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An Analytical Solution for a Nonlinear Differential Equation with Logarithmic Decay

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The problem is the ordinary differential equation $du/dt = -p\exp(-q/u)$ with $u(0) = \alpha$. We prove that the general three-parameter problem may be reduced through a group invariance to computing a single, parameter-free function $w(t)$. This solution may be most compactly expressed in the form $w = 1/\ln(t^*\ln^2(t^*)B(\tau))$, where $\tau \equiv \ln(\ln(t^*))$ and $t^* = t + \exp(1)$. We compute a Chebyshev series for $B(\tau)$ for $\tau \in [0,6]$ and show that $B(\tau) \sim 1 + (4\tau - 2)\exp(-\tau) + 4\tau^2\exp(-2\tau)$ to within 1 part in 10^4 for $\tau > 6$. The problem was motivated by the similar logarithmic decay of certain classes of quasi-solitary waves, such as the "breather" of the ϕ^4 field theory.

The problem posed in the abstract is tantalizingly simple. Nevertheless, the multiplicity of time scales—time, the logarithm of time, and even the logarithm of the logarithm—make it difficult to understand the asymptotic behavior and develop overlapping approximations which collectively are accurate for all time.

In this note, we compute explicit, analytic approximations to $u(t;\, p, q, \alpha)$ which are valid for all values of the parameters and t. One enormous simplification is that this three-parameter problem is reduced to the computation of a *single*, *universal* function through the following.

THEOREM. *Let $w(t)$ solve the problem*

$$dw/dt = -\exp(-1/w) \qquad \textit{and} \qquad w(0) = 1. \tag{1}$$

Then the solution of the general, three-parameter problem

$$du/dt = -p\exp(-q/u) \qquad \textit{and} \qquad u(0) = \alpha \tag{2}$$

may be expressed in terms of $w(t)$ as

$$u(t;\, p, q, \alpha) = qw([p/q](t - \psi)), \tag{3}$$

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where

$$\psi = -[q/p]w^{-1}(\alpha/q). \tag{4}$$

Proof. Direct substitution of (3) and (4) into (2), cancellation of common factors of the parameters, and invocation of (1).

Since the RHS of (1) is negative definite, it follows that $w(t)$ (and $u(t;\, p, q, \alpha)$) are *monotonically decreasing* for all t. This implies that the inverse function $w^{-1}(t)$ is singly-branched for real t so that the theorem may be applied for all t and all parameter values. Because of this group invariance theorem, the rest of this note will concentrate on solving the reduced problem (1) for $w(t)$.

A useful hint comes from observing that the related problem

$$dy/dt = -y^2\exp(-1/y) \qquad \text{and} \qquad y(0) = 1 \tag{5}$$

has the exact solution

$$y = 1/\ln(t^*), \tag{6}$$

where

$$t^* \equiv t + \exp(1). \tag{7}$$

Because y^2 is monotonically decreasing as y decreases, it follows that $w(t)$ will decay faster than $y(t)$ as $t \to \infty$. However, because y^2 varies slowly in comparison to $\exp(-1/y)$, which is the factor common to both problems, it follows that $w(t)$ should decay in a fashion that closely resembles $1/\ln(t^*)$. This suggests introducing a new variable $A(t^*)$ to "modulate" the logarithmic decay: we define A via

$$w(t) \equiv 1/\ln(t^*A). \tag{8}$$

Introducing the logarithmic time

$$T \equiv \ln(t^*), \tag{9}$$

we find from (1) that A must solve

$$dA/dT = -A + T^2 + 2T\ln(A) + \ln^2(A) \qquad \text{and} \qquad A(1) = 1. \tag{10}$$

Inspection of (10) shows that for $T \gg 1$, $A(T)$ will grow or decay exponentially fast—violating our intuition that A represents a slow modulation of the basic logarithmic decay—unless the first two terms on the RHS approximately cancel, implying that

$$A(T) \sim T^2 \qquad T \to \infty. \tag{11}$$

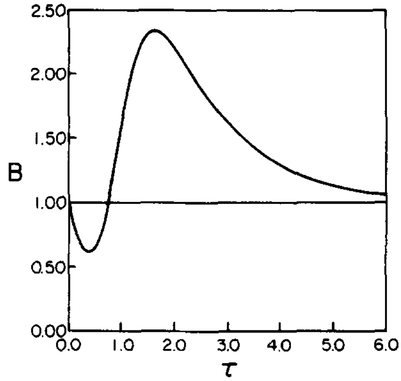

FIG. 1. $B(\tau)$, where $\tau = \ln(\ln[t + \exp(1)])$.

This suggests introducing both a new time scale

$$\tau \equiv \ln(T) = \ln[\ln(t^*)] \tag{12}$$

and a new unknown $B(\tau)$ to describe the slow modulations of the modulation function $A(T)$, where

$$w(t) \equiv 1/\ln[t^* T^2 B(\tau)] \tag{13}$$

and $B(\tau)$ satisfies

$$dB/d\tau = (1 - B)\, e^{\tau} - 2B + 4\tau + \ln(B^2) + e^{-\tau}[2\tau + \ln(B)]^2$$

and

$$B(0) = 1. \tag{14}$$

The reasoning that led from (1) to (14) is heuristic and intuitive. Nonetheless, we can give an *a posteriori* justification for the change of variables by numerically integrating (14) using a fourth-order Runge–Kutta method. Figure 1 shows that the function $B(\tau)$ does indeed vary smoothly between narrow limits. It is noteworthy that the right-hand limit of the graph, $\tau = 6$, corresponds to $T \approx 403$ and $t \approx 1.61 \times 10^{175}$. Integrating to so large a time with a fixed time step would take longer than the age of the universe, even on a supercomputer! Through the transformations, however, we have captured the behavior of $w(t)$ over the whole of this enormous interval.

Although no conventional analytic solution of (1) is known, we can nevertheless write down a simple, analytical expression for $w(t)$ by computing a Chebyshev series for $B(\tau)$. Since the Chebyshev interpolation points are unevenly spaced, it is convenient to recall that a Chebyshev series is merely a Fourier cosine series in disguise. Introducing the change of variable

$$\tau = 3(1 - \cos(\theta)) \tag{15}$$

with the explicit inverse

$$\theta = \arccos(1 - \tau/3), \tag{16}$$

we rewrite (14) in terms of the new coordinate θ and then solve the initial value problem using an evenly spaced grid in θ so as to obtain $B(\tau[\theta])$ at the points

$$\tau_i \equiv 3(1 - \cos[\pi(2i + 1)/(2N + 2)], \qquad i = 0, \ldots, N. \tag{17}$$

Then

$$B(\tau) \approx \sum_{n=0}^{N} a_n \cos(n\theta[\tau]), \qquad \tau \in [0, 6], \tag{18}$$

where the Fourier coefficients are, with $d_n = 1$ except for $d_0 = 2$,

$$a_n = (2/(d_n[N + 1])) \sum_{i=0}^{N} B(\tau_i)\cos(n\theta[\tau_i]), \qquad n = 0, \ldots, N. \tag{19}$$

The theory of Chebyshev series is fully discussed in [1], so we omit details. The first 31 coefficients, computed using $N = 40$, are given in Table I. The approximation has a maximum absolute error of less than 0.0001.

The choice of $\tau = 6$ as the upper limit of the expansion interval is somewhat arbitrary, but Fig. 1 shows that the modulation function is asymptotic to 1 from above. This immediately gives an explicit asymptotic approximation for $B(\tau)$ because a term on the RHS of (14) such as $(-2B + 4\tau)$ will create rapid growth in B unless approximately cancelled by other terms. Elementary algebra then gives

$$B(\tau) \sim 1 + (4\tau - 2)\, e^{-\tau} + 4\tau^2 e^{-2\tau}, \qquad \tau \gg 1. \tag{20}$$

Table II shows that the approximation is extremely accurate even for moderate τ.

The accuracy of (20) plus the Chebyshev series is roughly 1 part in 10,000 for all $t > 0$. In contrast, approximating $w(t)$ by $1/\ln(t^*)$ is a mediocre approximation even when t is huge. If we write

$$\ln(t^* T^2 B(\tau)) = \ln(t^*) + \ln[\ln^2(t^*)] + \ln\{B(\ln[\ln(t^*)]\} \tag{21}$$

TABLE I
Chebyshev–Fourier Coefficients for $B(\tau)$

n	a_n	n	a_n
0	1.29371	16	−0.002928
1	0.036521	17	−0.002653
2	−0.466685	18	−0.001319
3	−0.326637	19	−0.000205
4	0.045723	20	0.000355
5	0.242301	21	0.000480
6	0.206837	22	0.000366
7	0.073008	23	0.000170
8	−0.030902	24	−0.0
9	−0.060218	25	−0.000091
10	−0.037641	26	−0.000101
11	−0.004572	27	−0.000065
12	0.013605	28	−0.000021
13	0.014275	29	0.000010
14	0.006738	30	0.000022
15	−0.000108		

we find that even for $t = 1{,}000{,}000$, the three terms on the right are respectively 13.8, 5.25, and 1.81. It is impossible to compute $w(t)$ other than crudely without accurately evaluating the second modulation function, $B(\ln[\ln(t^*)])$.

Strictly speaking, to complete a global solution for $u(t;\, p, q, \alpha)$ we need to also solve for $w(t)$ for negative t. It is trivial to show that, defining

TABLE II
$B(\tau)$ and Absolute Errors in the Asymptotic Approximation to $B(\tau)$

τ	T	B	Error
1.	2.71	1.58231	−0.69
2.	7.39	2.20295	0.098
3.	20.1	1.61134	0.024
4.	54.6	1.28209	0.0042
5.	148	1.12653	0.00070
6.	403.	1.05553	0.00012
7.	1096.	1.02389	0.000019
8.	2981.	1.0101	0.0000029
9.	8103.	1.0042	0.00000046
10.	22026.	1.00173	0.000000070

$t' \equiv t + \sigma$, where the constant σ (≈ -0.40) is determined by matching the condition $w(0) = 1$,

$$w(t) \sim -t' - \ln(-t') + O(\ln[t']/t') \qquad \text{as} \qquad t \to -\infty \tag{22}$$

By using the methods of Boyd [2], it is possible to compute a Chebyshev expansion on the semi-infinite interval $t \in [-\infty, 0]$, but since the physical interest is in the decay, we omit the details.

In summary, we see that by introducing successively slower time scales and changing variables so that successively higher "modulation" functions become the unknowns, we may tame this nonlinear differential equation and solve it for arbitrary positive t. The combination of Chebyshev expansion with asymptotic approximation is very powerful because together they give a "global" solution.

The inspiration for this problem is the "breather" of the ϕ^4 field theory [3], which also decays with time at a rate which is proportional to the inverse exponential of the amplitude. From this example, we may anticipate a logarithmic time decay accompanied by modulation on a log–log time scale. However, the ϕ^4 problem is a *partial* rather than an ordinary differential equation, and is obviously much harder. Only future work will determine whether some of the ideas developed here can be applied to that much more difficult problem and others like it.

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