3)$			4 0 0 0 $(i<p)$	$4(l-q-p)$ 3 $4(q-p)$ 0 $(i=p)$	ε_i $(1 \leqslant i \leqslant p)$
$C_{2l}^{l,p}(\mathrm{II}_b, \mathrm{II}_a, \varepsilon_i)$ $(1 \leqslant p \leqslant l)$ $(l \geqslant 2)$			4 0 0 0 $(i<p)$	$4(l-p)$ 3 $4(l-p)$ 0 $(i=p)$	ε_i $(1 \leqslant i \leqslant p-1)$
$D_l^{q,p}(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_i)$ $(1 \leqslant p < q \leqslant l-1)$ $(l \geqslant 4)$			1 0 0 0 $(i<p)$	$(2l-q-p)$ 0 $(q-p)$ 0 $(i=p)$	ε_i $(1 \leqslant i \leqslant p)$
$D_l^{p,l}(\mathrm{I}_a, \mathrm{I}_b, \varepsilon_i)$ $(1 \leqslant p \leqslant l-1)$ $(l \geqslant 4)$			1 0 0 0 $(i<p)$	$(l-p)$ 0 $(l-p)$ 0 $(i=p)$	ε_i $(1 \leqslant i \leqslant p-1)$
$D_{2l}^{l,2p}(\mathrm{III}_a, \mathrm{I}_a)$ $(1 \leqslant p \leqslant l-1)$ $(l \geqslant 2)$			2 0 2 0 $(i<p)$	$4(l-p)$ 1 $4(l-p)$ 0 $(i=p)$	
$D_{2l+1}^{l,2p}(\mathrm{III}_b, \mathrm{I}_a)$ $(1 \leqslant p \leqslant l-1)$ $(l \geqslant 2)$			2 0 2 0 $(i<p)$	$2(2l-2p+1)$ 1 $2(2l-2p+1)$ 0 $(i=p)$	

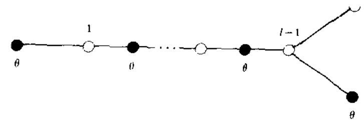
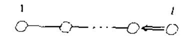
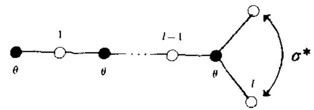
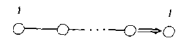
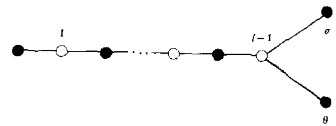
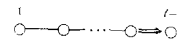
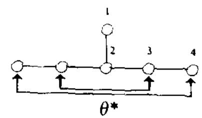
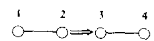
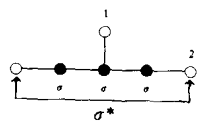
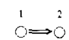
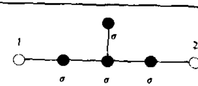
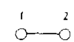
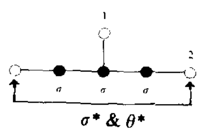
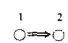
$D_{2l}^{l,2l}(\mathrm{III}_a, \mathrm{I}_b, \varepsilon_i)$ $(l \geq 2)$			2 0 2 0 $(i<l)$	1 0 0 0 $(i=l)$	ε_l
$D_{2l+1}^{l,2l+1}(\mathrm{III}_b, \mathrm{I}_b)$ $(l \geq 2)$			2 0 2 0 $(i<l)$	2 0 2 1 $(i=l)$	
$D_{2l}^{l,l}(\mathrm{III}_a, \mathrm{III}_a, \varepsilon_i)$ $(l \geq 2)$			4 0 0 0 $(i<l-1)$	4 1 4 0 $(i=l-1)$	ε_i $(1 \leq 2i \leq l-1)$
$E_6^{6,4}(\mathrm{I}, \mathrm{II}, \varepsilon_i)$			1 0 0 0 $(i=1,2)$	1 0 1 0 $(i=3,4)$	ε_1
$E_6^{6,2}(\mathrm{I}; \mathrm{III})$			3 0 3 0 $(i=1)$	4 0 4 1 $(i=2)$	
$E_6^{6,2}(\mathrm{I}, \mathrm{IV})$			4 0 4 0		
$E_6^{4,2}(\mathrm{II}, \mathrm{III}, \varepsilon_i)$			4 0 2 0 $(i=1)$	4 1 4 0 $(i=2)$	ε_1

Table continued

TABLE IV — *Continued*

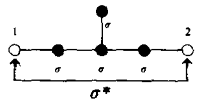
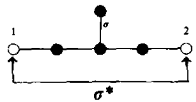
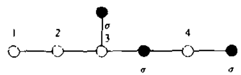
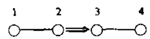
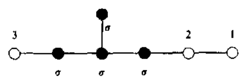
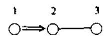
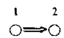
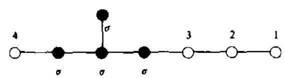
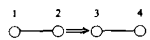
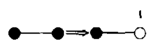
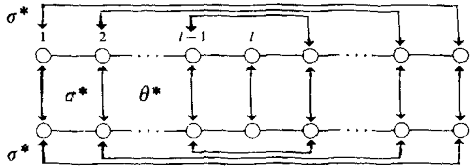
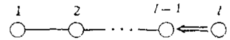
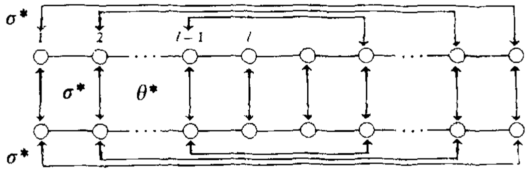
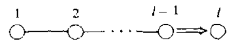
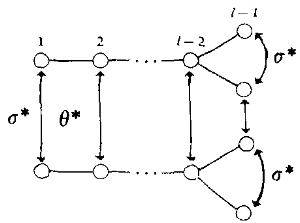
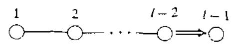
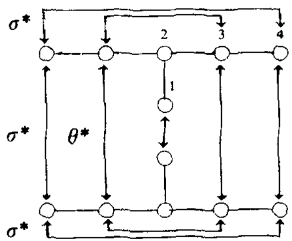
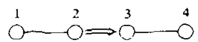
Type(σ, θ Int(ε_i))	(σ, θ)-index	$\bar{\Lambda}_{\sigma,\theta}$	$m^+(\lambda, \sigma\theta)$ $m^-(\lambda, \sigma\theta)$	$m^+(2\lambda, \sigma\theta)$ $m^-(2\lambda, \sigma\theta)$	Quadratic elements
$E_6^{4,2}$(II, IV)			8 3 8 5		
$E_6^{2,2}$(III, IV)			8 7 8 1		
$E_7^{7,4}$(V, VI, ε_i)			1 0 0 0 ($i = 1, 2$)	2 0 2 0 ($i = 3, 4$)	ε_1
$E_7^{7,3}$(V, VII, ε_i)			1 0 0 0 ($i = 1$)	4 0 4 0 ($i = 3, 4$)	ε_1
$E_7^{4,3}$(VI, VII, ε_i)			6 0 2 0 ($i = 1$)	8 1 8 0 ($i = 2$)	ε_1
$E_8^{8,4}$(VIII, IX, ε_i)			1 0 0 0 ($i = 1, 2$)	4 0 4 0 ($i = 3, 4$)	ε_1
$F_4^{4,1}$(I, II)			4 3 4 4		

TABLE V

Type$(\sigma, \theta\,\mathrm{Int}(\varepsilon_j))$	(σ, θ)-index	$\bar{\Delta}_{\sigma,\theta}$	$m^+(\lambda, \sigma\theta)$ $m^-(\lambda, \sigma\theta)$	$m^+(2\lambda, \sigma\theta)$ $m^-(2\lambda, \sigma\theta)$	Quadratic elements
$({}^2A_{2l-1} \times {}^2A_{2l-1})(\varepsilon_j)$			2 0 2 0 $(j<l)$	2 0 0 0 $(j=l)$	ε_l
$({}^2A_{2l} \times {}^2A_{2l})$			2 0 2 0 $(j<l)$	2 0 2 2 $(j=l)$	
$({}^2D_l \times {}^2D_l)(\varepsilon_j)$			2 0 0 0 $(j<l-1)$	2 0 2 0 $(j=l-1)$	ε_j $(1 \leqslant j \leqslant l-2)$
$({}^2E_6 \times {}^2E_6)(\varepsilon_j)$			2 0 0 0 $(j=1,2)$	2 0 2 0 $(j=3,4)$	ε_1

In summarizing, we have obtained the following result:

7.19. THEOREM. *Assume X is semisimple and of adjoint type. Then the irreducible indices of admissible pairs of commuting involutions (σ, θ) of (X, Φ), where $\sigma, \theta \neq \mathrm{id}$ are exhausted by the induces in* (7.18.1)–(7.18.6).

7.20. $\mathrm{Aut}(X, \Phi)$*-isomorphism classes of admissible pairs of involutions.* Some of the ordered pairs (σ, θ) are isomorphic under $\mathrm{Aut}(X, \Phi)$ to their dual (θ, σ), using the diagram automorphism. These are $D_{2l}^{l,l}(\mathrm{III}_a, \mathrm{III}_a, \varepsilon_i)$ and the pairs in (7.18.4). The group $\mathrm{Aut}(X, \Phi)$ identifies none of the pairs of involutions whose diagrams are not the same.

8. CLASSIFICATION OF THE QUADRATIC ELEMENTS REPRESENTING THE CLASSES IN $\mathscr{C}(\sigma, \theta)$

In this section we shall determine a set of quadratic elements, which represent the isomorphism classes within a set $\mathscr{C}(\sigma, \theta)$.

8.1. Let (σ, θ) be a pair of involutorial automorphisms of G, A a maximal (σ, θ)-split torus of G, and $T \supset A$ a (σ, θ)-stable maximal torus of G such that T_θ^- resp. T_σ^- is a maximal θ-split resp. σ-split torus of G (see (5.13)). We shall write Φ for $\Phi(T)$, X for $X^*(T)$ and W for $W(T)$.

For a closed subgroup H of G, we call two pairs (σ_1, θ_1) and (σ_2, θ_2) in $\mathscr{F}$ *isomorphic* under H if there exists $h \in H$ such that $\mathrm{Int}(h)\, \sigma_1\, \mathrm{Int}(h)^{-1} = \sigma_2$ and $\mathrm{Int}(h)\, \theta_1\, \mathrm{Int}(h)^{-1} = \theta_2$.

In (5.16) we showed that any pair in $\mathscr{F}(\sigma, \theta)$ is isomorphic to a pair $(\sigma, \theta\, \mathrm{Int}(a)) \in \mathscr{F}_A(\sigma, \theta)$. As for the possible isomorphisms between these pairs, we can restrict ourselves to $N_G(A)$:

8.2. LEMMA. *Two pairs $(\sigma, \theta\, \mathrm{Int}(a_1))$ and $(\sigma, \theta\, \mathrm{Int}(a_2))$ in $\mathscr{F}_A(\sigma, \theta)$ are isomorphic under G if and only if they are isomorphic under $N_G(A)$.*

Proof. It suffices to show the "only if" statement. Assume $g \in G$ such that $\mathrm{Int}(g)\, \sigma\, \mathrm{Int}(g)^{-1} = \sigma$ and $\mathrm{Int}(g) \theta\, \mathrm{Int}(a_1 g^{-1}) = \theta\, \mathrm{Int}(a_2)$. Since both gAg^{-1} and A are maximal $(\sigma, \theta\, \mathrm{Int}(a_2))$-split tori of G, there exists by (5.15) $h \in (G_\sigma \cap G_{\theta \mathrm{Int}(a_2)})^0$ such that $hg \in N_G(A)$. This proves the assertion.

8.3. *Remark.* The question whether two pairs in $\mathscr{F}_A(\sigma, \theta)$ are isomorphic under $N_G(A)$ can be reduced to the case where G is adjoint. Henceforth we assume this for the remaining part of this section.

8.4. *Action on $W(A)$ on $\mathscr{F}_A(\sigma, \theta)$.* The action of $N_G(A)$ on $\mathscr{F}_A(\sigma, \theta)$ can be split in an action of $W(A)$ on $\mathscr{F}_A(\sigma, \theta)$ and an action of $Z_G(A)$ on $\mathscr{F}_A(\sigma, \theta)$. That $W(A)$ acts on the pairs in $\mathscr{F}_A(\sigma, \theta)$ can be seen as follows.

Let (σ, θ) be a standard pair. Then by (6.15) every $w \in W(A)$ has a representative $h \in (G_\sigma \cap G_\theta)^0$. So if $(\sigma, \theta\,\mathrm{Int}(a)) \in \mathscr{F}_A(\sigma, \theta)$, $w \in W(A)$, and $h \in (G_\sigma \cap G_\theta)^0$ a representative of w, then $\mathrm{Int}(h)\,\sigma\,\mathrm{Int}(h)^{-1} = \sigma$ and $\mathrm{Int}(h)\,\theta\,\mathrm{Int}(a)\,\mathrm{Int}(h)^{-1} = \theta\,\mathrm{Int}(hah^{-1}) = \theta\,\mathrm{Int}(w(a))$.

Denote the set of quadratic elements of A by $F(A)$. In (8.13) we shall describe a set of representatives of the $W(A)$-conjugacy classes in $F(A)$. We first deal with the question of when two pairs in $\mathscr{F}_A(\sigma, \theta)$ are isomorphic under $Z_G(A)$.

8.5. PROPOSITION. *Let T, A be as in* (8.1). *Then two pairs* $(\sigma, \theta\,\mathrm{Int}(a_1))$ *and* $(\sigma, \theta\,\mathrm{Int}(a_2))$ *in* $\mathscr{F}_A(\sigma, \theta)$ *are isomorphic under* $Z_G(A)$ *if and only if there exists $t \in T$ such that $\sigma(t) = t$ and $a_1 a_2 = \theta(t)\,t^{-1}$.*

Proof. If $t \in T$ satisfies the above conditions, then $\mathrm{Int}(t)$ is the desired isomorphism. So assume there is $g \in Z_G(A)$ such that $\mathrm{Int}(g)\sigma\,\mathrm{Int}(g)^{-1} = \sigma$ and $\mathrm{Int}(g)\,\theta\,\mathrm{Int}(a_1 g)^{-1} = \theta\,\mathrm{Int}(a_2)$. As in (1.4) let $T_{\sigma\theta}^- = \{t \in T \mid \sigma\theta(t) = t^{-1}\}$. Now $T_{\sigma\theta}^-$ and $g(T_{\sigma\theta}^-)\,g^{-1}$ are both maximal $\sigma\theta$-split tori of $Z_G(A)$, so by (1.5), (1.6), and (5.3) there is $h \in (Z_G(A) \cap G_\sigma \cap G_\theta)^0$ such that $hg \in Z_{Z_G(A)}(T_{\sigma\theta}^-) = Z_G(AT_{\sigma\theta}^-) = Z_G(T_\sigma^- T_\theta^-)$.

Now T and $hgTg^{-1}h^{-1}$ are maximal tori in $Z_G(AT_{\sigma\theta}^-)$ and since the derived group of $Z_G(AT_{\sigma\theta}^-)$ is contained in $(Z_G(A) \cap G_\sigma \cap G_\theta)^0$, there exists $k \in (Z_G(A) \cap G_\sigma \cap G_\theta)^0$ such that $t = khg \in T$. Since $a_1, a_2 \in A$ and $kh \in (Z_G(A) \cap G_\sigma \cap G_\theta)^0$ we have $\sigma(t) = t$ and $a_2 = a_1\theta(t)\,t^{-1}$, which proves the result.

For a standard pair we can prove an even stronger result:

8.6. COROLLARY. *Assume (σ, θ) is a standard pair. Let T, A be as in* (8.1) *and $a \in F(A)$. Then (σ, θ) and $(\sigma, \theta\,\mathrm{Int}(a))$ are isomorphic if and only if there is $t \in T$ such that $\sigma(t) = t$ and $a = \theta(t)\,t^{-1}$.*

Proof. The result follows immediately from (8.5), (8.2) and the fact that by (6.15) any $w \in W(A)$ has a representative in $(G_\sigma \cap G_\theta)^0$.

It is possible to characterize these quadratic elements $\theta(t)\,t^{-1}$ occurring in (8.5) as a product of a quadratic element in $(T_\sigma^-)_\theta^+$ and one in $(T_\theta^-)_\sigma^+$. This is useful for checking whether in an explicit example two pairs in $\mathscr{F}_A(\sigma, \theta)$ are isomorphic under $Z_G(A)$. We will state this result here, but we shall not use it for the classification.

8.7. COROLLARY. *Let σ, θ, T, and A be as in* (8.5) *and $a \in F(A)$. Then the following statements are equivalent*:

(1) *There is a $t \in T$ such that $\sigma(t) = t$ and $a = \theta(t)\,t^{-1}$.*

(2) *$a = xy$ where $x \in (T_\sigma^-)_\theta^+$, $y \in (T_\theta^-)_\sigma^+$, $x^2 = y^2 = e$.*

(3) *There is a $t \in T$ such that $\theta(t) = t$ and $a = \sigma(t)\,t^{-1}$.*

Proof. (1) $\Rightarrow$ (2): Assume $t \in T$ such that $\sigma(t) = t$ and $a = \theta(t)\, t^{-1}$. Write $t = t_1 t_2 t_3 t_4$ with $t_1 \in (T_\sigma^+)_\theta^+$, $t_2 \in (T_\theta^-)_\sigma^+$, $t_3 \in (T_\sigma^-)_\theta^+$, $t_4 \in A$. Then $t_2 t_3 t_4$ satisfies the same conditions, so we may assume $t = t_2 t_3 t_4$. From $\sigma(t) = t$ we see that $(t_3 t_4)^2 = e$ and since $\theta(t_3 t_4)(t_3 t_4)^{-1} = t_4^{-2}$, we obtain $t_4^4 = e$ and also $t_3^4 = e$. Now $a = \theta(t)\, t^{-1} = t_2^{-2} t_4^{-2} = t_2^{-2} t_3^{-2}$, so it follows that $t_2^4 = e$. Taking $x = t_2^2$, $y = t_3^2$ the result follows.

(2) $\Rightarrow$ (1): Assume now that $a = xy$ as in (2). Let $t_1 \in (T_\theta^-)_\sigma^+$ be such that $t_1^2 = x$ and let $t_2 \in (T_\sigma^-)_\theta^+$ be such that $t_2^2 = y$. Since $y \in T_\theta^- \cap T_\sigma^-$ and $T_\theta^- \cap T_\sigma^- = A$ (see also (8.9) below) there exists a $t_3 \in A$ such that $t_3^2 = y$. If $t = t_1 t_2 t_3$, then $\sigma(t) = t_1 t_2^{-1} t_3^{-1} = t_1 t_2 t_3\, y^2 = t$ and $\theta(t)\, t^{-1} = t_1^{-2} t_3^{-2} = xy = a$, which proves (1).

The equivalence of (2) and (3) follows by symmetry.

8.8. *A characterization of the quadratic elements of A.* We can describe the quadratic elements of A as a product of quadratic elements of T. Let Δ be a strong (σ, θ)-basis of Φ and $\bar{\Delta}_{\sigma,\theta}$ the corresponding basis of $\Phi(A)$. Since the elements of both Δ and $\bar{\Delta}_{\sigma,\theta}$ are linearly independent, we can find for each $\alpha \in \Delta$ an $\omega_\alpha \in X_*(T)$ such that $\langle \alpha, \omega_\beta \rangle = \delta_{\alpha,\beta}$ for $\alpha, \beta \in \Delta$. Similarly for each $\lambda \in \bar{\Delta}_{\sigma,\theta}$ let $\gamma_\lambda \in X_*(A)$ be such that $\langle \lambda, \gamma_\mu \rangle = \delta_{\lambda,\mu} (\lambda, \mu \in \bar{\Delta}_{\sigma,\theta})$. In (6.6) we defined for $\lambda \in \bar{\Delta}_{\sigma,\theta}$ the quadratic elements $\varepsilon_\lambda = \gamma_\lambda(-1) \in A$. Since $(\sigma, \theta) \mid T$ is a pair of basic involutions of (X, Φ) we can describe ε_λ also in terms of the one parameter subgroups ω_α $(\alpha \in \Delta)$.

8.9. LEMMA. *Let $\lambda \in \bar{\Delta}_{\sigma,\theta}$. For $\Phi(T, \lambda) \cap A$ we have the following possibilities*:

(1) $\Phi(T, \lambda) \cap \Delta = \{a\}$ *with* $\alpha = \sigma^*(\alpha) = \theta^*(\alpha)$;

(2) $\Phi(T, \lambda) \cap \Delta = \{\alpha, \theta^*(\alpha)\}$ *or* $\{\alpha, \sigma^*(\alpha)\}$ *with* $\sigma^*(\alpha) = \alpha$, $\theta^*(\alpha) = \alpha$, *or* $\sigma^*\theta^*(\alpha) = \alpha$;

(3) $\Phi(T, \lambda) \cap \Delta = \{\alpha, \sigma^*(\alpha), \theta^*(\alpha), \sigma^*\theta^*(\alpha)\}$.

In these three cases we have for ε_λ, respectively,

(1) $\varepsilon_\lambda = \omega_\alpha(-1)$;

(2) $\varepsilon_\lambda = (\omega_\alpha \omega_{\theta^*(\alpha)})(-1)$ *or* $(\omega_\alpha \omega_{\sigma^*(\alpha)})(-1)$;

(3) $\varepsilon_\lambda = (\omega_\alpha \omega_{\sigma^*(\alpha)} \omega_{\theta^*(\alpha)} \omega_{\sigma^*\theta^*(\alpha)})(-1)$.

Proof. The first statement follows immediately from Lemma 7.10. As for the other statements note first that, since G is adjoint, $\{\omega_\alpha\}_{\alpha \in \Delta}$ is a basis of $X_*(T)$. But then every quadratic element of T is a product of the quadratic elements $\omega_\alpha(-1)$ $(\alpha \in \Delta)$. In particular there is a subset $\Delta_1 \subset \Delta$ such that $\varepsilon_\lambda = \prod_{\alpha \in \Delta_1} \omega_\alpha(-1)$.

Since for $\alpha \in \Delta$ we have $\alpha(\varepsilon_\lambda) = \alpha(\gamma_\lambda(-1)) = (-1)^{\langle \pi(\alpha), \gamma_\lambda \rangle}$, the result follows from the definition of the ω_α ($\alpha \in \Delta$) and the first statement.

8.10. *Remark.* If (σ, θ) is normally related to T, then it follows from this result, (3.5) and the fact that $\Delta_0(\sigma, \theta) = \Delta_0(\sigma) \cup \Delta_0(\theta)$, that $T^-_{\sigma,\theta} = T^-_\sigma \cap T^-_\theta$. Since we will not need this in the sequel we leave the proof for the reader.

For arbitrary quadratic elements of A we note:

8.11. LEMMA. *Let $\varepsilon \in F(A)$. Then there exists a subset Δ_1 of $\bar{\Delta}_{\sigma,\theta}$* such that $\varepsilon = \prod_{\lambda \in \Delta_1} \varepsilon_\lambda$. *In particular, $F(A)$ is completely determined by the set of indecomposable roots $\Phi(A)'$.*

This follows immediately from the fact that $\bar{\Delta}_{\sigma,\theta}$ is a $\mathbb{Z}$-basis of $X^*(A)$.

8.11.1. *Remark.* Since also $W(A)$ is generated by the reflections s_λ with $\lambda \in \bar{\Delta}_{\sigma,\theta} \subset \Phi(A)'$, it follows that for determining a set of representatives of the $W(A)$-conjugacy classes in $F(A)$ we may restrict ourselves to $\Phi(A)'$, which is reduced. Henceforth we will assume that $\Phi(A) = \Phi(A)'$.

8.12. *Action of the affine Weyl group on $F(A)$.* Assume G is semisimple, $\Phi(A)$ is reduced and Δ, $\bar{\Delta}_{\sigma,\theta}$ are as in (8.8). Write $X_*(A)$ additively and let $E = X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$. For $x \in E$, let $t(x)$ denote the translation of E along the vector x and let Q denote the group of the translations $t(v)$, where $v = \sum_{\lambda \in \bar{\Delta}_{\sigma,\theta}} m_\lambda \lambda^\vee$ with $m_\lambda \in \mathbb{Z}$ and $\lambda^\vee \in \Phi(A)^\vee$ a coroot. If $W^a(A)$ denotes the affine Weyl group of $\Phi(A)$, then $W^a(A)$ is the semidirect product of $W(A)$ and Q (see Bourbaki [5, Chap. VI, No. 2.1]). Let

$$\Lambda_2 = \left\{ \frac{1}{2} \left(\sum_{\lambda \in \bar{\Delta}_{\sigma,\theta}} m_\lambda \gamma_\lambda \right) \,\middle|\, m_\lambda \in \mathbb{Z} \right\}$$

and let $\xi_2 \in F^*$ be a primitive 2th root of unity. Now define $\phi_2 \colon \Lambda_2 \to A$ by

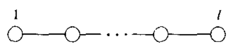
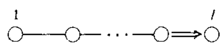
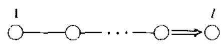
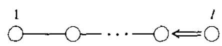
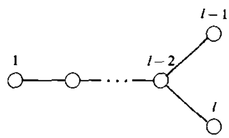
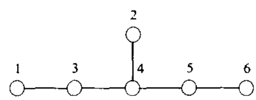
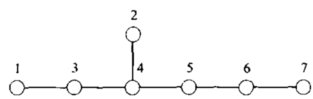
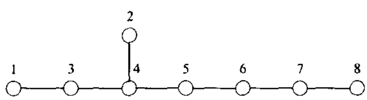
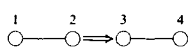
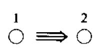
$$\frac{1}{2} \left(\sum_{\lambda \in \bar{\Delta}_{\sigma,\theta}} m_\lambda \gamma_\lambda \right) \to \left(\sum_{\lambda \in \bar{\Delta}_{\sigma,\theta}} m_\lambda \gamma_\lambda \right)(\xi_2).$$

Q acts transitively on the fibers of ϕ_2 and $\phi_2(\Lambda_2) = F(A)$. Moreover, since $W^a(A) = Q \cdot W(A)$, the orbits of Λ_2 under the action of $W^a(A)$ correspond one to one to the $W(A)$-conjugacy classes in $F(A)$. Let C be the chamber of E with respect to $\bar{\Delta}_{\sigma,\theta}$ and P_0 the unique fundamental region in C containing the origin in its closure. Denote the closures of C resp. P_0 by $\bar{C}$ resp. $\bar{P}_0$. Now any $W^a(A)$ orbit in E meets $\bar{P}_0$ exactly once (see Bourbaki [5, Chap. VI, No. 2.1]). So if $R = \Lambda_2 \cap \bar{P}_0$, then $\phi_2(R)$ is a set of representatives of the $W(A)$-conjugacy classes in $F(A)$. One easily sees that R consists of at most $|\bar{\Delta}_{\sigma,\theta}|$ vectors. Eventually after applying still a Weyl group element, we obtain the following result:

8.13. PROPOSITION. *Assume $\Phi(A)$ is irreducible. Then any element of $F(A)$ is conjugate under $W(A)$ to one of the ε_λ, $\lambda \in \bar{\Delta}_{\sigma,\theta}$, as given in Table* VI.

For more details on the proof we refer to Borel and Siebenthal [4] who also derived this result in a slightly different context. They work with compact groups, but this specific result depends only on the action of the affine Weyl group. See also Oshima and Sekiguchi [18].

TABLE VI

Φ	Dynkin diagram	Quadratic elements
$A_l\,(l \geqslant 1)$		$\varepsilon_j\,(2j \leqslant l+1)$
$B_l\,(l \geqslant 2)$		$\varepsilon_j\,(j \leqslant l)$
$BC_l\,(l \geqslant 1)$		$\varepsilon_j\,(j \leqslant l)$
$C_l\,(l \geqslant 3)$		$\varepsilon_j\,(2j \leqslant l)$ ε_l
$D_l\,(l \geqslant 4)$		$\varepsilon_j\,(2j \leqslant l)$ ε_{l-1} ε_l
E_6		ε_1 ε_2
E_7		ε_1 ε_2 ε_7
E_8		ε_1 ε_8
F_4		ε_1 ε_4
G_2		ε_1

We still need to determine which of these ε_λ in Table VI are isomorphic under $Z_G(A)$. This depends only on the signatures of the simple roots:

8.14. THEOREM. *Assume G is semisimple and let (σ, θ) be a pair of commuting involutorial automorphisms of G. If T, A, Δ and $\bar{\Delta}_{\sigma,\theta}$ are as in (8.8), then, for any $\lambda \in \bar{\Delta}_{\sigma,\theta}$ with $m^+(\lambda, \sigma\theta) = m^-(\lambda, \sigma\theta)$, the pair $(\sigma, \theta\,\mathrm{Int}(\varepsilon_\lambda))$ is isomorphic to (σ, θ).*

It is possible to prove this result by checking the condition (2) of (8.7) for all the irreducible (σ, θ) indices in (7.18.1)–(7.18.6). We shall give another proof by proving five lemmas, which deal with all, except four, cases.

For (8.15)–(8.19) we assume that G, T, σ, θ, Δ are as in (8.14).

8.15. LEMMA. *Assume $\alpha \in \Delta - \Delta_0(\sigma, \theta)$ such that $\sigma^*\theta^*(\alpha) \neq \alpha$ and either $\sigma^*(\alpha) = \alpha$ or $\theta^*(\alpha) = \alpha$. If $\lambda = \pi(\alpha)$ then $(\sigma, \theta\,\mathrm{Int}(\varepsilon_\lambda))$ is isomorphic to (σ, θ).*

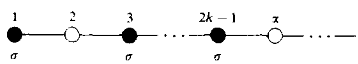
Proof. Let $t = \omega_\alpha(-1) \in T$. Then $\beta(t) = 1$ if $\beta \in \Delta - \{\alpha\}$ and $\alpha(t) = -1$. Let $u = t\sigma\theta(t^{-1})$. Then $u \in T^-_{\sigma\theta}$ (see (1.4)), so for $\beta \in \Delta$ we have $\beta(u) = \beta(t)\,\sigma^*\theta^*(\beta)(t^{-1})$, because $\sigma\theta(\beta) = \sigma^*\theta^*(\beta) + \gamma$, where γ lies in the $\mathbb{Z}$-span of $\Delta_0(\sigma, \theta)$.

Now since $\beta(u) = 1$ if $\beta \neq \alpha$ or $\sigma^*\theta^*(\alpha)$, it follows that $\mathrm{Int}(\varepsilon_\lambda) = \mathrm{Int}(u)$, so by (8.5) we are done.

8.16. LEMMA. *Assume $\alpha \in \Delta - \Delta_0(\sigma, \theta)$, such that $\sigma^*\theta^*(\alpha) \neq \alpha$, $\sigma^*(\alpha) \neq \alpha$, and $\theta^*(\alpha) \neq \alpha$. If $\lambda = \pi(\alpha)$ then $(\sigma, \theta\,\mathrm{Int}(\varepsilon_\lambda))$ is isomorphic to (σ, θ).*

Proof. Let $t = \omega_\alpha(-1)$, $x = t\theta(t^{-1})$ and $u = x\sigma\theta(x^{-1}) = t\sigma(t^{-1})\,\sigma\theta(t^{-1})\,\theta(t)$. Similarly to (8.15) we have for $\beta \in \Delta$: $\beta(u) = \beta(t)\,\sigma^*(\beta)(t)\,\theta^*(\beta)(t^{-1})\,\sigma^*\theta^*(\beta)(t^{-1})$. So $\beta(u) = \alpha(t) = -1$ if β equals one of α, $\sigma^*(\alpha)$, $\theta^*(\alpha)$, $\sigma^*\theta^*(\alpha)$ and $\beta(u) = 1$ for the other roots in Δ. It follows that $\mathrm{Int}(u) = \mathrm{Int}(\varepsilon_\lambda)$ and the result now follows from (8.5).

8.17. LEMMA. *Assume $\alpha \in \Delta - \Delta_0(\sigma, \theta)$ such that $\theta^*(\alpha) = \sigma^*(\alpha) = \alpha$ and let $\lambda = \pi(\alpha)$. If α is contained in a subdiagram of the (σ, θ)-index of the form*



Then $(\sigma, \theta\,\mathrm{Int}(\varepsilon_\lambda))$ is isomorphic to (σ, θ).

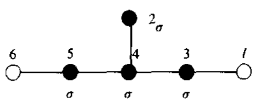
Proof. Assume that the roots of Δ are numbered as in the above diagram and that $\alpha = \alpha_{2k}$. For $i = 1, \ldots, k$ let $t_i = \omega_{\alpha_{2i-1}}(-1)$. Take

$t=\prod_{i=1}^{k} t_i$ and let $u=t\sigma\theta(t^{-1})$. Then for $i=1,...,k$ we have $\alpha_{2i-1}(u)=\alpha_{2i-1}(t)^2=1$. Moreover, for $i=1,...,k-1$ we have

$$\begin{aligned}\alpha_{2i}(u)&=\alpha_{2i}(t)\,\sigma\theta(\alpha_{2i})(t^{-1})\\&=(\alpha_{2i-1}\alpha_{2i}\alpha_{2i+1})(t^{-1})=\alpha_{2i-1}(t^{-1})\,\alpha_{2i+1}(t^{-1})=1.\end{aligned}$$

Since $\alpha_{2k}(u)=\alpha_{2k}(t)\,\sigma\theta(t^{-1})=w_0(\sigma)\,w_0(\theta)(\alpha_{2k})(t^{-1})=\alpha_{2k-1}(t^{-1})=-1$ and $\beta(u)=1$ for $\beta\in\Delta$, $\beta\neq\alpha_j$ $(j=1,...,2k)$ it follows that $\mathrm{Int}(u)=\mathrm{Int}(\varepsilon_\lambda)$, which proves the result.

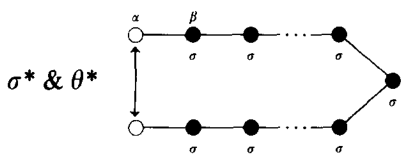
8.18. LEMMA. *Assume that the* (σ,θ)*-index has a subdiagram of the form*



and $\Phi_0(\sigma)=\varnothing$, $\Phi_0(\theta)$ *of type* D_4. *If* $\lambda_1=\pi(\alpha_1)$ *and* $\lambda_2=\pi(\alpha_6)$, *then for* $i=1,2$ $(\sigma,\theta\,\mathrm{Int}(\varepsilon_{\lambda_i}))$ *is iomorphic to* (σ,θ).

Proof. Let $t_1=\omega_{\alpha_1}(-1)\,\omega_{\alpha_2}(-1)$ and $t_2=\omega_{\alpha_6}(-1)\,\omega_{\alpha_2}(-1)$. Put $u_1=t_1\sigma\theta(t_1^{-1})$ and $u_2=t_2\sigma\theta(t_2^{-1})$. Similarly as in (8.17) it follows that $\mathrm{Int}(u_1)=\mathrm{Int}(\varepsilon_{\lambda_1})$ and $\mathrm{Int}(u_2)=\mathrm{Int}(\varepsilon_{\lambda_2})$, hence the result follows from (8.5).

8.19. LEMMA. *Assume the* (σ,θ)*-index has a subdiagram of the form*:

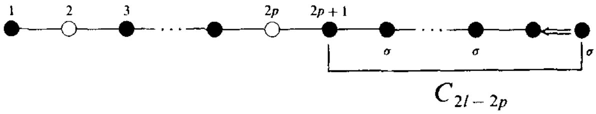


If $\lambda=\pi(\alpha)$, *then* $(\sigma,\theta\,\mathrm{Int}(\varepsilon_\lambda))$ *is isomorphic to* (σ,θ).

Proof. Put $t_1=\omega_\beta(-1)$, $t_2=\omega_{\sigma^*(\beta)}(-1)$, $t=t_1t_2$ and $u=t\sigma\theta(t^{-1})$. Similarly as in (8.17) one verifies that $\mathrm{Int}(u)=\mathrm{Int}(\varepsilon_\lambda)$, so the result follows by (8.5).

8.20. *Proof of Theorem* (8.14). Applying the Lemmas 8.15–8.19 to the irreducible (σ,θ)-indices in (7.18.1)–(7.18.6), we are left with four cases, which do not satisfy any of the conditions in one of these lemmas. For each of them we shall give an element $t\in T$ satisfying the condition of (8.5). As an example we shall treat the following case in more detail.

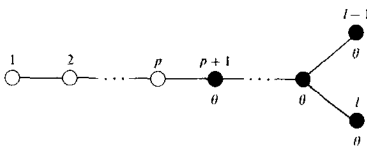
(1) $C_{2l}^{l,p}(\mathrm{II}_b, \mathrm{II}_a)$ with $\alpha = \alpha_{2p}$. We number the roots according to the Dynkin diagram below.



Let $\lambda = \pi(\alpha) = \lambda_p$, $t = \omega_{\alpha_{2p+1}}(-1)$ and $u = t\sigma\theta(t^{-1})$. Then $\mathrm{Int}(u) = \mathrm{Int}(\varepsilon_\lambda)$, namely $w_0(\theta)(\alpha) = s_{\alpha_{2p-1}} w_1(\alpha)$, where w_1 is the longest element of $W(C_{2l-2p})$ with respect to $\Delta \cap C_{2l-2p}$. Let $\beta \in C_{2l-2p}$ be such that $-\beta$ is the highest short root of C_{2l-2p} with respect to $\Delta \cap C_{2l-2p}$ (see Bourbaki [5, Chap. VI, No. 1.8]). Then $\beta(t) = -1$.

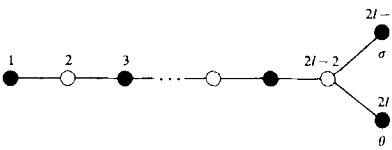
Extend $\{\alpha_{2p+1}, \beta\}$ to a maximal orthogonal set of roots as in (4.13) and write w_1 as a product of the corresponding reflections. Then $w_1(\alpha_{2p}) = \alpha_{2p} + \alpha_{2p+1} + \beta$. So $w_0(\theta)(\alpha_{2p}) = \alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1} + \beta$. Now $\sigma\theta(\alpha_{2p})(t) = w_0(\sigma)\, w_0(\theta)(\alpha_{2p})(t) = s_{\alpha_{2p+1}} s_{\alpha_{2p-1}}(\alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1} + \beta)(t) = \beta(t) = -1$. So $\alpha_{2p}(u) = \beta(t) = -1$. Since clearly $\gamma(u) = 1$ for $\gamma \in \Delta - \{\alpha_{2p}\}$ we are done.

(2) $D_l^{p,l}(\mathrm{I}_a, \mathrm{I}_b)$ with $\alpha = \alpha_p$.



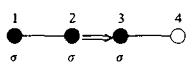
Let $\lambda = \pi(\alpha) = \lambda_p$, $t = \omega_{\alpha_{p+1}}(-1)\,\omega_{\alpha_{l-1}}(-1)$, and $u = t\sigma\theta(t^{-1})$. Then $\mathrm{Int}(u) = \mathrm{Int}(\varepsilon_\lambda)$.

(3) $D_{2l}^{l,l}(\mathrm{III}_a, \mathrm{III}_a, \varepsilon_i)$ with $\alpha = \alpha_{2l-2}$.



Let $\lambda = \pi(\alpha) = \lambda_{l-1}$, $t = \omega_{\alpha_{2l-1}}(-1)$ and $u = t\theta(t^{-1})$. Then $\mathrm{Int}(u) = \mathrm{Int}(\varepsilon_\lambda)$.

(4) $F_4^{4,1}(\mathrm{I}, \mathrm{II})$ with $\alpha = \alpha_4$.



Let $\alpha = \alpha_4$, $\lambda = \pi(\alpha)$, $t = \omega_{\alpha_3}(-1)$, and $u = t\sigma\theta(t^{-1})$. Then $\mathrm{Int}(u) = \mathrm{Int}(\varepsilon_\lambda)$. This completes the result.

8.21. COROLLARY. *The standard pair of a family $\mathscr{F}_A(\sigma, \theta)$ is unique up to isomorphism.*

Proof. The result follows immediately from (8.14), (6.13), and (6.14).

We are left with the following problem: which of the remaining ε_λ give rise to isomorphic pairs in $\mathscr{F}_A(\sigma, \theta)$. That they are not standard can be seen by looking at the corresponding rank one subgroups.

8.22. *Restricted rank one subgroups.* Let (σ, θ) be a standard pair and assume T, A are as in (8.1). For $\lambda \in \Phi(A)$ let $\Phi(\lambda) = \{\alpha \in \Phi(T) \mid \alpha \mid A = m\lambda, m \in \mathbb{Z}\}$. This is a closed symmetric subset of $\Phi(T)$. Let now $G(\lambda)$ denote the (closed) subgroup of G generated by T and the root subgroups U_β, with $\beta \in \Phi(\lambda)$. It follows from Borel and Tits [3, p. 74, Prop. 2.2, and p. 65, 2.3 Remark], that $G(\lambda)$ is reductive and that $\Phi(\lambda) = \Phi(T, G(\lambda))$.

Since $\theta(\lambda) = \sigma(\lambda) = -\lambda$, we see that $\Phi(\lambda)$ and $G(\lambda)$ are σ- and θ-stable. Moreover, if $\sigma_1 = \sigma \mid G(\lambda)$, $\theta_1 = \theta \mid G(\lambda)$, then (σ_1, θ_1) is normally related to T and $G(\lambda)$ has restricted (σ_1, θ_1)-rank one (i.e., rank $\overline{\Phi(\lambda)}_{\sigma_1, \theta_1} = 1$).

8.23. LEMMA. *Let (σ, θ) be a standard pair and T, A as in* (8.1). *Let Δ be a strong (σ, θ) basis of Φ and $\bar{\Delta}_{\sigma,\theta}$ the corresponding basis of $\Phi(A)$. Let ε_λ $(\lambda \in \bar{\Delta}_{\sigma,\theta})$ be as in* (8.8). *If $\lambda \in \bar{\Delta}_{\sigma,\theta}$ is such that $m^+(\lambda, \sigma\theta) \neq m^-(\lambda, \sigma\theta)$ then, $(\sigma, \theta \operatorname{Int}(\varepsilon_\lambda))$ and (σ, θ) are not isomorphic.*

Proof. Assume (σ, θ) and $(\sigma, \theta \operatorname{Int}(\varepsilon_\lambda))$ are isomorphic $(\lambda \in \bar{\Delta}_{\sigma,\theta})$. From (8.6) it follows that also $(\sigma, \theta) \mid G(\lambda)$ and $(\sigma, \theta \operatorname{Int}(\varepsilon_\lambda)) \mid G(\lambda)$ are isomorphic. So we may assume $G = G(\lambda)$. Now

$$\mathfrak{g} = Z_\mathfrak{g}(A) \oplus \mathfrak{g}(A, \lambda) \oplus \mathfrak{g}(A, -\lambda) \oplus \mathfrak{g}(A, 2\lambda) \oplus \mathfrak{g}(A, -2\lambda).$$

On the other hand, $\sigma\theta \mid Z_G(A) = \sigma\theta \operatorname{Int}(\varepsilon_\lambda) \mid Z_G(A)$ and

$$\sigma\theta(\mathfrak{g}(A, \pm m\lambda)) = \sigma\theta \operatorname{Int}(\varepsilon_\lambda)(\mathfrak{g}(A, \pm m\lambda)) = \mathfrak{g}(A, \pm m\lambda), \qquad m = 1, 2.$$

Comparing the dimensions of the eigenspaces of $\sigma\theta$ and $\sigma\theta \operatorname{Int}(\varepsilon_\lambda)$ in $\mathfrak{g}$, we conclude that if (σ, θ) and $(\sigma, \theta \operatorname{Int}(\varepsilon_\lambda))$ are isomorphic, we have $m^+(\lambda, \sigma\theta) = m^-(\lambda, \sigma\theta)$. The lemma is proved.

8.24. LEMMA. *Let (σ, θ) be a standard pair. Let $a \in F(A)$ and $\Delta_1 \subset \bar{\Delta}_{\sigma,\theta}$ be such that $a = \prod_{\lambda \in \Delta_1} \varepsilon_\lambda$. Then $(\sigma, \theta \operatorname{Int}(a))$ is isomorphic to (σ, θ) if and only if $m^+(\lambda, \sigma\theta) = m^-(\lambda, \sigma\theta)$ for all $\lambda \in \Delta_1$.*

Proof. Assume $\lambda \in \Delta_1$ such that $m^+(\lambda, \sigma\theta) \neq m^-(\lambda, \sigma\theta)$. If $(\sigma, \theta \operatorname{Int}(a))$ and (σ, θ) are isomorphic, then by (8.6) also their restrictions to $G(\lambda)$ are isomorphic. But since $(\sigma, \theta \operatorname{Int}(a)) \mid G(\lambda) = (\sigma, \theta \operatorname{Int}(\varepsilon_\lambda)) \mid G(\lambda)$ the result follows from (8.23).

We can now prove the following results:

8.25. PROPOSITION. *Let* (σ, θ) *be a standard pair. Let* $\lambda_1, \lambda_2 \in \bar{\Delta}_{\sigma,\theta}$ *be such that* $m^+(\lambda_i, \sigma\theta) \neq m^-(\lambda_i, \sigma\theta)$ $(i = 1, 2)$. *Then* $(\sigma, \theta \operatorname{Int}(\varepsilon_{\lambda_1}))$ *and* $(\sigma, \theta \operatorname{Int}(\varepsilon_{\lambda_2}))$ *are isomorphic if and only if* ε_{λ_1} *and* ε_{λ_2} *are conjugate under* $W(A)$.

Proof. The "if" statement being clear, assume $(\sigma, \theta \operatorname{Int}(\varepsilon_{\lambda_1}))$ and $(\sigma, \theta \operatorname{Int}(\varepsilon_{\lambda_2}))$ are isomorphic. By (8.4) there exists $w \in W(A)$ such that $(\sigma, \theta \operatorname{Int}(w\varepsilon_{\lambda_1}))$ and $(\sigma, \theta \operatorname{Int}(\varepsilon_{\lambda_2}))$ are isomorphic under $Z_G(A)$. Let $a \in A$ be such that $\varepsilon_{\lambda_2} = w(\varepsilon_{\lambda_1})a$ and let $\Delta_1 \subset \bar{\Delta}_{\sigma,\theta}$ be such that $a = \prod_{\lambda \in \Delta_1} \varepsilon_\lambda$. Then by (8.5) (σ, θ) and $(\sigma, \theta \operatorname{Int}(a))$ are isomorphic. Hence by (8.24) $m^+(\lambda, \sigma\theta) = m^-(\lambda, \sigma\theta)$ for all $\lambda \in \Delta_1$. We shall now show that a must equal e.

Checking the signatures for the simple roots for the irreducible (σ, θ)-indices in (7.18.1)–(7.18.6) (see also Tables II–V), it follows that only the following four cases occur for $\bar{\Phi}_{\sigma,\theta}$ irreducible:

(1) $m^+(\lambda, \sigma\theta) = m^-(\lambda, \sigma\theta)$ for all $\lambda \in \bar{\Delta}_{\sigma,\theta}$. Then by (8.14) $\mathscr{C}(\sigma, \theta)$ consists of a single isomorphism class.

(2) $m^+(\lambda, \sigma\theta) \neq m^-(\lambda, \sigma\theta)$ for all $\lambda \in \bar{\Delta}_{\sigma,\theta}$. Then $a = e$.

(3) $m^+(\lambda, \sigma\theta) \neq m^-(\lambda, \sigma\theta)$ for exactly one $\lambda \in \bar{\Delta}_{\sigma,\theta}$. In this case $\lambda_1 = \lambda_2$, so there is nothing to prove.

(4) $\bar{\Phi}_{\sigma,\theta}$ is of type B_n or BC_n and $m^+(\lambda, \sigma\theta) \neq m^-(\lambda, \sigma\theta)$ for all long roots in $\bar{\Delta}_{\sigma,\theta}$ and $m^+(\lambda, \sigma\theta) = m^-(\lambda, \sigma\theta)$ for the single short root $\mu \in \bar{\Delta}_{\sigma,\theta}$.

In this case $a = e$ or $a = \varepsilon_\mu$. Assume $a = \varepsilon_\mu$. So $w(\varepsilon_{\lambda_1}) = \varepsilon_{\lambda_2}\varepsilon_\mu$. Here $\lambda_1, \lambda_2 \in \bar{\Delta}_{\sigma,\theta} - \{\mu\}$. On the other hand, $\varepsilon_\mu \varepsilon_{\lambda_2}$ is conjugate under $W(A)$ to ε_μ, what implies that ε_{λ_1} is conjugate under $W(A)$ to ε_μ. So by (8.23) and (8.14) we obtain a contradiction. Hence $a = e$. Since (1)–(4) exhaust all the possibilities for $\bar{\Phi}_{\sigma,\theta}$ irreducible, the result is proved.

We shall say that $\mathscr{C}(\sigma, \theta)$ is irreducible if the index of the corresponding admissible pair of commuting involutions of (X, Φ) is irreducible (i.e., $\bar{\Phi}_{\sigma,\theta}$ is irreducible). Summarizing the above results we have obtained the following characterization of the isomorphism classes in $\mathscr{C}$:

8.26. THEOREM. *Asume* G *is semisimple and* T *a maximal torus of* G. *Then the classes* $\mathscr{C}(\sigma, \theta)$ *in* $\mathscr{C}$ *correspond bijectively to the isomorphism classes of the indices of the corresponding admissible pair of commuting involutions of* (X, Φ). *The isomorphism classes contained in* $\mathscr{C}(\sigma, \theta)$ *are represented by quadratic elements of a fixed maximal* (σ, θ)*-split torus* A *of* G. *For* $\mathscr{C}(\sigma, \theta)$ *irreducible these are given in Tables* II–IV.

Note that, in cases (7.18.5)–(7.18.6), $\mathscr{C}(\sigma, \theta)$ consists of a single isomorphism class of commuting involutorial automorphisms of G.

9. Aut(G)-Isomorphism Classes and Associated Pairs

In this section we will discuss the action of the full automorphism group on the ordered pairs of commuting involutions. Moreover, we will give the associated pairs (see (9.5)), which are of importance in the study of semisimple symmetric spaces.

9.1. Definition. Two pairs of commuting involutions (σ_1, θ_1) and (σ_2, θ_2) are Aut(G)-*isomorphic* if there exists a $\phi \in \mathrm{Aut}(G)$ such that $\phi\sigma_1\phi^{-1} = \sigma_2$ and $\phi\theta_1\phi^{-1} = \theta_2$.

Denote the set of Aut(G)-isomorphism classes in $\mathscr{F}$ by $\mathscr{C}^a$ and for a maximal torus T denote the set of Aut(G, T)-isomorphism classes of ordered pairs of commuting involutions of $(X^*(T), \Phi(T))$ by $\mathscr{C}^a(T)$. Similarly as in (5.19) we can define a mapping

$$\rho^a : \mathscr{C}^a \to \mathscr{C}^a(T).$$

Write $\mathscr{A}^a(T) = \rho^a(\mathscr{C}^a)$ and $\mathscr{C}^a(\sigma, \theta) = (\rho^a)^{-1} \rho^a(\sigma, \theta)$.

From (7.20) it follows that Aut(T) only identifies the isomorphism classes of some dual pairs of commuting involutions (σ, θ) and (θ, σ). The classification of $\mathscr{A}^a(T)$ is immediate from this.

As for the classification of $\mathscr{C}^a(\sigma, \theta)$ it suffices to consider the action of Aut(T) on the quadratic elements representing the classes in $\mathscr{C}(\sigma, \theta)$. This can be seen as follows. Let (σ, θ) be a standard pair, A a maximal (σ, θ)-split torus of G and

$$N^a = \{\phi \in \mathrm{Aut}(G) \mid \phi(A) \subset A\}.$$

Similarly as in (8.2) we can restrict to the action of N^a on $\mathscr{F}_A(\sigma, \theta)$.

9.2. Lemma. *Two pairs* $(\sigma, \theta\,\mathrm{Int}(a_1))$ *and* $(\sigma, \theta\,\mathrm{Int}(a_2))$ *in* $\mathscr{F}_A(\sigma, \theta)$ *are isomorphic under* Aut(G) *if and only if they are isomorphic under* N^a.

9.3. The group N^a differs at most some diagram automorphism from $N_G(A)$. Namely let $T \supset A$ be a (σ, θ)-stable maximal torus of G as in (5.13) and for a strong (σ, θ)-basis Δ of $\Phi(T)$ define

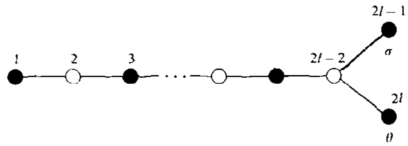
$$D(A, \Delta) = \{\phi \in \mathrm{Aut}(G, T) \mid \phi(A) = A,\ \phi(\Delta) = \Delta\}.$$

We now have

$$N^a = D(A, \Delta) \cdot N_G(A).$$

For $\phi \in D(A, \Delta)$, the automorphisms ϕ, σ and θ do not need to commute. So the action of N^a on $\mathscr{F}_a(\sigma, \theta)$ does not split in an action of $N_G(A)$ on $\mathscr{F}_A(\sigma, \theta)$ and an action of $D(A, \Delta)$ on $\mathscr{F}_A(\sigma, \theta)$.

9.4. If $\Phi(T)$ is irreducible, then $D(A, \Delta) \neq \mathrm{id}$ if $\Phi(T)$ is of type A_l $(l \geq 2)$, D_l $(l \geq 4)$, or E_6. Using the characterization of the quadratic elements of (8.9) it easily follows that N^a does not identify any of the quadratic elements representing the classes of $\mathscr{C}(\sigma, \theta)$, except in the case of $D_{2l}^{l,l}(\mathrm{III}_a, \mathrm{III}_a, \varepsilon_i)$.



Let ε_i $(i = 1, ..., l-2)$ be the quadratic elements representing the classes in $\mathscr{C}(\sigma, \theta)$. Then by (8.9) $\varepsilon_i = \omega_{2i}(-1)$ $(i = 1, ..., l-2)$. Let $\Phi^0 = \{\alpha \in \Phi(T) \mid (\alpha, \alpha_{2l-1}) = (\alpha, \alpha_{2l}) = 0\}$. It is easy to check that Φ^0 is of type D_{2l-2} and $\alpha_1, ..., \alpha_{2l-3}$ can be extended with a root $\beta \in \Phi^0$ to a basis Δ^0 of Φ^0. Let $-\alpha_H \in \Phi^0$ be the longest root with respect to Δ^0. Define $\phi^\circ \in \mathrm{Aut}(\Phi^0)$ by:

$$\alpha_i \mapsto \alpha_{2l-2-i}, \qquad (i = 1, ..., 2l-3)$$
$$\alpha_H \mapsto \beta$$

Extend ϕ^0 to an automorphism $\phi \in \mathrm{Aut}(\Phi)$ by $\phi(\alpha_{2l-1}) = \alpha_{2l-1}$ and $\phi(\alpha_{2l}) = \alpha_{2l}$. Then $\phi(\varepsilon_{l-i-1}) = \varepsilon_i \varepsilon_{l-1}$, hence ε_i and ε_{l-i-1} are isomorphic under $Z_G(A)$.

9.5. *Associated pairs.* Let (σ, θ) be a standard pair, A a maximal (σ, θ)-split torus of G and $\varepsilon_0 = e$. For a pair $(\sigma, \theta\,\mathrm{Int}(\varepsilon_i))$ $(i = 0, ..., \mathrm{rank}(A))$, representing a class in $\mathscr{C}(\sigma, \theta)$, we define the associated pairs as $(\sigma, \sigma\theta\,\mathrm{Int}(\varepsilon_i))$ and $(\theta, \sigma\theta\,\mathrm{Int}(\varepsilon_i))$.

It will appear in Section 10 that these pairs correspond to the natural associated pairs in the case of semisimple symmetric spaces. For the absolutely irreducible (σ, θ)-indices the associated pairs are listed in Table VII. We have omitted those pairs which are simultaneously self-associated and self-dual. The remaining cases are easily derived.

9.6. *Remark.* The determination of these associated pairs is mainly a matter of determining $\sigma\theta\,\mathrm{Int}(\varepsilon_i)$, which consists of the three commuting involutions σ, θ, and $\mathrm{Int}(\varepsilon_i)$. A classification of all triples of commputing involutions is not simple, because it is not only a combinatorial matter but also a topological one. We will deal with a classification of these in future papers.

TABLE VII

$(\sigma, \theta\,\mathrm{Int}(\varepsilon_i))$		$(\sigma, \sigma\theta\,\mathrm{Int}(\varepsilon_i))$	$(\theta, \sigma\theta\,\mathrm{Int}(\varepsilon_i))$
$A^{2l-1,l}_{2l-1}(\mathrm{I}, \mathrm{III}_b, \varepsilon_l)$		$A^{2l-1}_{2l-1}(\mathrm{I}, \varepsilon_l)$	$A^{2l-1,l}_{2l-1}(\mathrm{III}_b, \mathrm{I}, \varepsilon_l)$
$A^{2l,2l-1}_{4l-1}(\mathrm{II}, \mathrm{III}_b, \varepsilon_i)$	$i=0$ $i=l$	$A^{2l}_{4l-1}(\mathrm{II}, \varepsilon_l)$ $A^{4l-1,2l}_{4l-11}(\mathrm{I}, \mathrm{III}_b, \varepsilon_0)$	$A^{2l,2l-1}_{4l-1}(\mathrm{III}_b, \mathrm{II}, \varepsilon_0)$ $A^{4l-1,2l-1}_{4l-1}(\mathrm{I}, \mathrm{II})$
$A^{2l,2l+1}_{4l+1}(\mathrm{II}, \mathrm{III}_b)$		$A^{4l+1,2l+1}_{4l+1}(\mathrm{I}, \mathrm{III}_b, \varepsilon_0)$	$A^{4l+1,2l}_{4l+1}(\mathrm{I}, \mathrm{II})$
$A^{l,p}_l(\mathrm{I}, \mathrm{III}_a)$ $(1 \leqslant 2p \leqslant l)$		$A^l_l(\mathrm{I}, \varepsilon_p)$	$A^{p,l}_l(\mathrm{III}_a, \mathrm{I})$
$A^{2l-1,2p}_{4l-1}(\mathrm{II}, \mathrm{III}_a)$		$A^{2l-1}_{4l-1}(\mathrm{II}, \varepsilon_p)$	$A^{2p,2l-1}_{4l-1}(\mathrm{III}_a, \mathrm{II})$
$A^{2l,2p}_{4l+1}(\mathrm{II}, \mathrm{III}_a)$		$A^{2l}_{4l+1}(\mathrm{II}, \varepsilon_p)$	$A^{2p,2l}_{4l+1}(\mathrm{III}_a, \mathrm{II})$
$A^{l,p}_{2l-1}(\mathrm{III}_b, \mathrm{III}_a, \varepsilon_i)$ $(1 \leqslant p < l)$ $(0 \leqslant i \leqslant p-1)$	$l-p+2i \leqslant p$ $p < l-p+2i \leqslant l$ $p \leqslant l+p-2i < l$ $l+p-2i < l$	$A^{l,l-p+2i}_{2l-1}(\mathrm{III}_b, \mathrm{III}_a, \varepsilon_i)$ $A^{l,l-p+2i}_{2l-1}(\mathrm{III}_b, \mathrm{III}_a, \varepsilon_{p-i})$ $A^{l,l+p-2i}_{2l-1}(\mathrm{III}_b, \mathrm{III}_a, \varepsilon_{l-i})$ $A^{l,l+p-2i}_{2l-1}(\mathrm{III}_b, \mathrm{III}_a, \varepsilon_{l-i})$	$A^{p,l-p+2i}_{2l-1}(\mathrm{III}_a, \mathrm{III}_b, \varepsilon_{l-p+i})$ $A^{p,l-p+2i}_{2l-1}(\mathrm{III}_a, \mathrm{III}_b, \varepsilon_{p-i})$ $A^{p,l+p-2i}_{2l-1}(\mathrm{III}_a, \mathrm{III}_b, \varepsilon_i)$ $A^{p,l+p-2i}_{2l-1}(\mathrm{III}_a, \mathrm{III}_b, \varepsilon_{l-i})$
$A^{q,p}_l(\mathrm{III}_a, \mathrm{III}_a, \varepsilon_i)$ $1 \leqslant p < q \leqslant \frac{1}{2}(l+1)$ $(0 \leqslant i \leqslant p)$	$q-p+2i \leqslant p$ $p < q-p+2i \leqslant \frac{1}{2}(l+1)$ $q \leqslant l+1-q+p-2i < \frac{1}{2}(l+1)$ $p \leqslant l+1-q+p-2i < q$ $l+1-q+p-2i < q$	$A^{q,q-p+2i}_l(\mathrm{III}_a, \mathrm{III}_a, \varepsilon_i)$ $A^{q,q-p+2i}_l(\mathrm{III}_a, \mathrm{III}_a, \varepsilon_{p-i})$ $A^{q,l+1-q+p-2i}_l(\mathrm{III}_a, \mathrm{III}_a, \varepsilon_{q-p+i})$ $A^{q,l+1-q+p-2i}_l(\mathrm{III}_a, \mathrm{III}_a, \varepsilon_{l+1-q-i})$ $A^{q,l+1-q+p-2i}_l(\mathrm{III}_a, \mathrm{III}_a, \varepsilon_{l+1-q-i})$	$A^{p,q-p+2i}_l(\mathrm{III}_a, \mathrm{II}_a, \varepsilon_{q-p+i})$ $A^{p,q-p+2i}_l(\mathrm{III}_a, \mathrm{III}_a, \varepsilon_{p-i})$ $A^{p,l+1-q+p-2i}_l(\mathrm{III}_a, \mathrm{III}_a, \varepsilon_i)$ $A^{p,l+1-q+p-2i}_l(\mathrm{III}_a, \mathrm{III}_a, \varepsilon_i)$ $A^{p,l+1-q+p-2i}_l(\mathrm{III}_a, \mathrm{III}_a, \varepsilon_{l+1-q-i})$
$B^{q,p}_l(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_i)$ $(1 \leqslant p < q \leqslant l)$ $(0 \leqslant i \leqslant p)$	$q-p+2i \leqslant p$ $p < q-p+2i \leqslant l$ $q \leqslant 2l+1-q+p-2i < l$ $p \leqslant 2l+1-q+p-2i < q$ $2l+1-q+p-2i < q$	$B^{q,q-p+2i}_l(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_i)$ $B^{q,q-p+2i}_l(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_{p-i})$ $B^{q,2l+1-q+p-2i}_l(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_{q-p+i})$ $B^{q,2l+1-q+p-2i}_l(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_{2l+1-q-i})$ $B^{q,2l+1-q+p-2i}_l(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_{2l+1-q-i})$	$B^{p,q-p+2i}_l(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_{q-p+i})$ $B^{p,q-p+2i}_l(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_{p-i})$ $B^{p,2l+1-q+p-2i}_l(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_i)$ $B^{p,2l+1-q+p-2i}_l(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_i)$ $B^{p,2l+1-q+p-2i}_l(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_{2l+1-q-i})$

$C_l^{l,p}(\mathrm{I}, \mathrm{II}_a)$ $(2p \leqslant l)$		$C_l^l(\mathrm{I}, \varepsilon_p)$	$C_l^{p,l}(\mathrm{II}_a, \mathrm{I})$
$C_{2l}^{2l,l}(\mathrm{I}, \mathrm{II}_b, \varepsilon_i)$	$i=0$ $i=l$	$C_{2l}^l(\mathrm{II}_b, \varepsilon_l)$ $C_{2l}^{2l}(\mathrm{I}, \varepsilon_l)$	$C_{2l}^{l,2l}(\mathrm{II}_b, \mathrm{I})$ $C_{2l}^{l,2l}(\mathrm{II}_b, \mathrm{I}, \varepsilon_l)$
$C_l^{q,p}(\mathrm{II}_a, \mathrm{II}_a, \varepsilon_i)$ $(1 \leqslant p < q \leqslant \frac{1}{2}l)$ $(0 \leqslant i \leqslant p)$	$q-p+2i \leqslant p$ $p < q-p+2i \leqslant \frac{1}{2}l$ $q \leqslant l-q+p-2i \leqslant \frac{1}{2}l$ $p \leqslant l-q+p-2i < q$ $l-q+p-2i < q$	$C_l^{q,q-p+2i}(\mathrm{II}_a, \mathrm{II}_a, \varepsilon_i)$ $C_l^{q,q-p+2i}(\mathrm{II}_a, \mathrm{II}_a, \varepsilon_{p-i})$ $C_l^{q,l-q+p-2i}(\mathrm{II}_a, \mathrm{II}_a, \varepsilon_{q-p+i})$ $C_l^{q,l-q+p-2i}(\mathrm{II}_a, \mathrm{II}_a, \varepsilon_{l-q-i})$ $C_l^{q,l-q+p-2i}(\mathrm{II}_a, \mathrm{II}_a, \varepsilon_{l-q-i})$	$C_l^{p,q-p+2i}(\mathrm{II}_a, \mathrm{II}_a, \varepsilon_{q-p+i})$ $C_l^{p,q-p+2i}(\mathrm{II}_a, \mathrm{II}_a, \varepsilon_{p-i})$ $C_l^{p,l-q+p-2i}(\mathrm{II}_a, \mathrm{II}_a, \varepsilon_i)$ $C_l^{p,l-q+p-2i}(\mathrm{II}_a, \mathrm{II}_a, \varepsilon_i)$ $C_l^{p,l-q+p-2i}(\mathrm{II}_a, \mathrm{II}_a, \varepsilon_{l-q-i})$
$C_{2l}^{l,p}(\mathrm{II}_b, \mathrm{II}_a, \varepsilon_i)$ $(1 \leqslant p \leqslant l)$ $(0 \leqslant i \leqslant p-1)$	$l-p+2i \leqslant p$ $p < l-p+2i \leqslant l$ $p \leqslant l+p-2i < l$ $l+p-2i < l$	$C_{2l}^{l,l-p+2i}(\mathrm{II}_b, \mathrm{II}_a, \varepsilon_i)$ $C_{2l}^{l,l-p+2i}(\mathrm{II}_b, \mathrm{II}_a, \varepsilon_{p-i})$ $C_{2l}^{l,l+p-2i}(\mathrm{II}_b, \mathrm{II}_a, \varepsilon_{l-i})$ $C_{2l}^{l,l+p-2i}(\mathrm{II}_b, \mathrm{II}_a, \varepsilon_{l-i})$	$C_{2l}^{p,l-p+2i}(\mathrm{II}_a, \mathrm{II}_b, \varepsilon_{l-p+i})$ $C_{2l}^{p,l-p+2i}(\mathrm{II}_a, \mathrm{II}_b, \varepsilon_{p-i})$ $C_{2l}^{p,l+p-2i}(\mathrm{II}_a, \mathrm{II}_b, \varepsilon_i)$ $C_{2l}^{p,l+p-2i}(\mathrm{II}_a, \mathrm{II}_b, \varepsilon_{l-i})$
$D_l^{q,p}(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_i)$ $(1 \leqslant p < q \leqslant l-1)$ $(0 \leqslant i \leqslant p-1)$	$q-p+2i \leqslant p$ $p < q-p+2i \leqslant l$ $q \leqslant 2l-q+p-2i < l$ $p \leqslant 2l-q+p-2i < q$ $2l-q+p-2i < q$	$D_l^{q,q-p+2i}(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_i)$ $D_l^{q,q-p+2i}(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_{p-i})$ $D_l^{q,2l-q+p-2i}(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_{q-p+i})$ $D_l^{q,2l-q+p-2i}(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_{2l-q-i})$ $D_l^{q,2l-q+p-2i}(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_{2l-q-i})$	$D_l^{p,q-p+2i}(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_{q-p+i})$ $D_l^{p,q-p+2i}(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_{p-i})$ $D_l^{p,2l-q+p-2i}(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_i)$ $D_l^{p,2l-q+p-2i}(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_i)$ $D_l^{p,2l-q+p-2i}(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_{2l-q-i})$
$D_l^{l,p}(\mathrm{I}_b, \mathrm{I}_a, \varepsilon_i)$ $(1 \leqslant p < l)$ $(0 \leqslant i \leqslant p-1)$	$l-p+2i \leqslant p$ $p < l-p+2i \leqslant l$ $p \leqslant l+p-2i < l$ $l+p-2i < l$	$D_l^{l,l-p+2i}(\mathrm{I}_b, \mathrm{I}_a, \varepsilon_i)$ $D_l^{l,l-p+2i}(\mathrm{I}_b, \mathrm{I}_a, \varepsilon_{p-i})$ $D_l^{l,l+p-2i}(\mathrm{I}_b, \mathrm{I}_a, \varepsilon_{l-i})$ $D_l^{l,l+p-2i}(\mathrm{I}_b, \mathrm{I}_a, \varepsilon_{l-i})$	$D_l^{p,l-p+2i}(\mathrm{I}_a, \mathrm{I}_b, \varepsilon_{l-p+i})$ $D_l^{p,l-p+2i}(\mathrm{I}_a, \mathrm{I}_b, \varepsilon_{p-i})$ $D_l^{p,l+p-2i}(\mathrm{I}_a, \mathrm{I}_b, \varepsilon_i)$ $D_l^{p,l+p-2i}(\mathrm{I}_a, \mathrm{I}_b, \varepsilon_{l-i})$
$D_{2l}^{l,2p}(\mathrm{III}_a, \mathrm{I}_a)$ $(1 \leqslant 2p \leqslant 2l-1)$		$D_{2l}^{l,l}(\mathrm{III}_a, \mathrm{III}_a, \varepsilon_p)$	$D_{2l}^{2p,l}(\mathrm{I}_a, \mathrm{III}_a)$
$D_{2l+1}^{l,2p}(\mathrm{III}_b, \mathrm{I}_a)$ $(1 \leqslant 2p \leqslant 2l)$		$D_{2l+1}^{l}(\mathrm{III}_b, \varepsilon_p)$	$D_{2l+1}^{2p,l}(\mathrm{I}_a, \mathrm{III}_b)$

Table continued

TABLE VII — *Continued*

$(\sigma, \theta\,\mathrm{Int}(\varepsilon_i))$		$(\sigma, \sigma\theta\,\mathrm{Int}(\varepsilon_i))$	$(\theta, \sigma\theta\,\mathrm{Int}(\varepsilon_i))$
$D_{2l}^{l,2l}(\mathrm{III}_a, \mathrm{I}_b, \varepsilon_i)$	$i=0$	$D_{2l}^{l}(\mathrm{III}_a, \varepsilon_l)$	$D_{2l}^{2l,l}(\mathrm{I}_b, \mathrm{III}_a)$
	$i=l$	$D_{2l}^{2l}(\mathrm{I}_b, \varepsilon_l)$	$D_{2l}^{l,2l}(\mathrm{III}_a, \mathrm{I}_b, \varepsilon_l)$
$D_{2l+1}^{l,2l+1}(\mathrm{III}_b, \mathrm{I}_b)$		$D_{2l+1}^{2l+1}(\mathrm{I}_b, \varepsilon_l)$	$D_{2l+1}^{2l+1,l}(\mathrm{I}_b, \mathrm{III}_b)$
$E_6^{6,4}(\mathrm{I}, \mathrm{II}, \varepsilon_i)$	$i=0$	$E_6^{6,2}(\mathrm{I}, \mathrm{IV})$	$E_6^{4,2}(\mathrm{II}, \mathrm{IV})$
	$i=1$	$E_6^{6}(\mathrm{I}, \varepsilon_2)$	$E_6^{4,6}(\mathrm{II}, \mathrm{I}, \varepsilon_1)$
$E_6^{6,2}(\mathrm{I}, \mathrm{III})$		$E_6^{6}(\mathrm{I}, \varepsilon_1)$	$E_6^{2,6}(\mathrm{III}, \mathrm{I})$
$E_6^{4,2}(\mathrm{II}, \mathrm{III}, \varepsilon_i)$	$i=0$	$E_6^{2}(\mathrm{III}, \varepsilon_1)$	$E_6^{2,4}(\mathrm{III}, \mathrm{II})$
	$i=1$	$E_6^{4}(\mathrm{II}, \varepsilon_4)$	$E_6^{2,4}(\mathrm{III}, \mathrm{II}, \varepsilon_1)$
$E_6^{2,2}(\mathrm{III}, \mathrm{IV})$		$E_6^{2}(\mathrm{IV}, \varepsilon_1)$	$E_6^{2,2}(\mathrm{IV}, \mathrm{III})$
$E_7^{7,4}(\mathrm{V}, \mathrm{VI}, \varepsilon_i)$	$i=0$	$E_7^{7,3}(\mathrm{V}, \mathrm{VII})$	$E_7^{4,3}(\mathrm{VI}, \mathrm{VII}, \varepsilon_1)$
	$i=1$	$E_7^{7}(\mathrm{V}, \varepsilon_1)$	$E_7^{4,7}(\mathrm{VI}, \mathrm{V}, \varepsilon_1)$
$E_7^{7,3}(\mathrm{V}, \mathrm{VII}, \varepsilon_1)$		$E_7^{7}(\mathrm{V}, \varepsilon_7)$	$E_7^{4,7}(\mathrm{VI}, \mathrm{V}, \varepsilon_1)$
$E_7^{4,3}(\mathrm{VI}, \mathrm{VII})$		$E_7^{3}(\mathrm{VII}, \varepsilon_3)$	$E_7^{3,4}(\mathrm{VII}, \mathrm{VI})$
$E_8^{8,4}(\mathrm{VIII}, \mathrm{IX}, \varepsilon_i)$	$i=0$	$E_8^{4}(\mathrm{IX}, \varepsilon_4)$	$E_8^{4,8}(\mathrm{IX}, \mathrm{VIII})$
	$i=1$	$E_8^{8}(\mathrm{VIII}, \varepsilon_8)$	$E_8^{4,8}(\mathrm{IX}, \mathrm{VIII}, \varepsilon_1)$
$F_4^{4,1}(\mathrm{I}, \mathrm{II})$		$F_4^{4}(\mathrm{I}, \varepsilon_1)$	$F_4^{4}(\mathrm{II}, \mathrm{I})$

10. Classification of Semisimple Symmetric Spaces

In this section we shall show that there is a bijection between the set of isomorphism classes of locally semisimple symmetric spaces and the isomorphism classes of (ordered) pairs of commuting involutions as treated in the Sections 7–8. Moreover, the fine structure as developed for pairs of commuting involutions, transfers directly to the corresponding symmetric space.

10.1. Let G_0 be a real semisimple connected Lie group and denote its Lie algegra by $\mathfrak{g}_0$. Let $\sigma \in \mathrm{Aut}(G_0)$ be an involutorial automorphism and let H be a closed subgroup of G_0 satisfying $(G_0)_\sigma^0 \subset H \subset (G_0)_\sigma$. If $\mathfrak{h}$ denotes the Lie algebra of H (or $(G_0)_\sigma$), then the pair (G_0, H) is called a *semisimple symmetric pair* and $(\mathfrak{g}_0, \mathfrak{h})$ a *semisimple locally symmetric pair.* We shall write also $(\mathfrak{g}_0, \sigma)$ instead of $(\mathfrak{g}_0, \mathfrak{h})$. The symmetric space G_0/H is called an *affine symmetric space.* There is a bijection between the set of locally semisimple symmetric pairs and the set of affine symmetric spaces $G_0/(G_0)_\sigma^0$. We will restrict our analysis to the semisimple locally symmetric pairs. Two semisimple locally symmetric pairs $(\mathfrak{g}_0, \mathfrak{h}_1)$ and $(\mathfrak{g}_0, \mathfrak{h}_2)$ are *isomorphic under an inner* (resp. *outer*) *automorphism* if there exists $\phi \in \mathrm{Aut}(G)^0$ (resp. $\mathrm{Aut}(G)$) such that $\phi(\mathfrak{g}_0) = \mathfrak{g}_0$ and $\phi(\mathfrak{h}_1) = \mathfrak{h}_2$.

Let $\mathfrak{g}$ denote the complexification of $\mathfrak{g}_0$ and let $G = \mathrm{Aut}(\mathfrak{g})^0$. An semisimple symmetric pair determines a pair of commuting involutions of $\mathfrak{g}$.

10.2. Proposition. *Let $(\mathfrak{g}_0, \sigma)$ be a semisimple locally symmetric pair. Then there exists a Cartan involution θ of $\mathfrak{g}_0$ such that $\sigma\theta = \theta\sigma$.*

This is proved in Berger [2].

If $\theta_1, \theta_2 \in \mathrm{Aut}(\mathfrak{g}_0)$ are Cartan involutions satisfying $\theta_i\sigma = \sigma\theta_i$ $(i = 1, 2)$, then there exists $Y \in \mathfrak{h}$ such that $\exp Y\theta_1 \exp - Y = \theta_2$ (see Matsuki [16]). In other words, if we lift σ, θ_1 and θ_2 to involutions of $\mathfrak{g}$, then the pairs (σ, θ_1) and (σ, θ_2) are isomorphic in the sense of (5.15). Conversely, starting with a pair of commuting involutions of $\mathfrak{g}$, we obtain a semisimple locally symmetric pair, This follows from the following result.

10.3. Lemma. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\theta_1, ..., \theta_n$ commuting involutorial automorphisms of $\mathfrak{g}$. Then there exists a compact real form $\mathfrak{u}$ of $\mathfrak{g}$, with conjugation τ, such that $\theta_i\tau = \tau\theta_i$ for $i = 1, ..., n$.*

This result is discussed in the thesis of B. Hoogenboom [13]. Another proof goes as follows. Let R denote the subgroup of $\mathrm{Aut}(\mathfrak{g})$ spanned by $\theta_1, ..., \theta_n$. Since R is a compact subgroup of $\mathrm{Aut}(\mathfrak{g})$, there exists a maximal compact subgroup U of $\mathrm{Aut}(\mathfrak{g})$ containing R. Since U is maximal compact, also $U \cap \mathrm{Aut}(\mathfrak{g})^0$ is a maximal compact subgroup of $\mathrm{Aut}(\mathfrak{g})^0$ and its Lie algebra $\mathfrak{u}$ satisfies the above properties.

10.4. *Dual and associated pairs.* Let (θ, σ) be a pair of commuting involutions of the complex Lie algebra $\mathfrak{g}$ and $\mathfrak{u}$ a (θ, σ)-stable compact real form of $\mathfrak{g}$ with conjugation τ. Denote $\theta\tau$ by $\bar{\theta}$ and $\sigma\tau$ by $\bar{\sigma}$.

For a pair (θ, σ), the first involution determines a real form. Then $(\mathfrak{g}^{\bar{\theta}}, \sigma \mid \mathfrak{g}^{\bar{\theta}})$ is a locally semisimple symmetric pair, corresponding to (θ, σ). Here $\mathfrak{g}^{\bar{\theta}}$ is the set of fixed points of the conjugation $\bar{\theta}$ in $\mathfrak{g}$. The set of fixed points of σ in $\mathfrak{g}^{\bar{\theta}}$ will be denoted by $\mathfrak{g}^{\bar{\theta}}_{\sigma}$. It follows from Helgason [11, Chap. X, 1.4] that the isomorphism class of $(\mathfrak{g}^{\bar{\theta}}, \sigma \mid \mathfrak{g}^{\bar{\theta}})$ does not depend on the choice of the (σ, θ)-stable compact real form $\mathfrak{u}$ of $\mathfrak{g}$.

The pair (σ, θ) is called the *dual pair* of (θ, σ) and the corresponding semisimple locally symmetric pair $(\mathfrak{g}^{\bar{\sigma}}, \mathfrak{g}^{\bar{\sigma}}_{\theta})$ is called the *dual pair* of $(\mathfrak{g}^{\bar{\theta}}, \mathfrak{g}^{\bar{\theta}}_{\sigma})$. Similarly the pair $(\theta, \sigma\theta)$ will be called the *associated pair* of (θ, σ) and $(\mathfrak{g}^{\bar{\theta}}, \mathfrak{g}^{\bar{\theta}}_{\theta\sigma})$ *the associated symmetric pair* of $(\mathfrak{g}^{\bar{\theta}}, \mathfrak{g}^{\bar{\theta}}_{\sigma})$.

10.5. Let (θ, σ), $\bar{\theta}$ and $\mathfrak{g}^{\bar{\theta}}$ be as in (9.4). We can lift (θ, σ) to a pair of commuting involutions of $G = \mathrm{Aut}(\mathfrak{g})^0$, which we denote also by (θ, σ). The pairs of commuting involutions of G correspond bijectively with the pairs of commuting involutions of $\mathfrak{g}$.

The tori occurring in Sections 1–8 correspond to the following subspaces of $\mathfrak{g}^{\bar{\theta}}$. Let $\mathfrak{g}^{\bar{\theta}} = \mathfrak{k} \oplus p$ be the usual decomposition in eigenspaces of θ (i.e., a Cartan decomposition of $\mathfrak{g}^{\bar{\theta}}$). Likewise let $\mathfrak{g}^{\bar{\theta}} = \mathfrak{h} \oplus \mathfrak{q}$ be the decomposition in eigenspaces of $\sigma \mid \mathfrak{g}^{\bar{\theta}}$. Now θ-split (resp. σ-split and (σ, θ)-split) tori of G correspond to Cartan subspaces of $\mathfrak{p}$ (resp. $\mathfrak{q}$ and $\mathfrak{p} \cap \mathfrak{q}$).

The characterization of the pairs of commuting involutions of G in Sections 5.8 gives a characterization of locally semisimple symmetric pairs in terms of a (σ, θ)-stable Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}^{\bar{\theta}}$, such that $\mathfrak{t} \cap \mathfrak{p}$ (resp. $\mathfrak{t} \cap \mathfrak{q}$, resp. $\mathfrak{t} \cap \mathfrak{p} \cap \mathfrak{q}$) is maximal abelian in $\mathfrak{p}$ (resp. $\mathfrak{q}$, resp. $\mathfrak{p} \cap \mathfrak{q}$). These Cartan subalgebras of $\mathfrak{g}^{\bar{\theta}}$ are frequently used in the analysis on semisimple symmetric spaces (see [18]).

A symmetric pair is called *irreducible* if the adjoint representation of $\mathfrak{h}$ on $\mathfrak{q}$ is irreducible. This is equivalent to the notion of irreducibility defined in (7.17). From (8.26) we obtain now.

10.6. THEOREM. *The inner (resp. outer) isomorphism classes of the semisimple locally symmetric pairs* $(\mathfrak{g}_0, \sigma)$ *correspond bijectively to the inner (resp. outer) isomorphism classes of ordered pairs of commuting involutions* (θ, σ) *of* $\mathfrak{g}$ *or* $\mathrm{Aut}(\mathfrak{g})^0$. *Here* $\mathfrak{g}$ *denotes the complexification of* $\mathfrak{g}_0$ *and* $\theta \mid \mathfrak{g}_0$ *a Cartan involution of* $\mathfrak{g}_0$ *commuting with* σ. *In particular, a pair* (S, ε_λ), *where* S *is an admissible irreducible* (θ, σ)*-index and* ε_λ *one of the quadratic elements occurring with this* (θ, σ)*-index in* (7.18.1)–(7.18.6), *represents the isomorphism class of an irreducible semisimple locally symmetric pair and its dual.*

Berger [2] only classified the semisimple locally symmetric pairs under the action of the full automorphism group. As one can see from (9.4) some of the inner isomorphism classes are identified under an outer automorphism. In order to identify the above results with those of Berger [2], we listed in Tables II and VIII the subalgebras $\mathfrak{g}_\sigma^{\bar\theta}(\varepsilon_i)$, where $\mathfrak{g}_\sigma^{\bar\theta}(\varepsilon_i)$ denotes the set of fixed points of $\sigma\,\mathrm{Int}(\varepsilon_i)$ in $\mathfrak{g}^{\bar\theta}$.

Finally the associated pairs can be derived from Table VII. These are among others of importance in the Fourier analysis on symmetric spaces and also in descriptions of orbits of semisimple symmetric spaces under the action of minimal parabolic subgroups (see Matsuki [16]).

10.7. A pair (θ, σ) is called *self-dual* if (θ, σ) is isomorphic to (σ, θ) and *self-associated* if (θ, σ) is isomorphic to $(\theta, \sigma\theta)$ or equivalently if the associated dual pair is self-dual. These pairs can be characterized as follows.

10.8. Lemma. *Let $(\theta, \sigma) \in \mathrm{Aut}(G)$ be a pair of commuting involutions. Then the following are equivalent:*

(1) *(θ, σ) is self-dual,*

(2) *θ is isomorphic to σ,*

(3) *there is a maximal θ-split torus A of G and a quadratic element $\varepsilon \in A$ such that $(\theta, \sigma) = (\theta, \theta\,\mathrm{Int}(\varepsilon))$.*

Proof. Since $(3) \Rightarrow (1)$ follows immediately from (5.21(ii)) and since $(1) \Rightarrow (2)$ is obvious it suffices to prove $(2) \Rightarrow (3)$. If $g \in G$ such that $\mathrm{Int}(g)\sigma\,\mathrm{Int}(g^{-1}) = \theta$, then $\theta = \sigma\,\mathrm{Int}(\sigma(g)\,g^{-1})$. By a result of Richardson [20, 6.3] there is a maximal σ-split torus A of G such that $\sigma(g)\,g^{-1} \in A$. Since A is also maximal θ-split, the result is clear.

10.9. *Remark.* If $\theta \in \mathrm{Aut}(G)$ is an involution, A a minimal θ-split torus of G and $\varepsilon \in A$, $\varepsilon^2 = e$, then the pair $(\theta, \theta\,\mathrm{Int}(\varepsilon))$ corresponds to a symmetric pair of type K_ε, as introduced by Oshima and Sekiguchi [18]. From the above result it is now clear that a symmetric pair $(\mathfrak{g}^{\bar\theta}, \sigma)$ is of type K_ε if and only if $\mathfrak{g}_\theta$ and $\mathfrak{g}_\sigma$ are isomorphic.

In [2, 19] a semisimple symmetric pair is called self-dual if it is isomorphic to its dual under an outer automorphism. So in this case the pairs in (7.20) and (9.5) are also self-dual.

10.10. *Remark.* Berger [2] gives a description of how one can obtain the affine symmetric spaces from the semisimple locally symmetric pairs and the fundamental groups. Together with more recent, detailed descriptions of the fundamental groups (see Takeuchi [28] and Goto and Kobayashi [10]) a complete description of the global pairs can be obtained.

TABLE VIII

(σ, θ)-index		$\mathfrak{g}_\theta^{\bar\sigma}(\varepsilon_i)$	$\mathfrak{g}_\sigma^{\bar\theta}(\varepsilon_i)$
$A_{2l+1}^{2l+1,l}(\mathrm{I}, \mathrm{II})$		$\mathfrak{sp}(l+1, \mathbb{R})$	$\mathfrak{so}^*(2l+2)$
$A_{2l-1}^{2l-1,l}(\mathrm{I}, \mathrm{III}_b, \varepsilon_i)$	$i=0$	$\mathfrak{sl}(l, \mathbb{C}) + \mathfrak{so}(2)$	$\mathfrak{so}^*(2l)$
	$i=l$	$\mathfrak{sl}(l, \mathbb{R}) + \mathfrak{sl}(l, \mathbb{R}) + \mathbb{R}$	$\mathfrak{so}(l, l)$
$A_{4l-1}^{2l,2l-1}(\mathrm{III}_b, \mathrm{II}, \varepsilon_i)$	$i=0$	$\mathfrak{sp}(l, l)$	$\mathfrak{su}^*(2l) + \mathfrak{su}^*(2l) + \mathbb{R}$
	$i=l$	$\mathfrak{sl}(2l, \mathbb{R})$	$\mathfrak{sl}(2l, \mathbb{C}) + \mathfrak{so}(2)$
$A_{4l+1}^{2l+1,2l}(\mathrm{III}_b, \mathrm{II})$		$\mathfrak{sp}(2l+1, \mathbb{R})$	$\mathfrak{sl}(2l+1, \mathbb{C}) + \mathfrak{so}(2)$
$A_l^{l,p}(\mathrm{I}, \mathrm{III}_a)$		$\mathfrak{sl}(p, \mathbb{R}) + \mathfrak{sl}(l-p+1, \mathbb{R}) + \mathbb{R}$	$\mathfrak{so}(p, l-p+1)$
$A_{4l-1}^{2p,2l-1}(\mathrm{III}_a, \mathrm{II})$		$\mathfrak{sp}(p, 2l-p)$	$\mathfrak{su}^*(2p) + \mathfrak{su}^*(4l-2p) + \mathbb{R}$
$A_{4l+1}^{2p,2l}(\mathrm{III}_a, \mathrm{II})$		$\mathfrak{sp}(p, 2l+1-p)$	$\mathfrak{su}^*(2p) + \mathfrak{su}^*(4l+2-2p) + \mathbb{R}$
$A_{2l-1}^{l,p}(\mathrm{III}_b, \mathrm{III}_a, \varepsilon_i)$	$0 \leqslant i \leqslant p-1$	$\mathfrak{su}(i, p-i) + \mathfrak{su}(l-i, l-p+i) + \mathfrak{so}(2)$	$\mathfrak{su}(i, l-i) + \mathfrak{su}(p-i, l-p+i) + \mathfrak{so}(2)$
$A_l^{q,p}(\mathrm{III}_a, \mathrm{III}_a, \varepsilon_i)$	$0 \leqslant i \leqslant p$	$\mathfrak{su}(l+1-q-i, q-p+i)$ $+ \mathfrak{su}(i, p-i) + \mathfrak{so}(2)$	$\mathfrak{su}(i, l+1-q-i)$ $+ \mathfrak{su}(p-i, q-p+i) + \mathfrak{so}(2)$
$B_l^{q,p}(\mathrm{I}, \mathrm{I}, \varepsilon_i)$	$0 \leqslant i \leqslant p$	$\mathfrak{so}(i, p-i) + \mathfrak{so}(2l+1-q-i, q-p+i)$	$\mathfrak{so}(i, 2l+1-q-i) + \mathfrak{so}(p-i, q-p+i)$
$C_l^{l,p}(\mathrm{I}, \mathrm{II}_a)$		$\mathfrak{sp}(p, \mathbb{R}) + \mathfrak{sp}(l-p, \mathbb{R})$	$\mathfrak{su}(p, l-p) + \mathfrak{so}(2)$
$C_{2l}^{2l,l}(\mathrm{I}, \mathrm{II}_b, \varepsilon_i)$	$i=0$	$\mathfrak{sp}(l, \mathbb{C})$	$\mathfrak{su}(l, l) + \mathrm{R}$
	$i=l$	$\mathfrak{sp}(l, \mathbb{R}) + \mathfrak{sp}(l, \mathbb{R})$	$\mathfrak{su}(l, l) + \mathfrak{so}(2)$
$C_l^{q,p}(\mathrm{II}_a, \mathrm{II}_a, \varepsilon_i)$	$0 \leqslant i \leqslant p$	$\mathfrak{sp}(i, p-i) + \mathfrak{sp}(l-q-i, q-p+i)$	$\mathfrak{sp}(i, l-q-i) + \mathfrak{sp}(p-i, q-p+i)$
$C_{2l}^{l,p}(\mathrm{II}_b, \mathrm{II}_a, \varepsilon_i)$	$0 \leqslant i \leqslant p-1$	$\mathfrak{sp}(i, p-i) + \mathfrak{sp}(l-i, l-p+i)$	$\mathfrak{sp}(i, l-i) + \mathfrak{sp}(p-i, l-p+i)$
$D_l^{q,p}(\mathrm{I}_a, \mathrm{I}_a, \varepsilon_i)$	$0 \leqslant i \leqslant p$	$\mathfrak{so}(i, p-i) + \mathfrak{so}(2l-q-i, q-p+i)$	$\mathfrak{so}(i, 2l-q-i) + \mathfrak{so}(p-i, q-p+i)$
$D_l^{p,l}(\mathrm{I}_a, \mathrm{I}_b, \varepsilon_i)$	$0 \leqslant i \leqslant p-1$	$\mathfrak{so}(i, l-i) + \mathfrak{so}(p-i, l-p+i)$	$\mathfrak{so}(i, p-i) + \mathfrak{so}(l-i, l-p+i)$

$D_{2l}^{l,2p}(\mathrm{III}_a, \mathrm{I}_a)$		$\mathfrak{so}^*(2p)+\mathfrak{so}^*(4l-2p)$	$\mathfrak{su}(p, 2l-p)+\mathfrak{so}(2)$
$D_{2l+1}^{l,2p}(\mathrm{III}_b, \mathrm{I}_a)$		$\mathfrak{so}^*(2p)+\mathfrak{so}^*(4l-2p+2)$	$\mathfrak{su}(p, 2l+1-p)+\mathfrak{so}(2)$
$D_{2l}^{l,2l}(\mathrm{III}_a, \mathrm{I}_b, \varepsilon_i)$	$i=0$ $i=l$	$\mathfrak{so}^*(2l)+\mathfrak{so}^*(2l)$ $\mathfrak{so}(2l, \mathbb{C})$	$\mathfrak{su}(l, l)+\mathfrak{so}(2)$ $\mathfrak{sl}(2l, \mathbb{R})+\mathbb{R}$
$D_{2l+1}^{l,2l+1}(\mathrm{III}_b, \mathrm{I}_b)$		$\mathfrak{so}(2l+1, \mathbb{C})$	$\mathfrak{sl}(2l+1, \mathbb{R})+\mathbb{R}$
$D_{2l}^{l,l}(\mathrm{III}_a, \mathrm{III}_a, \varepsilon_i)$	$0\leqslant 2i\leqslant l-1$	$\mathfrak{su}(2i+1, 2l-2i-1)$	$\mathfrak{su}(2i+1, 2l-2i-1)$
$E_6^{6,4}(\mathrm{I}, \mathrm{II}, \varepsilon_i)$	$i=0$ $i=1$	$\mathfrak{su}^*(6)+\mathfrak{su}(2)$ $\mathfrak{sl}(6, \mathbb{R})+\mathfrak{sl}(2, \mathbb{R})$	$\mathfrak{sp}(3, 1)$ $\mathfrak{sp}(4, \mathbb{R})$
$E_6^{6,2}(\mathrm{I}, \mathrm{III})$		$\mathfrak{so}(5, 5)+\mathbb{R}$	$\mathfrak{sp}(2, 2)$
$E_6^{6,2}(\mathrm{I}, \mathrm{IV})$		$F\mathrm{I}$	$\mathfrak{sp}(2, 2)$
$E_6^{4,2}(\mathrm{II}, \mathrm{III}, \varepsilon_i)$	$i=0$ $i=1$	$\mathfrak{so}^*(10)+\mathfrak{so}(2)$ $\mathfrak{so}(6, 4)+\mathfrak{so}(2)$	$\mathfrak{su}(5, 1)+\mathfrak{sl}(2, \mathbb{R})$ $\mathfrak{su}(4, 2)+\mathfrak{su}(2)$
$E_6^{4,2}(\mathrm{II}, \mathrm{IV})$		$F\mathrm{I}$	$\mathfrak{su}^*(6)+\mathfrak{su}(2)$
$E_6^{2,2}(\mathrm{III}, \mathrm{IV})$		$F\mathrm{II}$	$\mathfrak{so}(9, 1)+\mathbb{R}$
$E_7^{7,4}(\mathrm{V}, \mathrm{VI}, \varepsilon_i)$	$i=0$ $i=1$	$\mathfrak{so}^*(12)+\mathfrak{su}(2)$ $\mathfrak{so}(6, 6)+\mathfrak{sl}(2, \mathbb{R})$	$\mathfrak{su}(6, 2)$ $\mathfrak{su}(4, 4)$
$E_7^{7,3}(\mathrm{V}, \mathrm{VII}, \varepsilon_i)$	$i=0$ $i=1$	$E\mathrm{II}+\mathfrak{so}(2)$ $E\mathrm{I}+\mathbb{R}$	$\mathfrak{su}(6, 2)$ $\mathfrak{su}^*(8)$
$E_7^{4,3}(\mathrm{VI}, \mathrm{VII}, \varepsilon_i)$	$i=0$ $i=1$	$E\mathrm{III}+\mathfrak{so}(2)$ $E\mathrm{II}+\mathfrak{so}(2)$	$\mathfrak{so}(10, 2)+\mathfrak{sl}(2, \mathbb{R})$ $\mathfrak{so}^*(12)+\mathfrak{su}(2)$
$E_8^{8,4}(\mathrm{VIII}, \mathrm{IX}, \varepsilon_i)$	$i=0$ $i=1$	$E\mathrm{VI}+\mathfrak{su}(2)$ $E\mathrm{V}+\mathfrak{sl}(2, \mathbb{R})$	$\mathfrak{so}(12, 4)$ $\mathfrak{so}^*(16)$
$F_4^{4,1}(\mathrm{I}, \mathrm{II})$		$\mathfrak{so}(5, 4)$	$\mathfrak{sp}(2, 1)+\mathfrak{su}(2)$

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